WORKS PUBLISHED BY THE ROYAL SOCIETY.

Demy 8vo, pp. 243. Price 5s.

THE YEAR-BOOK OF THE ROYAL SOCIETY, 1901.

Containing an Account of the Chief Events and Arrangements of the Year.

Ready shortly. Demy 8vo, pp. 427.

THE RECORD OF THE ROYAL SOCIETY.

Second Edition. 1901.

Containing an Account of the Foundation, History, &c., of the Society, with a Chronological List of the Fellows from the foundation.

CONTENTS.

(A)

VOL. 197.

[iv]

[**v**]

LIST OF ILLUSTRATIONS.

ADVERTISEMENT.

THE Committee appointed by the *Royal Society* to direct the publication of the *Philosophical Transactions* take this opportunity to acquaint the public that it fully appears, as well from the Council-books and Journals of the Society as from repeated declarations which have been made in several former *Transactions*, that the printing of them was always, from time to time, the single act of the respective Secretaries till the Forty-seventh Volume; the Society, as a Body, never interesting themselves any further in their publication than by occasionally recommending the revival of them to some of their Secretaries, when, from the particular circumstances of their affairs, the *Transactions* had happened for any length of time to be intermitted. And this seems principally to have been done with a view to satisfy the public that their usual meetings were then continued, for the improvement of knowledge and benefit of mankind : the great ends of their first institution by the Royal Charters, and which they have ever since steadily pursued.

But the Society being of late years greatly enlarged, and their communications more numerous, it was thought advisable that a Committee of their members should be appointed to reconsider the papers read before them, and select out of them such as they should judge most proper for publication in the future *Transactions*; which was accordingly done upon the 26th of March, 1752. And the grounds of their choice are, and will continue to be, the importance and singularity of the subjects, or the advantageous manner of treating them ; without pretending to answer for the certainty of the facts, or propriety of the reasonings contained in the several papers so published, which must still rest on the credit or judgment of their respective authors.

It is likewise necessary on this occasion to remark, that it is an established rule of the Society, to which they will always adhere, never to give their opinion, as a Body,

upon any subject, either of Nature or Art, that comes before them. And therefore the thanks, which are frequently proposed from the Chair, to be given to the authors of such papers as are read at their accustomed meetings, or to the persons through whose hands they received them, are to be considered in no other light than as a matter of civility, in return for the respect shown to the Society by those communications. The like also is to be said with regard to the several projects, inventions, and curiosities of various kinds, which are often exhibited to the Society; the authors whereof, or those who exhibit them, frequently take the liberty to report, and even to certify in the public newspapers, that they have met with the highest applause and approbation. And therefore it is hoped that no regard will hereafter be paid to such reports and public notices; which in some instances have been too lightly credited, to the dishonour of the Society.

PHILOSOPHICAL TRANSACTIONS.

I. *The Integration of the Equations of Propagation of Electric Waves.*

By A. E. H. LOVE, *F.R.S., Sedleian Professor of Natural Philosophy in the University of Oxford.*

1. IN the older forms of the Undulatory Theory of Light, the propagation of the waves was traced by means of HUYGENS' principle; each element of a wave front was regarded as becoming a source of disturbance from which secondary waves are emitted. The principle is indefinite, inasmuch as the nature and intensity of the sources of secondary waves are unrestricted, save by the conditions that the secondary waves must combine in advance so as to give rise to the disturbance actually propagated, and must interfere in rear so as to give rise to no disturbance. That these conditions are insufficient, for the complete determination of the nature and intensity of the sources in question, is proved by observing that different writers, proceeding by different methods, have arrived at different expressions for "the law of disturbance in secondary waves," all these expressions satisfying the imposed conditions.[*]

In the more modern forms of the theory, the propagation of the waves is traced by means of a system of partial differential equations. This system has the same form, whether we regard the luminiferous medium as similar in its mode of action to an elastic solid, transmitting transverse waves, or regard light as an electromagnetic disturbance obeying the fundamental equations of the electric field. In both cases it appears that all the components of the vector quantities which represent the disturbance satisfy a partial differential equation of the form $\partial^2\phi/\partial t^2 = c^2 \nabla^2\phi$.

2. This equation is the same as occurs in the Theory of Sound. It has been integrated in two ways. POISSON[†] expressed the value of ϕ, at any point, at time t, in terms of the initial values of ϕ and $\partial\phi/\partial t$ on a sphere of radius ct, with its centre at the point. KIRCHHOFF[‡] obtained a more general integral, in which the value of ϕ at any point is expressed in terms of the values of ϕ, $\partial\phi/\partial\nu$ and $\partial\phi/\partial t$ on a closed surface S, separating the point from the singularities of the function ϕ, $d\nu$ being the

---

[*] Lord RAYLEIGH, "Wave Theory of Light," 'Encycl. Brit.,' 9th ed., vol. 24, pp. 429, 453.

[†] 'Mém. de l'Institut,' vol. 3 (1820), p. 121. *Cf.* Lord RAYLEIGH, 'Theory of Sound,' vol. 2, ch. 14

[‡] 'Wied. Ann.,' vol. 18 (1883), p 663; or 'Vorlesungen u. math. Optik' (Leipz. 1891), pp. 23 *et seq.* The result is given explicitly in equation (2) of § 7, *infra.*

element of the normal to S. KIRCHHOFF'S integral can be shown to include POISSON'S by taking, for S, a sphere, of radius ct, with its centre at the point. In the case of sound, or for any scalar disturbance, KIRCHHOFF'S integral is directly interpretable in terms of imagined sources of disturbance situated on the surface S ; for all the quantities that occur can be interpreted in terms of condensations and velocities. It might thus be regarded as providing an exact equivalent of HUYGENS' principle,* if the disturbance involved were of a scalar character. Its application to light is open to criticism (see § 8. *infra*).

Besides satisfying the partial differential equation $\partial^2\phi/\partial t^2 = c^2 \nabla^2\phi$, the components of a vector quantity, propagated by transverse waves, are also subject to the circuital condition ; and the problem of integrating the system of equations is, accordingly, not the same as the problem of integrating the single equation satisfied by the several components. Sir G. STOKES† has attacked the more general problem, by extending and transforming POISSON'S solution of the single equation. He has shown that the components of the disturbance at any point O, and at any time t, can be expressed as the sums of two parts, one depending on initial values on a sphere of radius ct, with its centre at the point O, and the other depending on initial values in space outside this sphere.‡ The latter part is relatively unimportant when, as in the applications made by Sir G. STOKES, the radius of the sphere is great, compared with the wave-length of the disturbance ; the former part has precisely the character required for representing transverse vector disturbances, and it admits of transformation to a form in which it expresses the radiation received at the point O as due to secondary waves sent out from surfaces other than spheres with their centres at O. The transformation to a plane wave front was given in the paper above quoted, and the results, which were deduced from this form of the integrals of the system of equations, have had a very important bearing on the development of the theory.

3. The object of this paper is to present an investigation of a new system of integrals of the system of equations that govern the propagation of transverse vector disturbances, and to exemplify the use that can be made of such integrals. The components of the vectors that constitute the disturbance ought to be expressible, as in KIRCHHOFF'S solution, in terms of surface values on an arbitrary surface ; the elements of the integrals ought to be quantities characteristic of transverse vector disturbances, as in Sir G. STOKES'S solution ; and the results ought to admit of interpretation in terms of sources of disturbance of definite types, as KIRCHHOFF'S result does when applied to sound waves. It is shown that the method developed by KIRCHHOFF can be adapted to the system of equations in such a way as to lead to

---

* It is so regarded by KIRCHHOFF (*loc. cit.*), by POINCARÉ, 'Théorie math. de la Lumière,' vol. 2 (Paris, 1892), ch. 7, and by DRUDE, 'Lehrbuch d. Optik.' (Leips , 1900), p. 167.

† "On the Dynamical Theory of Diffraction," 'Camb. Phil. Soc. Trans.,' vol. 9 (1849), 'Papers,' vol. 2, p. 241.

‡ See equations (29) and (30) of the paper above quoted, 'Papers,' vol. 2, p. 268.

results of the kind described, and in particular that the disturbances can be regarded as due to sources of two definite types. A source of one of these types is similar to an infinitesimal Hertzian vibrator. The character of the most important part of the radiation from such a vibrator is well known; it is periodic, with a damping coefficient, and is related in a definite way to a particular axis.* The radiation from a source of the corresponding type is related in the same way to the axis, but its mode of dependence upon time is arbitrary. The assumption of infinite trains of simple harmonic radiation, with or without damping coefficients, is an unnecessary restriction of the mathematical formulæ, and is inadequate to represent many phenomena. The other type of sources is arrived at by interchanging the *rôles* of the electric and magnetic forces in the type that is similar to Hertzian vibrators. There is a theorem that disturbances, which can be represented as due to sources of both types, may also be represented as due to sources of a single type, just as acyclic irrotational motions of incompressible fluid may be regarded as due to sources and double sources, or to sources only.†

4. A very general system of integrals of the system of equations that govern the propagation of waves having been obtained, it is natural to inquire after an expression for the law of disturbance in a secondary wave that shall accord with these integrals. The expression arrived at is rather simpler than that given by Sir G. STOKES‡ as regards the intensity of the secondary waves, but rather more complicated as regards the orientation of the plane through the direction of displacement and the direction of propagation. This plane is either the plane of polarisation of the secondary wave, or else it is at right angles to that plane. At one time it might have been interesting to pursue the question further, and to determine the conclusion, as regards the relation of direction of displacement to plane of polarisation, that could be drawn from the new integrals; but the question is not now of importance, since it is certain, on many grounds, that the plane of polarisation of light contains the magnetic force, and is at right angles to the electric force.

5. [*Partly re-written March*, 1901.]—Apart from this question of the plane of polarisation of scattered waves, the chief use of a law of disturbance in secondary waves is found in the solution by elementary methods of problems of diffraction; this use is not affected§ by such differences as exist between the law here found and that obtained by Sir G. STOKES. But, in connexion with the application of any such law to problems of diffraction through apertures, there also arises the question of the distribution over the aperture of sources that would give rise to the transmitted

---

* The relation to the axis is the same for the forms given by HERTZ, ' Electric Waves,' p. 143, as for those given by K. PEARSON and ALICE LEE, 'Phil. Trans.' A, 193 (1899), p. 159 The forms given in § 13 include both.

† LAMB, ' Hydrodynamics,' pp. 66 and 67.

‡ ' Papers,' vol. 2, p. 286.

§ Lord RAYLEIGH, ' Wave Theory of Light,' p. 453.

radiation. When there is no screen, such sources are determined for any imagined bounding surface simply and directly by the incident radiation. [But, when there is a screen, the distribution of the sources is not determined in the same way by the portion of the incident radiation that would come to the aperture if the screen were away. It is proved below that the state of the medium on that side of the screen to which the incident radiation comes can be expressed by means of two superposed fields of electric and magnetic force. The forces of one of these fields are expressed in terms of integrals taken over the surface of the aperture ; and the corresponding disturbance is a system of standing waves, the amplitudes of which diminish rapidly as the distance from the aperture increases. This disturbance can be described as the "effect of the aperture." The forces of the other field are determined by the actual sources of radiation and the boundary conditions that hold over all the unperforated portion of the screen. This disturbance can be described as the "incident radiation, as modified by the action of the screen." It is proved that, when the latter disturbance is known, the system of standing waves, described as the effect of the aperture, is also known. Further, it is proved that the distribution, over the aperture, of sources that would give rise to the transmitted radiation, is determined by the incident radiation, as modified by the action of the screen, in the same way as, if there were no screen, it would be determined by the incident radiation, unmodified. The ordinary optical rule ignores the modification of the incident radiation by the action of the screen, and the success of this rule appears to show that the effect of this modification on the transmitted radiation is practically unimportant when the wave-length is short.]

6. The results obtained, in regard to the effect of an aperture, can be applied also to the problem of the communication of electrical vibrations from a condenser to the external medium, the outer conducting sheet of the condenser being perforated by a small aperture, for, in this case, full account has been taken of the boundary-conditions at the conducting surfaces in calculating the normal modes of vibration. The communication of electrical oscillations from an electrical vibrator to the surrounding medium presents a problem, which has hitherto been solved in a few very special cases. The best known example is that of a spherical conductor, over which, at some instant, charge is distributed otherwise than according to the equilibrium law. The waves emitted have definite periods, but they decay so rapidly as to be practically dead-beat.* Such a system sends out into the medium a *pulse* of radiation, rather than a *train* of radiation. The greater permanence of the vibrations of HERTZ's "resonators," and of condensing systems, has been connected with the existence of greater electrostatic capacity† in such systems ; but no

---

* The problem is solved by J J. THOMSON, 'Recent Researches,' pp. 361 *et seq.* The rate of decay of the oscillations is discussed on p. 370.

† J. J. THOMSON, 'Recent Researches,' p. 396. *Cf.* J. LARMOR, 'London Math. Soc. Proc.,' vol. 26 (1895), p. 123, footnote.

problem of the decay of oscillations of a system with large capacity, through the gradual transmission of the energy to a distance, has so far been solved. For a condenser with concentric spherical conducting surfaces, the outer conducting sheet being very thin, and having a small circular aperture, the problem can be solved by means of previously known analysis and of results obtained in this paper. It appears that, so long as the outer conducting sheet, and the size of the aperture, remain the same, the rate of decay of the oscillations diminishes, as the capacity diminishes; the oscillations of a condenser with small capacity, obtained by making the radius of the inner sheet small compared with that of the outer, are much more slowly damped than those of a condenser of large capacity, obtained by making the radii nearly equal. This result applies to the oscillations of high frequency, involving a large number of nodes, as well as to those of lower frequency; and it suggests that the comparative permanence of the oscillations of condensing systems is to be traced, rather to the screening action, than to the increase of capacity.* A further result, that the oscillations of high frequency and many nodal divisions, are more rapidly damped than those of lower frequency and fewer nodal divisions, is in accordance with the conclusions arrived at by Sir G. STOKES,† for the like problems concerning sound.

[*Added, March*, 1901.—Since the paper was sent in, I have found that a similar method of integration has been employed by V. CERRUTI, ' Rome, R. Acc. Lincei, Rend.,' 1879–80, for the equations of small motion of an elastic solid. The fundamental particular solutions, there used, are the same as (17) of § 11 *infra*; and the solution of the problem of the vibrations of a solid, with a given boundary, over which the displacements, or the tractions, have assigned values, is developed on the basis of an existence-theorem, of the same kind as that assumed in § 21 ; no application is made of the results to problems of radiation.]

KIRCHHOFF'S *Integral.*

7. Suppose that ϕ is a function with the following properties :

(1) Outside a given closed surface S, ϕ and its first and second diffe ntial coefficients, with respect to x, y, z, are everywhere finite and continuous ;

(2) ϕ vanishes at infinite distances from S ;

---

* *March*, 1901.—Mr. LARMOR has called my attention to the fact that the work in the paper does not show that *all* methods of increasing the capacity, without altering the outer conductor or the aperture, are accompanied by increased dissipation. For instance, the capacity of the condenser might be increased by displacing the inner conductor relatively to the outer, without altering its size and shape, or by replacing part of the dielectric plate by conducting material. In such cases there are some analogies with other physical problems, which suggest a diminished rate of dissipation. Against them must be set the analogy with the problem worked out in the present paper.

† ' Phil. Trans.,' vol. 158 (1868).

(3) ϕ satisfies the equation

$$\partial^2\phi/\partial t^2 = c^2 \nabla^2\phi \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

at all points outside S, c being a definite constant, and ∇^2 denoting the operator

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

The value of ϕ at any point O outside S, and at time t, can be expressed as a surface-integral taken over S ; in fact this value is

$$-\frac{1}{4\pi}\iint\left\{\frac{1}{r}\left[\frac{\partial\phi}{\partial\nu}\right] + \frac{1}{cr}\frac{\partial r}{\partial\nu}\left[\frac{\partial\phi}{\partial t}\right] - \frac{\partial r^{-1}}{\partial\nu}[\phi]\right\}dS \qquad . \quad . \quad . \quad . \quad (2),$$

where $d\nu$ denotes the element of the normal to dS drawn outwards (i.e., into the region of space where O is situated), and the expressions in square brackets [ ] are to be formed, for each point of the surface, at the time $t - r/c$, r being the distance of the point of the surface from the point O.

This theorem was obtained by KIRCHHOFF* by an application of GREEN's theorem† to the function ϕ and an auxiliary function V, which satisfies the equation (1) at all points except O, and has the form

$$\frac{\text{F}(r + ct)}{r}, \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

where F is a function to which suitable properties are assigned.

8. [*Partly re-written March,* 1901.]—If ϕ were the velocity potential of sound waves in air, the terms of (2) that contain $[\phi]$ and $[\partial\phi/\partial\nu]$ would be interpretable in terms of velocity, and those that contain $[\partial\phi/\partial t]$ would be interpretable in terms of condensation ; the expression (2) would represent the motion at any point as due to sources of definite types distributed over the surface S. But, if ϕ is one of the components of a vector quantity, propagated by transverse waves, $[\partial\phi/\partial\nu]$ has no physical significance, and the expression (2) cannot be interpreted in terms of appropriate sources of disturbance.

Again, the expression (2) may be interpreted as showing that every element of the surface S becomes the centre of diverging secondary waves. If ϕ is one of the components of a vector quantity, and the primary waves are transverse, the application of Kirchhoff's theorem is open to the criticism that the secondary waves are not always

---

* 'Vorlesungen u. math. Optik,' pp. 23–27.

† The theorem referred to is the one expressed by the equation

$$\iiint(\phi\nabla^2 V - V\nabla^2\phi)\,d\tau = \iint\left(V\frac{\partial\phi}{\partial\nu} - \phi\frac{\partial V}{\partial\nu}\right)dS.$$

transverse, although, when synthesised, the disturbance, to which they give rise, is transverse.

[The criterion of transversality of a vector disturbance, propagated by wave motion, is that the vector concerned is everywhere circuital; and this implies that, in the case of diverging waves, the direction of the vector tends, at great distances from the source, to be at right angles to the radius, drawn from the source. Now, if we take, for example, the electric radiation represented by the expressions in § 13 *infra*, and choose, as the surface S, a sphere, with its centre at the source Q, the magnetic force
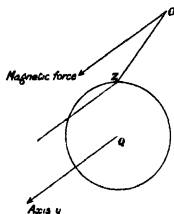
Fig. 1.

at the point Z ($x = 0$, $y = 0$) would be parallel to the axis y. KIRCHHOFF's integral would represent the magnetic force, at any point O, as made up of components, contributed by secondary waves, diverging from the elements of S; and, in the wave diverging from Z, the magnetic force would be everywhere parallel to the axis y. It can be verified readily, by forming the expression for this force, that it is not circuital; but it can be seen at once, without forming this expression, that the secondary wave is not transverse; for, at any distance, however great, it is not at right angles to the radius vector ZO, unless O is in the plane (x, z). The particular example is sufficient to substantiate the criticism; but a little reflexion shows that there is nothing peculiar to the example. In general, let (α, β, γ) be the vector, and suppose that at some point the direction of the vector is independent of the time, we may take the surface S to pass through the point, and take the axis y parallel to the direction of the vector at the point; then α and γ vanish; the equations $\alpha = 0$ and $\gamma = 0$ will represent two surfaces passing through the point, and we may take the direction of the normal to S, at the point, to be the line of intersection of these surfaces. Then α, $\partial\alpha/\partial t$, $\partial\alpha/\partial\nu$ and γ, $\partial\gamma/\partial t$, $\partial\gamma/\partial\nu$ vanish at the point, at all times. In the secondary wave sent out from the point, the vector is everywhere parallel to the axis y; and, accordingly, the secondary wave is not a transverse wave.]

Equations of Propagation of Electric Waves.

9. The equations of propagation of electric waves in free æther are

$$\left. \begin{aligned} \frac{1}{c} \frac{\partial}{\partial t}(X, Y, Z) &= \text{curl } (\alpha, \beta, \gamma) \\ -\frac{1}{c} \frac{\partial}{\partial t}(\alpha, \beta, \gamma) &= \text{curl } (X, Y, Z) \end{aligned} \right\} \quad \cdots \cdots \cdots (4),$$

where (X, Y, Z) denotes the electric force, measured electrostatically, (α, β, γ) the magnetic force, in electromagnetic measure, and c the velocity of propagation of electrical effects. I propose to adopt, as a means of formal simplification, and without attaching to it any definite physical meaning, the view* that (α, β, γ) may be regarded as a "generalised velocity," and to introduce the corresponding "generalised displacement" (u, v, w), so that

$$\frac{\partial}{\partial t}(u, v, w) = (\alpha, \beta, \gamma) \quad \cdots \cdots \cdots \cdots (5).$$

I also introduce the vector (f, g, h) by the equation

$$(f, g, h) = \text{curl } (u, v, w) \quad \cdots \cdots \cdots \cdots (6),$$

so that (f, g, h) is twice the "rotation" corresponding to the displacement (u, v, w). The first of equations (4) becomes

$$(X, Y, Z) = c (f, g, h) \quad \cdots \cdots \cdots \cdots (7),$$

and, according to the view above referred to, this equation may be regarded as expressing a purely kinematical relation, while the second of equations (4) gives the equations of motion of the æther   They are

$$\frac{\partial^2}{\partial t^2}(u, v, w) = c^2 \nabla^2 (u, v, w) \quad \cdots \cdots \cdots (8),$$

with the circuital relation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad \cdots \cdots \cdots \cdots (9).$$

Further, it is convenient to derive (u, v, w) from a vector potential (F, G, H) by the equation

$$(u, v, w) = \text{curl } (F, G, H) \quad \cdots \cdots \cdots (10),$$

* J. LARMOR, 'Phil. Trans.,' A, vol. 185, Part II (1894).

taking (F, G, H) to satisfy the same system of differential equations (8) and (9) as is satisfied by (u, v, w). The vectors (f, g, h) and (F, G, H) are connected by the equations

$$(f, g, h) = -\nabla^2(\text{F, G, H}) = -c^{-2}\frac{\partial^2}{\partial t^2}(\text{F, G, H}) \quad . \quad . \quad . \quad (11).$$

It is also convenient sometimes to quote the fundamental equations in the forms

$$\frac{\partial}{\partial t}(f, g, h) = \text{curl}\,(\alpha, \beta, \gamma) \Big\rangle$$

$$-\frac{1}{c^2}\frac{\partial}{\partial t}(\alpha, \beta, \gamma) = \text{curl}\,(f, g, h)$$

The quantity (f, g, h) will sometimes be called the "electric displacement"; it is the product by 4π of the quantity so denominated by MAXWELL; the quantity (u, v, w) will sometimes be called the "magnetic displacement." MAXWELL's vector potential would be expressed, in the above notation by $\frac{\partial}{\partial t}$ (F, G, H).

Special Types of Solution, Sources of Disturbance.

10. It has been pointed out that KIRCHHOFF's method of integration of equation (1) depends on the application of a certain reciprocal theorem to two solutions of that equation, one of the two having the form (3). When we seek to apply a similar method to the system of equations (8) and (9), we are met at the outset by the difficulty that no simultaneous solutions of the form (3) exist, and by the necessity of devising some forms of solution, which shall become infinite at the origin, and contain arbitrary functions. If we regard the form (3) as corresponding to the solution r^{-1} of the equation $\nabla^2\phi = 0$, the appropriateness of seeking for solutions of the system (8), which correspond to spherical harmonics of order different from zero, at once suggests itself.

If in equation (1) we put

$$\phi = \phi_n S_n,$$

where S_n is a spherical surface harmonic of order n, and ϕ_n is a function of r and t, we find for ϕ_n an equation which can be written

$$\left\{\frac{\partial^2}{\partial r^2} - \frac{n(n+1)}{r^2}\right\}(r\phi_n) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}(r\phi_n).$$

By using the relations[*]

---

[*] The method has been used by R. R. WEBB in the discussion of RICCATI's equation, it is indicated by J. W. L. GLAISHER, 'Phil. Trans.,' vol. 172 (1881), p. 804.

$$\left\{\frac{\partial^2}{\partial r^2} - \frac{n(n+1)}{r^2}\right\} \mathbf{F} = \left(\frac{\partial}{\partial r} - \frac{n}{r}\right)\left(\frac{\partial}{\partial r} + \frac{n}{r}\right) \mathbf{F} = \left(\frac{\partial}{\partial r} + \frac{n+1}{r}\right)\left(\frac{\partial}{\partial r} - \frac{n+1}{r}\right) \mathbf{F}$$

and

$$\left(\frac{\partial}{\partial r} - \frac{n}{r}\right)\left(\frac{\partial}{\partial r} - \frac{n-1}{r}\right) \cdots \left(\frac{\partial}{\partial r} - \frac{1}{r}\right) \mathbf{F} = r^{n+1}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^n \left(\frac{\mathbf{F}}{r}\right),$$

where $\mathbf{F}$ is any function of r, we find for ϕ the form

$$\phi = r^n \mathbf{S}_n \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^n \left\{\frac{\mathbf{F}(r+ct) + f(r-ct)}{r}\right\}, \quad \ldots \quad (13)$$

where $\mathbf{F}$ and f are arbitrary functions.

It is now easy to write down simultaneous solutions[*] of the system of equations (8) and (9). Taking ω_n to represent a spherical solid harmonic of positive degree n, and writing

$$\phi_0 = \mathbf{F}(r+ct) + f(r-ct), \quad \ldots \quad \ldots \quad (14),$$

a set of such solutions is given by the equation

$$\frac{1}{r}\left(\frac{\partial}{\partial r}\right)^n \left(\frac{\phi_0}{r}\right) \quad \quad \ldots \quad (15)$$

and a second set of such solutions is obtained by taking the curl of the first set. We should find for example, after a little reduction,

$$\frac{\partial w}{\partial y} - \frac{\partial c}{\partial z} = -\frac{n+1}{2n+1}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{n-1}\left(\frac{1}{r}\frac{\partial^2 \phi_0}{\partial r^2}\right) \cdot \frac{\partial \omega_n}{\partial r} - \frac{n}{2n+1}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{n+1}\left(\frac{\phi_0}{r}\right) \cdot r^{2n+3}\frac{\partial}{\partial r}\left(\frac{\omega_n}{r^{2n+1}}\right) \ldots \ldots (16).$$

The solutions given by (15) may be referred to as "solutions of the first type," and those given by equations such as (16), as "solutions of the second type."

11. For our immediate purpose it will be sufficient to take $n=1$ and $\omega_1 = x$. The components of a vector which yields a solution of the first type are

$$0, \quad z\left(\frac{1}{r^2}\frac{\partial \phi_0}{\partial r} - \frac{1}{r^3}\phi_0\right), \quad -y\left(\frac{1}{r^2}\frac{\partial \phi_0}{\partial r} - \frac{1}{r^3}\phi_0\right), \quad \ldots \quad (17)$$

and the components of the curl of this vector are

$$(y^2 + z^2)\left(-\frac{1}{r^3}\frac{\partial^2 \phi_0}{\partial r^2} + \frac{3}{r^4}\frac{\partial \phi_0}{\partial r} - \frac{3}{r^5}\phi_0\right) - 2\left(\frac{1}{r^2}\frac{\partial \phi_0}{\partial r} - \frac{1}{r^3}\phi_0\right), \\
-xy\left(-\frac{1}{r^3}\frac{\partial^2 \phi_0}{\partial r^2} + \frac{3}{r^4}\frac{\partial \phi_0}{\partial r} - \frac{3}{r^5}\phi_0\right), \\
-xz\left(-\frac{1}{r^3}\frac{\partial^2 \phi_0}{\partial r^2} + \frac{3}{r^4}\frac{\partial \phi_0}{\partial r} - \frac{3}{r^5}\phi_0\right).$$
$$\quad \ldots \quad (18).$$

---

[*] When the functions that occur are simple harmonic functions of the time, the solution of the simplified system of equations is well known. See LAMB, 'Hydrodynamics,' pp. 487 and 555 et seq.

In these expressions ϕ_0 is any solution of the equation

$$\frac{\partial^2 \phi_0}{\partial t^2} = c^2 \frac{\partial^2 \phi_0}{\partial r^2}. \qquad . \quad (19).$$

It is worth while to observe that the components of the curl of the vector represented by (18) are

$$0, \qquad -z\left(\frac{1}{r^2}\frac{\partial^2 \phi_0}{\partial r^2} - \frac{1}{r^3}\frac{\partial^2 \phi_0}{\partial r^2}\right), \qquad y\left(\frac{1}{r^2}\frac{\partial^2 \phi_0}{\partial r^2} - \frac{1}{r^3}\frac{\partial^2 \phi_0}{\partial r^2}\right) \quad . \quad . \quad (20).$$

12. We may use the results just obtained to describe two types of sources of electromagnetic disturbances. We shall take ϕ_0 to be a function of $ct - r$, say

$$\phi_0 = \phi(ct - r) = \phi \quad . \quad . \quad . \quad . \quad . \quad (21).$$

In the case of sources of the first type, the magnetic force is axial, and the lines of electric force are circles about the axis; when the axis is the axis of x, the vector potential has the form

$$0, \qquad \frac{z}{r^3}\left(\phi + \frac{r}{c}\dot{\phi}\right), \qquad -\frac{y}{r^3}\left(\phi + \frac{r}{c}\dot{\phi}\right); \quad . \quad . \quad . \quad (22)$$

the magnetic displacement has the form

$$-\left(\frac{2}{r^3} - 3\frac{y^2 + z^2}{r^5}\right)\left(\phi + \frac{r}{c}\dot{\phi}\right) + \frac{y^2 + z^2}{r^3 c^2}\ddot{\phi},$$

$$-\frac{ry}{r^2}\left(3\phi + 3\frac{r}{c}\dot{\phi} + \frac{r^2}{c^2}\ddot{\phi}\right),$$

$$-\frac{rz}{r^2}\left(3\phi + 3\frac{r}{c}\dot{\phi} + \frac{r^2}{c^2}\ddot{\phi}\right);$$

and the electric displacement has the form

$$0, \qquad -\frac{z}{r^2 c^2}\left(\ddot{\phi} + \frac{r}{c}\dddot{\phi}\right), \qquad \frac{y}{r^2 c^2}\left(\ddot{\phi} + \frac{r}{c}\dddot{\phi}\right); \quad . \quad . \quad . \quad (24).$$

In these formulæ the dots denote partial differentiation with respect to the time. The axis of the source is the axis of x, corresponding to $\omega_1 = x$ in § 11; when the axis of the source is in the direction (l', m', n'), the result will be obtained by adding the expressions for the component vectors due to sources in the directions of the axes, and given by putting $l'\phi$, $m'\phi$ and $n'\phi$ in place of ϕ (with cyclical interchanges of the letters x, y, z). If the source is at the point (x', y', z'), instead of the origin, we have to write $x - x'$, $y - y'$, $z - z'$ in place of x, y, z, and take r to be the distance of (x, y, z) from (x', y', z').

13. In the case of sources of the second type, the electric force is axial, and the lines of magnetic force are circles about the axis; when the axis is the axis of x, the vector potential has the form

$$\left(\frac{}{} - 3\frac{y^2 + z^2}{r^4} \right)\left(\phi + \frac{r}{c}\,\dot\phi \right) - \frac{y^2 + z^2}{r^3 c^2}\,\ddot\phi\,,$$

$$\frac{xy}{r^2}\left(3\phi + 3\frac{r}{c}\,\dot\phi + \frac{r^2}{c^2}\,\ddot\phi \right),$$

$$\frac{xz}{r^2}\left(3\phi + 3\frac{r}{c}\,\dot\phi + \frac{r^2}{c^2}\,\ddot\phi \right);$$

$$\left. \begin{array}{c} \\ \\ \\ \end{array} \right\} \quad . \quad . \quad . \quad . \quad . \quad (25),$$

the magnetic displacement has the form

$$0, \quad \frac{z}{r^3}\left(\chi + \frac{r}{c}\,\dot\chi \right), \quad -\frac{y}{r^3}\left(\chi + \frac{r}{c}\,\dot\chi \right), \quad . \quad . \quad . \quad . \quad (26),$$

where

$$\chi = \frac{1}{c^2}\,\dot\phi\,; \quad . \quad . \quad . \quad . \quad . \quad (27),$$

and the electric displacement has the form

$$-\left(\frac{2}{r^3} - 3\frac{y^2 + z^2}{r^5} \right)\left(\chi + \frac{r}{c}\,\dot\chi \right) + \frac{y^2 + z^2}{r^4 c^2}\,\ddot\chi\,,$$

$$-\frac{xy}{r^2}\left(3\chi + 3\frac{r}{c}\,\dot\chi + \frac{r^2}{c^2}\,\ddot\chi \right),$$

$$-\frac{xz}{r^2}\left(3\chi + 3\frac{r}{c}\,\dot\chi + \frac{r^2}{c^2}\,\ddot\chi \right).$$

$$\left. \begin{array}{c} \\ \\ \\ \end{array} \right\} \quad . \quad . \quad . \quad . \quad (28).$$

The most important part of the radiation due to a Hertzian vibrator appears to be of this type.[*]

The functions ϕ and χ, which figure in the expressions for the electric and magnetic displacements due to sources of the two types, will be referred to as the "radiation functions" for the sources.

In the expressions here obtained the source is at the origin, and its axis is the axis of x; the expressions for the displacements due to a source, of arbitrary position and direction, can be deduced as before.

The Reciprocal Theorem.

14. Let (u, v, w) be a possible system of magnetic displacements, and (f, g, h) the corresponding electric displacements, which are free from singularities in space bounded by one or more closed surfaces, denoted collectively by S. Then $u, \ldots$ are functions of x, y, z, t, which, with their first and second differential coefficients, are finite and continuous throughout this space. Denoting differentiation with respect to t by a dot, we observe that the equations of motion might be obtained by transforming the variation of the Action function[†]

$$\int dt \iiint \{u^2 + v^2 + w^2 - c^2(f^2 + g^2 + h^2)\} d\tau$$

* HERTZ, 'Electric Waves,' p. 143   † A factor $1/8\pi$ is omitted.

according to the rules of the Calculus of Variations. If then, in the variation

$$2 \int dt \iiint \{ u \delta u + v \delta v + w \delta w - c^2 (f \delta f + g \delta g + h \delta h) \} d\tau,$$

we replace $\delta u, \ldots$ by a second system* of possible displacements $u', \ldots$ we shall obtain a symmetrical expression

$$\iiint \{ u u' + v v' + w w' - c^2 (f f' + g g' + h h') \} d\tau \quad . \quad . \quad . \quad . \quad (29),$$

which admits of a similar transformation; and the result obtained, when simplified by means of the equations of motion, will consist of the volume integral of a perfect differential coefficient with respect to t, and a surface integral. The symmetry of the expression (29) then leads to the reciprocal theorem.

We have

$$\iiint (\dot{u} u' + \dot{v} v' + \dot{w} w') d\tau = \iiint \left\{ \frac{\partial}{\partial t} (u u' + v v' + w w') - (u \dot{u}' + v \dot{v}' + w \dot{w}') \right\} d\tau,$$

the volume integrations being taken through the space bounded by the surfaces S. Also, denoting by l, m, n the direction cosines of the normal to S drawn into this space, we have

$$\iiint (f f'' + g g' + h h') d\tau = - \iint \{ (gn - hm) u' + (hl - fn) v' + (fm - gl) w' \} dS$$
$$- \iiint \left\{ \left(\frac{\partial g}{\partial z} - \frac{\partial h}{\partial y} \right) u' + \left(\frac{\partial h}{\partial x} - \frac{\partial f}{\partial z} \right) v' + \left(\frac{\partial f}{\partial y} - \frac{\partial g}{\partial x} \right) w' \right\} d\tau.$$

Hence the expression (29) becomes

$$- \iiint \left[u' \left\{ \ddot{u} + c^2 \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \right) \right\} + v' \left\{ \ddot{v} + c^2 \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x} \right) \right\} + w' \left\{ \ddot{w} + c^2 \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \right\} \right] d\tau$$
$$+ \iiint \frac{\partial}{\partial t} (u u' + v v' + w w') d\tau + c^2 \iint \{ (gn - hm) u' + (hl - fn) v' + (fm - gl) w' \} dS.$$

The first line vanishes identically; and, from the symmetry of the expression (29), we deduce the reciprocal theorem

$$\iiint \frac{\partial}{\partial t} (u u' + v v' + w w') d\tau + c^2 \iint \{ (gn - hm) u' + (hl - fn) v' + (fm - gl) w' \} dS$$
$$= \iiint \frac{\partial}{\partial t} (u' u + v' v + w' w) d\tau + c^2 \iint \{ (g'n - h'm) u + (h'l - f'n) v + (f'm - g'l) w \} dS$$
$$. \quad . \quad . \quad (30).$$

* The second system, as well as the first, satisfies the fundamental equations.

We integrate this equation with respect to the time, between two fixed values t_0 and t_1, and we thus obtain the equation

$$\int_{t_0}^{t_1} dt \iint \{ u\,(g'n - h'm) + v\,(h'l - f'n) + w\,(f'm - g'l)$$
$$+ f\,(v'n - w'm) + g\,(w'l - u'n) + h\,(u'm - v'l) \}\,dS$$
$$= \frac{1}{c^2} \left[\iiint (\dot{u}u' - \dot{u}'u + \dot{v}v' - \dot{v}'v + \dot{w}w' - \dot{w}'w)\,d\tau \right]_{t_0}^{t_1} \quad . \quad . \quad . \quad . \quad . \quad (31).$$

If the functions involved are such that the volume integral on the right vanishes, and that the order of integrations on the left can be interchanged, we have

$$\iint dS \int_{t_0}^{t_1} \{ u(g'n - h'm) + v(h'l - f'n) + w(f'm - g'l) \}\,dt$$
$$= \iint dS \int_{t_0}^{t_1} \{ u'(gn - hm) + v'(hl - fn) + w'(fm - gl) \}\,dt \quad . \quad . \quad (32).$$

This equation plays the same part in the present theory as Green's equation

$$\iint \phi\,\frac{\partial V}{\partial \nu}\,dS = \iint V\,\frac{\partial \phi}{\partial \nu}\,dS,$$

V and ϕ being harmonic, plays in the Theory of Potential.

Integration of the General Equations.

15. We shall now suppose the boundaries of the region of space, to which the theorem of § 14 is applied, to be (1) a closed surface σ_1 containing none of the singularities of the functions u, v, w, (2) a small sphere σ_2 with its centre at a point O, inside σ_1. Then, taking O as origin, we shall assume for u', v', w', the expressions (17) of § 11, and for f', g', h' the corresponding expressions (18). For l, m, n at σ_2 we have to put x/r, y/r, z/r, and the contribution of σ_2 to the left-hand member of (31) becomes

$$\int_{t_0}^{t_1} dt \iint (vz - wy) \left\{ \frac{2}{r^3} \left(\frac{\partial \phi_0}{\partial r} - \frac{\phi_0}{r} \right) + \frac{1}{r^2} \left(\frac{\partial^2 \phi_0}{\partial r^2} - \frac{3}{r}\frac{\partial \phi_0}{\partial r} + \frac{3}{r^2}\phi_0 \right) \right\} d\sigma_2$$
$$+ \int_{t_0}^{t_1} dt \iint \left(f\frac{y^2 + z^2}{r^4} - g\frac{xy}{r^4} - h\frac{xz}{r^4} \right) \left(r\frac{\partial \phi_0}{\partial r} - \phi_0 \right) d\sigma_2 . \quad . \quad . \quad . \quad . \quad (33).$$

We take ϕ_0 to be of the form $\phi(r + ct)$, so that

$$\frac{\partial \phi_0}{\partial r} = \frac{1}{c}\frac{\partial \phi}{\partial t}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34),$$

and suppose* that the function ϕ is very nearly zero for all values of its argument ζ, except such as lie in the interval between $-\zeta_0$ and ζ_1, where ζ_0 and ζ_1 are two very small positive numbers; further, we suppose that between these values $\phi(\zeta)$ becomes so great that

$$\int^{\zeta_1}\phi(\zeta)\,d\zeta = 1 \quad . \quad . \qquad . \quad . \qquad . \quad . \quad (35).$$

We may then choose t_0 and t_1 so that, if r_0 is the radius of the sphere σ_2,

$$r_0 + ct_0 < -\zeta_0, \quad \text{and} \quad r_0 + ct_1 > \zeta_1;$$

then we shall have

$$\int_{t_0}^{t_1}\phi_0\,dt = \frac{1}{c}\int_{-\zeta_0}^{\zeta_1}\phi(\zeta)\,d\zeta = \frac{1}{c},$$

$$\int_{t_0}^{t_1} r_0\frac{\partial\phi_0}{\partial r}\,dt = \frac{r_0}{c}\int_{-\zeta_0}^{\zeta_1}\phi'(\zeta)\,d\zeta = 0,$$

and, provided r_0 is sufficiently small, these will hold for any negative value of t_0 and any positive value of t_1.

With this choice of ϕ_0, the second line of the expression (33) becomes $-\frac{2}{3}\,4\pi f_0 c^{-1}$ in the limit, when r_0 is indefinitely diminished, f_0 being the value of f for the point O and the time $t = 0$.

In the first line of the expression (33) we develope v and w in such forms as

$$v = (v)_0 + x\left(\frac{\partial v}{\partial x}\right)_0 + y\left(\frac{\partial v}{\partial y}\right)_0 + z\left(\frac{\partial v}{\partial z}\right)_0 + \text{terms of higher order,}$$

where $(\ )_0$ indicates that the value at O is to be taken; we observe that

$$\int_{t_0}^{t_1}\frac{\partial^2\phi_0}{\partial r^2}\,dt = \frac{1}{c}\int_{-\zeta_0}^{\zeta_1}\phi''(\zeta)\,d\zeta = 0,$$

and find that, when r_0 is diminished indefinitely, the limit of the first line is the same as that of

$$\iint dS\int_{t_0}^{t_1}\left\{z^2\left(\frac{\partial v}{\partial z}\right)_0 - y^2\left(\frac{\partial w}{\partial y}\right)_0\right\}\frac{\phi_0}{r^4}\,dt,$$

or it is $-\frac{1}{3}\,4\pi f_0 c^{-1}$.

---

* The process is adapted from KIRCHHOFF, 'Optik,' pp. 24, 25.

Accordingly, the contribution of σ_2 to the left-hand member of equation (31) is $-4\pi f_0 c^{-1}$.

In obtaining this result, we have interchanged the order of the integration with respect to t and the integration over the surface σ_2. This step is certainly permissible if the subject of integration is, in each case, a continuous function of x, y, z, t, for all the values that occur. Equation (35) and this condition can be satisfied in any number of ways, and, in particular, by taking a very large value of μ, and putting, after KIRCHHOFF,

$$\phi(\zeta) = \frac{\mu}{\sqrt{\pi}} e^{-\mu^2 \zeta^2} , \qquad . \qquad . \qquad . \qquad . \qquad (36),$$

provided we suppose that r_0 is small of order μ^{-6}.

With the same choice of ϕ_0, t_0, t_1, the right-hand member of equation (31) can have the limit zero. For this it is sufficient that, for all points between σ_1 and σ_2, the quantities u'. v', w' and u', v', w' should be ultimately zero when t is either t_0 or t_1. This is the case if $r + ct_0$ is negative and $r + ct_1$ is positive for all values of r that occur. Equation (31) then takes the form

$$\iint dS \int_{t_0}^{t_1} dt \{u'(mh - ng) + v'(nf - lh) + w'(lg - mf)$$

$$+ f'(mw - nv) + g'(nu - lw) + h'(lv - mu)\} = 4\pi f_0 c^{-1} \qquad . \qquad . \qquad (37),$$

where the surface integration is taken over σ_1 only.

The quantities u', v', w' and f', g', h' have values, which are not extremely near to zero, only when $r + ct$ is very near to zero, r being the distance of a point on the surface from the point O. The integration with respect to t, in the left-hand member of (37), can accordingly be carried out by observing the rules

$$\int_{t_0}^{t_1} \chi \phi_0 dt = \frac{1}{c} (\chi)_{t = -\frac{r}{c}} ,$$

$$\int_{t_0}^{t_1} \chi \frac{\partial \phi_0}{\partial r} dt = \frac{1}{c} \int_{t_0}^{t_1} \chi \frac{\partial \phi_0}{\partial t} dt = -\frac{1}{c} \int_{t_0}^{t_1} \phi_0 \frac{\partial \chi}{\partial t} dt = -\frac{1}{c^2} \left(\frac{\partial \chi}{\partial t}\right)_{t = -\frac{r}{c}} ,$$

$$\int_{t_0}^{t_1} \chi \frac{\partial^2 \phi_0}{\partial r^2} dt = \frac{1}{c^2} \int_{t_0}^{t_1} \chi \frac{\partial^2 \phi_0}{\partial t^2} dt = \frac{1}{c^2} \int_{t_0}^{t_1} \phi_0 \frac{\partial^2 \chi}{\partial t^2} dt = \frac{1}{c^2} \left(\frac{\partial^2 \chi}{\partial t^2}\right)_{t = -\frac{r}{c}} .$$

We thus find for the value of f, at the point O, and at the time $t = 0$, the equation

$$4\pi f_0 = \iint dS \left[\frac{y}{r^3} \left\{ (lg - mf) + \frac{r}{c} (l\dot{g} - m\dot{f}) \right\} - \frac{z}{r^3} \left\{ (nf - lh) + \frac{r}{c} (n\dot{f} - l\dot{h}) \right\} \right.$$

$$+ \left\{ \frac{2}{r^3} - \frac{3 (y^2 + z^2)}{r^5} \right\} \left\{ (mw - nv) + \frac{r}{c} (m\dot{w} - n\dot{v}) \right\}$$

$$- \frac{y^2 + z^2}{r^3 c^2} (m\ddot{w} - n\ddot{v})$$

$$+ \frac{xy}{r^2} \left\{ 3 (nu - lw) + 3 \frac{r}{c} (n\dot{u} - l\dot{w}) + \frac{r^2}{c^2} (n\ddot{u} - l\ddot{w}) \right\}$$

$$\left. + \frac{xz}{r^2} \left\{ 3 (lv - mu) + 3 \frac{r}{c} (l\dot{v} - m\dot{u}) + \frac{r^2}{c^2} (l\ddot{v} - m\ddot{u}) \right\} \right], \quad . \quad . \quad . \quad (38),$$

in which the values of u, $\dot{u}$, ... h, at any point of the surface, at time $t = -r/c$, are to be calculated, and the integral formed with these values.

16. In this equation, the point O is the origin, and the point (x, y, z) is on the surface. We can express the value of f, at any point (x, y, z), and at any time t, by suitable changes. We have to write $x' - x$, $y' - y$, $z' - z$ for x, y, z, and, in the expressions for u, $\dot{u}$ ... in terms of x', y', z' and t, we have to substitute for t, $t - r/c$, where r is the distance between the points (x, y, z) and (x', y', z'). Further, when the form of f has been obtained, the forms for g and h can be written down by symmetry, and the forms for u, v, w can be deduced from those for f, g, h, by writing F, G, H instead of u, v, w, and u, v, w instead of f, g, h.

It is convenient to have, for reference, the explicit expression of the results. For u, v, w we have

$$4\pi u = \iint dS \left[-\frac{y - y'}{r^3} \left\{ (lv - mu) + \frac{r}{c} (l\dot{v} - m\dot{u}) \right\} + \frac{z - z'}{r^3} \left\{ (nu - lw) + \frac{r}{c} (n\dot{u} - l\dot{w}) \right\} \right.$$

$$+ \left\{ \frac{2}{r^3} - 3 \frac{(y - y')^2 + (z - z')^2}{r^5} \right\} \left\{ (mH - nG) + \frac{r}{c} (m\dot{H} - n\dot{G}) \right\}$$

$$- \frac{(y - y')^2 + (z - z')^2}{r^3 c^2} (m\ddot{H} - n\ddot{G})$$

$$+ \frac{(x - x')(y - y')}{r^5} \left\{ 3 (nF - lH) + 3 \frac{r}{c} (n\dot{F} - l\dot{H}) + \frac{r^2}{c^2} (n\ddot{F} - l\ddot{H}) \right\}$$

$$\left. + \frac{(x - x')(z - z')}{r^5} \left\{ 3 (lG - mF) + 3 \frac{r}{c} (l\dot{G} - m\dot{F}) + \frac{r^2}{c^2} (l\ddot{G} - m\ddot{F}) \right\} \right],$$

$$4\pi v = \iint dS \left[-\frac{z - z'}{r^3} \left\{ (mw - nv) + \frac{r}{c} (m\dot{w} - n\dot{v}) \right\} + \frac{x - x'}{r^3} \left\{ (lv - mu) + \frac{r}{c} (l\dot{v} - m\dot{u}) \right\} \right.$$

$$+ \frac{(x - x')(y - y')}{r^5} \left\{ 3 (mH - nG) + 3 \frac{r}{c} (m\dot{H} - n\dot{G}) + \frac{r^2}{c^2} (m\ddot{H} - n\ddot{G}) \right\}$$

$$+ \left\{ \frac{2}{r^3} - 3 \frac{(z - z')^2 + (x - x')^2}{r^5} \right\} \left\{ (nF - lH) + \frac{r}{c} (n\dot{F} - l\dot{H}) \right\}$$

$$- \frac{(z - z')^2 + (x - x')^2}{r^3 c^2} (n\ddot{F} - l\ddot{H})$$

$$\left. + \frac{(y - y')(z - z')}{r^5} \left\{ 3 (lG - mF) + 3 \frac{r}{c} (l\dot{G} - m\dot{F}) + \frac{r^2}{c^2} (l\ddot{G} - m\ddot{F}) \right\} \right],$$

$$4\pi w = \iint dS\left[-\frac{x-x'}{r^3}\left\{(nu - lw) + \frac{r}{c}(n\dot{u} - l\dot{w})\right\} + \frac{y-y'}{r^3}\left\{(mw - nv) + \frac{r}{c}(m\dot{w} - n\dot{v})\right\}\right.$$

$$+ \frac{(x-x')(z-z')}{r^5}\left\{3(mH - nG) + 3\frac{r}{c}(m\dot{H} - n\dot{G}) + \frac{r^2}{c^2}(m\ddot{H} - n\ddot{G})\right\}$$

$$+ \frac{(y-y')(z-z')}{r^5}\left\{3(nF - lH) + 3\frac{r}{c}(n\dot{F} - l\dot{H}) + \frac{r^2}{c^2}(n\ddot{F} - l\ddot{H})\right\}$$

$$+ \left\{\frac{2}{r^3} - 3\frac{(x-x')^2 + (y-y')^2}{r^5}\right\}\left\{(lG - mF) + \frac{r}{c}(l\dot{G} - m\dot{F})\right\}$$

$$\left. - \frac{(x-x')^2 + (y-y')^2}{r^4c^2}(l\ddot{G} - m\ddot{F})\right].$$

And for f, g, h we have

$$4\pi f = \iint dS\left[-\frac{y-y'}{r^3}\left\{(lg - mf) + \frac{r}{c}(l\dot{g} - m\dot{f})\right\} + \frac{z-z'}{r^3}\left\{(nf - lh) + \frac{r}{c}(n\dot{f} - l\dot{h})\right\}\right.$$

$$+ \left\{\frac{2}{r^3} - 3\frac{(y-y')^2 + (z-z')^2}{r^5}\right\}\left\{(mw - nv) + \frac{r}{c}(m\dot{w} - n\dot{v})\right\}$$

$$- \frac{(y-y')^2 + (z-z')^2}{r^4c^2}(m\ddot{w} - n\ddot{v})$$

$$+ \frac{(x-x')(y-y')}{r^5}\left\{3(nu - lw) + 3\frac{r}{c}(n\dot{u} - l\dot{w}) + \frac{r^2}{c^2}(n\ddot{u} - l\ddot{w})\right\}$$

$$\left. + \frac{(x-x')(z-z')}{r^5}\left\{3(lv - mu) + 3\frac{r}{c}(l\dot{v} - m\dot{u}) + \frac{r^2}{c^2}(l\ddot{v} - m\ddot{u})\right\}\right],$$

$$4\pi g = \iint dS\left[-\frac{z-z'}{r^3}\left\{(mh - ng) + \frac{r}{c}(m\dot{h} - n\dot{g})\right\} + \frac{x-x'}{r^3}\left\{(lg - mf) + \frac{r}{c}(l\dot{g} - m\dot{f})\right\}\right.$$

$$+ \frac{(x-x')(y-y')}{r^5}\left\{3(mw - nv) + 3\frac{r}{c}(m\dot{w} - n\dot{v}) + \frac{r^2}{c^2}(m\ddot{w} - n\ddot{v})\right\}$$

$$+ \left\{\frac{2}{r^3} - 3\frac{(z-z')^2 + (x-x')^2}{r^5}\right\}\left\{(nu - lw) + \frac{r}{c}(n\dot{u} - l\dot{w})\right\}$$

$$- \frac{(z-z')^2 + (x-x')^2}{r^4c^2}(n\ddot{u} - l\ddot{w})$$

$$\left. + \frac{(y-y')(z-z')}{r^5}\left\{3(lv - mu) + 3\frac{r}{c}(l\dot{v} - m\dot{u}) + \frac{r^2}{c^2}(l\ddot{v} - m\ddot{u})\right\}\right],$$

$$4\pi h = \iint dS\left[-\frac{x-x'}{r^3}\left\{(nf - lh) + \frac{r}{c}(n\dot{f} - l\dot{h})\right\} + \frac{y-y'}{r^3}\left\{(mh - ng) + \frac{r}{c}(m\dot{h} - n\dot{g})\right\}\right.$$

$$+ \frac{(x-x')(z-z')}{r^5}\left\{3(mw - nv) + 3\frac{r}{c}(m\dot{w} - n\dot{v}) + \frac{r^2}{c^2}(m\ddot{w} - n\ddot{v})\right\}$$

$$+ \frac{(y-y')(z-z')}{r^5}\left\{3(nu - lw) + 3\frac{r}{c}(n\dot{u} - l\dot{w}) + \frac{r^2}{c^2}(n\ddot{u} - l\ddot{w})\right\}$$

$$+ \left\{\frac{2}{r^3} - 3\frac{(x-x')^2 + (y-y')^2}{r^5}\right\}\left\{(lv - mu) + \frac{r}{c}(l\dot{v} - m\dot{u})\right\}$$

$$\left. - \frac{(x-x')^2 + (y-y')^2}{r^4c^2}(l\ddot{v} - m\ddot{u})\right].$$

In all these $t - r/c$ is to be put for t before integration.

17. It may be verified without difficulty by using KIRCHHOFF's method that the integrals written down in § 16, when taken over a closed surface S, not containing the point (x, y, z), but containing all the singularities of the functions u, v, w, f, g, h, represent the values of $4\pi u, \ldots$ at the external point (x, y, z), provided the normal (l, m, n) is drawn towards the exterior of S. It may also be verified, in the same way, that, if the surface S contains the point (x, y, z) and all the singularities of the functions, the integrals in question vanish identically.

A particular case, which leads to a verification of the formulæ, is afforded by taking the surface S to be a sphere, of radius ct, having its centre at the point (x, y, z). For this we have

$$x - x' = lr, \quad y - y' = mr, \quad z - z' = nr, \quad r = c t,$$

and the values of the quantities $u, v, \ldots$, at points on S, are the initial values, $u_0, v_0, \ldots$, of these quantities. Now the terms of $4\pi u$ that contain u, v, w, explicitly are

$$r^{-2} \iint dS \{ u_0 - l(lu_0 + mv_0 + nw_0) \},$$

which $\quad = r^{-2} \iint dS u_0 - r^{-2} \iint dS (x - x')(lu_0 + mv_0 + nw_0)$

$\qquad = r^{-2} \iint dS u_0 - r^{-3} \iiint u_0 d\tau + r^{-3} \iiint (x - x') \left(\frac{\partial u_0}{\partial x'} + \frac{\partial v_0}{\partial y'} + \frac{\partial w_0}{\partial z'} \right) d\tau,$

where the volume integrations extend through the volume within S, and the last volume integral vanishes identically. Again, the terms that contain F, G, H explicitly are

$$r^{-3} \iint dS (n G_0 - m H_0),$$

which $\qquad = r^{-3} \iiint \left(\frac{\partial H_0}{\partial y'} - \frac{\partial G_0}{\partial z'} \right) d\tau, \ = r^{-3} \iiint u_0 d\tau.$

Further, the terms that contain $\dot{F}, \dot{G}, \dot{H}$ explicitly can be written

$$r^{-1} \iint dS (m h_0 - n g_0)$$

by observing that $\dot{F} = c^2 \nabla^2 F = - c^2 f, \ldots$

The integral last written is

$$r^{-1} \iint dS \left\{ m \left(\frac{\partial v_0}{\partial x'} - \frac{\partial u_0}{\partial y'} \right) - n \left(\frac{\partial u_0}{\partial z'} - \frac{\partial w_0}{\partial x'} \right) \right\}$$

$$= r^{-1} \iint dS \left\{ -\left(l \frac{\partial u_0}{\partial x'} + m \frac{\partial u_0}{\partial y'} + n \frac{\partial u_0}{\partial z'} \right) + \left(l \frac{\partial u_0}{\partial x'} + m \frac{\partial v_0}{\partial x'} + n \frac{\partial w_0}{\partial x'} \right) \right\}$$

$$= r^{-1} \iint dS \frac{\partial u_0}{\partial r} - r^{-1} \iiint \frac{\partial}{\partial x'} \left(\frac{\partial u_0}{\partial x'} + \frac{\partial v_0}{\partial y'} + \frac{\partial w_0}{\partial z'} \right) d\tau,$$

of which the volume integral vanishes identically. The terms which contain $\dot{u}$, $\dot{v}$, $\dot{w}$ explicitly, and those which contain $\dot{F}$, $\dot{G}$, $\dot{H}$ explicitly, may be transformed in the same way, and we have finally

$$u = \frac{1}{4\pi c^2 t^2} \left\{ t \iint \dot{u}_0 \, dS + \iint u_0 dS + ct \iint \frac{\partial u_0}{\partial r} \, dS \right\} ; \quad . \quad . \quad . \quad (39) ;$$

and this is identical with POISSON's integral* of the equation

$$\partial^2 u / \partial t^2 = c^2 \nabla^2 u$$

in terms of initial conditions.

18. The results can be interpreted in terms of sources of disturbance of the two types previously investigated. Any point of the surface S must be regarded as the seat of a source of the first type, and of a source of the second type. The axis of the source of the first type is at right angles to the direction of (F, G, H), and is tangential to the surface; its radiation function is the product of dS, the resultant of (F, G, H) and the sine of the angle, which the direction of this resultant makes with the normal to the surface. The source of the first type is equivalent to three sources, with their axes parallel to the coordinate axes, and with radiation functions equal to

$$- dS(m\mathrm{H} - n\mathrm{G}), \qquad - dS(n\mathrm{F} - l\mathrm{H}), \qquad - dS(l\mathrm{G} - m\mathrm{F}) ;$$

these expressions, for any point on the surface, are functions of t, and they take the characteristic form of radiation functions, when $t - r/c$ is substituted for t.

The axis of the source of the second type is at right angles to the direction of (u, v, w), and is tangential to the surface; its radiation function is the product of dS, the resultant of (u, v, w) and the sine of the angle, which the direction of this resultant makes with the normal to the surface. The source of the second type is equivalent to three sources, with their axes parallel to the coordinate axes, and with radiation functions equal to

$$- dS(mw - nv), \qquad - dS(nu - lw), \qquad - dS(lv - mu) ;$$

---

* The form of POISSON's integral usually given requires the performance of differentiation, with respect to t, upon an integral taken over a sphere of radius ct, and thus, when the differentiation is carried out, there will be three terms in the complete expression; it is easy to verify that these terms are precisely those given in equation (39).

these expressions are to be formed for any time t, and then $t - r/c$ is to be substituted for t in them.

Reduction to a Single Type of Sources.

19. We may seek to express our results in terms of sources of a single type, instead of using two types of sources. The method to be followed is analogous to that used by GREEN for the Theory of Potential.* If V is a function, which is harmonic at all points outside a closed surface S, the value of V at an external point O is

$$-\frac{1}{4\pi} \iint \left(\frac{1}{r} \frac{\partial V}{\partial \nu} - V \frac{\partial r^{-1}}{\partial \nu} \right) dS, \quad \ldots \ldots \ldots \quad (40),$$

where $d\nu$ is the element of the normal to S drawn outwards, and r is distance from O. If now V' is harmonic within S, and equal to V on the surface, this becomes

$$-\frac{1}{4\pi} \iint \frac{1}{r} \left(\frac{\partial V}{\partial \nu} + \frac{\partial V'}{\partial \nu'} \right) dS, \quad \ldots \ldots \ldots \quad (41),$$

where $d\nu'$ is the element of the normal to S drawn inwards. This result is obtained from the reciprocal theorem

$$\iint \frac{1}{r} \frac{\partial V'}{\partial \nu} dS = \iint V' \frac{\partial r^{-1}}{\partial \nu} dS,$$

both V' and r^{-1} being harmonic at all points within S. Further, we know that there cannot be two functions satisfying the conditions satisfied by V'; the theorem that there is one such function is the fundamental existence-theorem of the Theory of Potential. The expression (40) may be interpreted in terms of sources and doublets on S; the expression (41) admits of a similar interpretation in terms of sources only.†

20. In adapting this process to the present theory, we begin by proving that there cannot be two sets of related vectors which

(1) are free from singularities at all points within a closed surface S;

(2) satisfy the system of equations (12) of § 9 at all points within S;

(3) yield the same tangential components, for either of the two vectors, at all points on S;

(4) vanish throughout the space within S for some value t_0 of t.

If there were two such sets, their differences (α, β, γ) and (f, g, h) would satisfy

---

* GREEN, 'Math. Papers,' p. 29.

† LAMB, 'Hydrodynamics,' pp. 66, 67.

the conditions (1), (2) and (4), and either (α, β, γ) or (f, g, h) would be normal to S, at every point on S; so that we should have either

$$\alpha : \beta : \gamma = l : m : n,$$

or
$$f : g : h = l : m : n,$$

(l, m, n) being the direction cosines of the normal to S drawn inwards.

In both cases

$$\int_{t_0}^{t} dt \iint dS \{ l(\beta h - \gamma g) + m(\gamma f - \alpha h) + n(\alpha g - \beta f) \} = 0.$$

Now this integral is

$$- \int_{t_0}^{t} dt \iiint d\tau \left[\left\{ f \left(\frac{\partial \gamma}{\partial y} - \frac{\partial \beta}{\partial z} \right) + \cdot + \cdot \right\} - \left\{ \alpha \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \right) + \cdot + \cdot \right\} \right],$$

the integral being taken through the volume within S, and this is

$$- \frac{1}{2} \iiint d\tau \left[(f^2 + g^2 + h^2) + \frac{1}{c^2} (\alpha^2 + \beta^2 + \gamma^2) \right]_t,$$

since α, β, γ and f, g, h vanish when $t = t_0$. The expression last obtained cannot vanish, unless α, β, γ and f, g, h vanish, at all times, and at all points within S. This proves the theorem.

It follows from this theorem that, if either the tangential components of (u, v, w), or those of (f, g, h), are given at all points of S, the solution of equations (8) and (9) of § 9 is unique.

21. Now let (u_1, v_1, w_1) and (u_2, v_2, w_2) be two sets of possible magnetic displacements, for which there are no singularities within a closed surface S; and let $F_1 \ldots$ be the corresponding vector potentials, and $f_1 \ldots$ the corresponding electric displacements. Suppose further that, at a certain time t_0, all these vanish at all points within S. We shall apply the theorem of § 14 to these vectors. We identify the set with suffix 2 with the set previously accented (§ 15), so that $u_2 = u', \ldots$, the point O, which is the sole singularity of the accented set, being outside S. Then the right-hand member of equation (31) vanishes, and we have

$$\int_{t_0}^{t_1} dt \iint dS \{ u_1 (g'n - h'm) + v_1 (h'l - f'n) + w_1 (f'm - g'l) \}$$

$$= \int_{t_0}^{t_1} dt \iint dS \{ u'(g_1 n - h_1 m) + v'(h_1 l - f_1 n) + w'(f_1 m - g_1 l) \}.$$

If there is a set of magnetic and electric displacements, free from singularities within S, and making the tangential components of the vector potential on S the

same as those in the solution $(u, v, \ldots h)$ for external space, this set also makes the tangential components of electric displacement the same as those in the solution for external space ; we take this to be the set with suffix 1, so that, on S,

$$g_1 n - h_1 m = g n - h m, \qquad \mathrm{G}_1 n - \mathrm{H}_1 m = \mathrm{G} n - \mathrm{H} m,$$

Equation (37) of § 15 now becomes

$$4\pi c^{-1} f_0 = \iint dS \int_{t_0}^{t_1} dt \left[f' \{ m (w - w_1) - n (v - v_1) \} + g' \{ n (u - u_1) - l (w - w_1) \} \right.$$
$$\left. + h' \{ l (v - v_1) - m (u - u_1) \} \right] \quad . \quad . \quad (42).$$

The corresponding equation for u_0 would be

$$4\pi c^{-1} u_0 = \iint dS \int_{t_0}^{t_1} dt \left[u' \{ m (w - w_1) - n (v - v_1) \} + v' \{ n (u - u_1) - l (w - w_1) \} \right.$$
$$\left. + w' \{ l (v - v_1) - m (u - u_1) \} \right] \quad . \quad . \quad (43).$$

The results would thus be interpretable in terms of sources of the second type only, the radiation functions of the sources depending upon the surface values of $u - u_1$, $v - v_1$, $w - w_1$, in the same way as those in § 18 depended upon the surface values of u, v, w. We might, in a similar way, show how the general forms of the displacements could be expressed in terms of sources of the first type only.

It is to be noted that this reduction of the number of types of sources has depended upon the possibility of choosing a time, before which there is no disturbance at any point within the given closed surface, and also that it involves an existence-theorem, which has not been proved. For a sphere, the existence-theorem could be proved by help of the formulæ in § 10.

Law of Disturbance in a Secondary Wave.

22. As a first application of the general formulæ we may consider the law of disturbance in a secondary wave. We suppose that simple harmonic plane waves, of the simplest type, polarised in the plane (x, z), and propagated parallel to the axis of z, are to be resolved into secondary waves due to sources, situated on the wave front $z = z'$. Let the primary waves be given by

$$\left. \begin{array}{lll} \mathrm{F} & \mathrm{G} = - \dfrac{1}{\kappa} \sin \kappa (z - ct), & \mathrm{H} = 0, \\[2mm] u = \cos \kappa (z - ct), & v = 0, & w = 0, \\[2mm] f = 0, & g = - \kappa \sin \kappa (z - ct), & h = 0, \end{array} \right\} \quad . \quad . \quad (44).$$

The disturbance at any point, for which $z > z'$, is given by the following equations, in which l, m, n now denote the direction cosines of the line, of length r, drawn from the point (x', y', z') to the point (x, y, z) :—

$$4\pi u = \iint dx' dy' \left[\frac{n}{r^3} \left\{ \cos \kappa (z' - ct + r) + \kappa r \sin \kappa (z' - ct + r) \right\} \right.$$
$$+ \frac{2}{\kappa r^5} \left\{ \sin \kappa (z' - ct + r) - \kappa r \cos \kappa (z' - ct + r) \right\}$$
$$\left. - \frac{(m^2 + n^2)}{\kappa r^5} \left\{ (3 - \kappa^2 r^2) \sin \kappa (z' - ct + r) - 3\kappa r \cos \kappa (z' - ct + r) \right\} \right], (45).$$

$$4\pi v = \iint dx' dy' \frac{lm}{\kappa r^5} \left\{ (3 - \kappa^2 r^2) \sin \kappa (z' - ct + r) - 3\kappa r \cos \kappa (z' - ct + r) \right\}, \quad . \ (46).$$

$$4\pi w = \iint dx' dy' \left[- \frac{l}{r^3} \left\{ \cos \kappa (z' - ct + r) + \kappa r \sin \kappa (z' - ct + r) \right\} \right.$$
$$\left. + \frac{ln}{\kappa r^5} \left\{ (3 - \kappa^2 r^2) \sin \kappa (z' - ct + r) - 3\kappa r \cos \kappa (z' - ct + r) \right\} \right] (47).$$

Also

$$4\pi f = \iint dx' dy' \frac{lm}{r^3} \left\{ (3 - \kappa^2 r^2) \cos \kappa (z' + ct + r) + 3\kappa r \sin \kappa (z' - ct + r) \right\}, \quad . \ (48).$$

$$4\pi g = \iint dx' dy' \left[\frac{-n}{r^3} \kappa \left\{ \sin \kappa (z' - ct + r) - \kappa r \cos \kappa (z' - ct + r) \right\} \right.$$
$$+ \frac{2}{r^3} \left\{ \cos \kappa (z' - ct + r) + \kappa r \sin \kappa (z' - ct + r) \right\}$$
$$\left. - \frac{n^2 + l^2}{r^5} \left\{ (3 - \kappa^2 r^2) \cos \kappa (z' - ct + r) + 3\kappa r \sin \kappa (z' - ct + r) \right\} \right], (49).$$

$$4\pi h = \iint dx' dy' \left[\frac{m}{r^3} \kappa \left\{ \sin \kappa (z' - ct + r) - \kappa r \cos \kappa (z' - ct + r) \right\} \right.$$
$$\left. + \frac{mn}{r^5} \left\{ (3 - \kappa^2 r^2) \cos \kappa (z' - ct + r) + 3\kappa r \sin \kappa (z' - ct + r) \right\} \right] (50).$$

At a great distance, the contribution of the element of area dS to (u, v, w) is

$$(n + m^2 + n^2, \ - lm, \ - l - ln) \frac{\kappa dS}{4\pi r} \sin \kappa (z' - ct + r); \ . \quad . \ (51);$$

the magnitude of this contribution is

$$(1 + n) \frac{\kappa dS}{4\pi r} \sin \kappa (z' - ct + r), \quad . \quad . \quad . \quad . \quad . \ (52),$$

and its direction is at right angles to that of r, and makes with the plane (z, r) the same angle that this plane makes with the plane (z, x). This direction is shown by

the point P in a spherical figure (fig. 2), in which Z represents the direction of propagation of the primary waves, R the direction of r, NX is the great circle of which Z is pole, NI is the great circle of which R is pole, and P is on NI produced so that
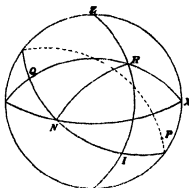
FIG. 2.

NP $=$ NX.   It is easy to verify that the direction cosines of the radius vector, drawn from the centre of the sphere to P, are

Again, at a great distance from the plane, the contribution of the element of area dS to (f, g, h) is

$$(- lm, \ n + l^2 + u^2, \ - m - mn)\frac{\kappa dS}{4\pi r}\kappa \cos \kappa (z' - ct + r) ; \quad . \quad . \quad (53):$$

and its magnitude is

$$(1 + n)\frac{\kappa dS}{4\pi r}\kappa \cos \kappa (z' - ct + r) ; \quad . \quad . \quad . \quad . \quad . \quad (54);$$

its direction might be shown by means of a construction similar to that used for the direction of the contribution to (u, v, w).

The result obtained by Sir G. STOKES* would be expressed, in the notation here employed, by the statements that the magnitude of the contribution of the element dS to (u, v, w) is

$$\sqrt{(m^2 + n^2)(1 + n)\frac{\kappa dS}{4\pi r}} \sin \kappa (z' - ct + r) \ . \quad . \quad . \quad . \quad . \quad (55),$$

and that its direction is that which would be shown by the point antipodal to Q, where RX and NI intersect (fig. 2). It has been pointed out by Lord RAYLEIGH† that

* 'Papers,' 2, p. 286.
† 'Wave Theory of Light,' pp. 452, 453.

such factors as $\sqrt{(m^2 + n^2)}$ and $\frac{1}{2}(1 + n)$ are of no importance in the ordinary appli-
cations of expressions for the law of disturbance in secondary waves, and that, in
fact, the enquiry after such a law involves a certain ambiguity. In the above deduc-
tion of such a law, we have used the general formulæ involving sources of two types;
if we could have used formulæ involving sources of one type only, the result would
probably have been different; this is the origin of the ambiguity referred to by Lord
RAYLEIGH.

23. There is another difficulty attending the deduction of a law of disturbance in
secondary waves from formulæ applicable to the propagation of a system of plane
waves, viz.: that integrals such as (45) taken over an infinite plane are not con-
vergent. The disturbances in the secondary waves ought to combine to give rise to
the disturbance actually propagated, or the result of the integration ought to be to
reproduce the displacements in the primary wave. If we form such an integral as
(45) for a portion of the plane (x', y'), and afterwards extend the boundaries of this
portion indefinitely, we do not arrive at a definite limit. Let O be the point at which
the disturbance is to be estimated, O' the foot of the perpendicular from O on the
plane $z = z'$ (fig. 3), and let the portion of the plane be bounded by a circle, of
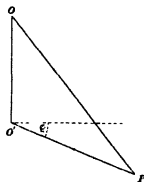
Fig. 3.

radius R', with its centre at O'. We introduce plane polar coordinates r', ϕ, with
origin at O', and put $l = \sin \theta \cos \phi$, $m = \sin \theta \sin \phi$, $n = \cos \theta$; then the value of
the expression (46) for v is

$$\frac{1}{4\pi} \int_0^{2\pi} d\phi \int_0^{R'} r' dr' \frac{\sin^2 \theta \sin \phi \cos \phi}{\kappa r^3} \{(3 - \kappa^2 r^2) \sin \kappa (z' - ct + r) - 3\kappa r \cos \kappa (z' - ct + r)\},$$

and this vanishes identically, however great R' may be, on account of the symmetry
of the circular boundary; it would not have vanished if we had taken a boundary of
a different shape, or a circular boundary with its centre away from O'. With the
boundary chosen as above, we could show that w, f, h vanish. To form the expression
for v, we put

$$\sin \theta = \frac{r'}{r}, \quad \cos \theta = \frac{z - z'}{r}, \quad r^2 = r'^2 + (z - z')^2, \quad R^2 = R'^2 + (z - z')^2,$$

$$\psi = \kappa (z' - ct + r)$$

and remember that z and z' are constants in the integration; we find

$$u = \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_{z-z'}^{R} dr \left[\frac{z-z'}{r^3} (\cos\psi + \kappa r \sin\psi) + \frac{2}{\kappa r^3} (\sin\psi - \kappa r \cos\psi) \right.$$
$$\left. - \frac{1}{\kappa r^3} \left[\frac{(z-z')^3}{r^3} + \sin^2\phi \left\{ 1 - \frac{(z-z')^2}{r^2} \right\} \right] \{(3 - \kappa^2 r^2) \sin\psi - 3\kappa r \cos\psi\} \right]$$
$$= \tfrac{1}{2} \int_{z-z'}^{R} dr \left[\frac{z-z'}{r^2} (\cos\psi + \kappa r \sin\psi) + \frac{2}{\kappa r^3} (\sin\psi - \kappa r \cos\psi) \right.$$
$$\left. - \frac{1}{2\kappa r^3} \left\{ 1 + \frac{(z-z')^2}{r^2} \right\} \{(3 - \kappa^2 r^2) \sin\psi - 3\kappa r \cos\psi\} \right].$$

This is immediately integrable, and we find[*]

$$u = -\tfrac{1}{4} \left[\left(1 + \frac{z-z'}{R} \right)^2 \cos\kappa(z'-ct+R) + \frac{1}{\kappa R} \left\{ 1 - \frac{(z-z')^2}{R^2} \right\} \sin\kappa(z'-ct+R) \right]$$
$$+ \cos\kappa(z-ct); \quad . \quad . \quad . \quad (56);$$

and, when R is very great, this is approximately equal to

$$\cos\kappa(z-ct) - \tfrac{1}{4}\cos\kappa(z'-ct+R). \quad . \quad . \quad . \quad . \quad (57).$$

Thus the value of u for the primary wave is reproduced by the secondary waves sent out from the parts of the plane, which are not at a very great distance. In like manner we should find for g the value

$$-\kappa \sin\kappa(z-ct)$$
$$+ \frac{\kappa}{4} \left[\left(1 + \frac{z-z'}{R} \right)^2 \sin\kappa(z'-ct+R) - \frac{1}{\kappa R} \left\{ 1 - \frac{(z-z')^2}{R^2} \right\} \cos\kappa(z'-ct+R) \right],$$

giving, when R is great, the approximate value

$$-\kappa \sin\kappa(z-ct) + \tfrac{1}{4}\kappa \sin\kappa(z'-ct+R); \quad . \quad . \quad . \quad . \quad (58);$$

and, as before, the value for the primary wave is reproduced by the secondary waves sent out from the parts of the plane, which are not at a very great distance. Both for u and for g, the distant parts of the plane contribute something finite to the disturbance; just as, in the ordinary elementary theory, there may remain a portion of a HUYGENS' zone uncompensated; such portions are always disregarded.[†]

The difficulty here considered arises entirely from our having applied to an infinite plane, formulæ, which were obtained on the express supposition that the surface, to which they are applied, lies entirely within a finite distance of the point, at which

---

[*] Equation (56) determines the intensity, at a point on the axis, of light diffracted through a circular aperture, the incident light being parallel, and the ordinary optical rule being assumed to hold (§§ 24, 30).

[†] *Cf.* BASSET, 'Physical Optics,' p. 46, or Lord RAYLEIGH, 'Wave Theory of Light,' p. 429.

the disturbance is estimated. The difficulty could not be evaded by adopting a different law of disturbance in secondary waves, and one aspect of it has been noticed by Sir G. STOKES[*] in connexion with the law obtained by him. The difficulty would not arise if we took a system of diverging *spherical* waves, and resolved the disturbance at a point O, outside some particular spherical wave front, into secondary waves due to a distribution of sources over this front. The difficulties of integration are, however, in this case considerable; when the point O is at a great distance from the sphere, the integrals can be evaluated approximately, and it can be verified that the disturbance corresponding to the primary wave is reproduced.

Passage of Waves through an Aperture.

24. The general problem of the passage of radiation across an aperture in a screen would involve a solution of the general equations (4) or (12) of § 9, subject to boundary conditions holding all over both faces of the screen; and, unless the incident radiation and the shape of the edge have very simple characters, this cannot at present be attempted.[†] In the theory of diffraction, it is customary to assume that the disturbance at points of the aperture, to which the disturbance on the further side is due, is that which would be found at those points if there were no screen, and also that the elements of the surface of the screen contribute nothing to the disturbance on the further side.[‡] In the Theory of Sound, HELM-HOLTZ[§] has justified the use of a somewhat similar assumption in the problem of the open pipe. In the present theory the question may be formulated as follows : —A train of radiation is propagated on one side of a surface S towards the surface ; there is an aperture in the surface, and the transmitted radiation is to be represented as due to sources situated in the aperture ; how must such sources be distributed ?

[25. (*Re-written March*, 1901.)—We simplify the general question by means of two suppositions :—(1.) that the incident radiation is represented by simple harmonic functions of the time, with period $2\pi/\kappa c$; (2.) that the surface S is plane. The first of these enables us to eliminate all vector potentials, by the rule

$$(F, G, H) = \kappa^{-2}(f, g, h) \quad . \quad . \quad . \quad . \quad . \quad . \quad (59).$$

It will appear later that the second supposition constitutes a practically unimportant restriction, when the aperture is small. We shall take the plane S to be given by the equation $z = z'$, and shall suppose that the incident radiation is propagated on the nearer side $(z < z')$. The transmitted radiation, on the further side $(z > z')$,

* ' Papers,' 2, p. 288. *Cf.* Lord RAYLEIGH, ' Wave Theory of Light,' p. 429.

† *Cf.* A. SOMMERFELD, ' Math. Theorie d. Diffraction,' Math. Ann , vol. 47 (1896).

‡ Lord RAYLEIGH, ' Wave Theory of Light,' p. 430 ; or ' Theory of Sound,' vol. 2, § 291.

§ ' J. f. Math. (Crelle),' 57 (1859) ; or ' Wiss. Abh.,' vol. 1, p 303.

being regarded as due to imagined sources, situated in the aperture, can be calculated directly from the formulæ of § 16, by first assigning certain functions of x', y', t as the forms of u, . . . under the sign of integration, then substituting $t - r/c$ for t, and finally integrating over the aperture. We shall take the forms, that are to be substituted, for u, . . . under the sign of integration, to be given by the equations

$$u = \bar{u}_1 \cos \kappa ct + \bar{u}_2 \sin \kappa ct,$$
$$\cdots$$
$$f = \bar{f}_1 \cos \kappa ct + \bar{f}_2 \sin \kappa ct,$$
$$\cdots$$
$$\qquad\qquad (60),$$

where $\bar{u}_1$, . . . are functions of x', y', for which

$$\bar{h}_1 = \frac{\partial \bar{n}_1}{\partial x'} - \frac{\partial \bar{u}_1}{\partial y'}, \qquad \cdots \qquad (61),$$

and similarly for $\bar{h}_2$. Further, we shall denote the values of u, . . . , resulting from the integrations, by u_+, . . . The answer to the general question of § 24, will thus lie in the determination of the functions $\bar{u}_1$, . . . These functions can be regarded as the values, at certain times, and at points within the aperture, of a certain system of magnetic and electric displacements.]

26. Before proceeding it will be convenient to record the forms for u_+, . . . in terms of the functions $\bar{u}_1$, . . . It will be sufficient to put down the terms that contain $\cos \kappa ct$. We observe that in the formulæ of § 16

$$x - x' = -\frac{\partial r^{-1}}{\cdots} = \frac{\partial r^{-1}}{\cdots}$$

and we also observe that, when the surface S is a portion of a plane ($z = z'$), we must have $l = 0$, $m = 0$, $n = 1$, the point (x, y, z) being on the side $z > z'$. We can therefore write down the formulæ for u_+, . . . f_+, . . . as follows :—

$$4\pi u_+ = \iint dx' dy' \left[-\frac{\partial r^{-1}}{\partial x} \bar{u}_1 \{ \cos \kappa (ct - r) - \kappa r \sin \kappa (ct - r) \} \right.$$
$$- \frac{\partial_2 r^{-1}}{\partial z^2} \kappa^{-2} \bar{g}_1 \{ \cos \kappa (ct - r) - \kappa r \sin \kappa (ct - r) \}$$
$$+ \tfrac{1}{3} \frac{\partial_3 r^{-1}}{\partial x \partial y} \kappa^{-2} \bar{f}_1 \{ (3 - \kappa^2 r^2) \cos \kappa (ct - r) - 3\kappa r \sin \kappa (ct - r) \}$$
$$\left. - \frac{(y - y')^2 + (z - z'^2)}{r^3} \bar{g}_1 \{ \cos \kappa (ct - r) \} \right], \qquad \cdots \qquad (63),$$

$$4\pi v_+ = \iint dx'dy'\left[-\frac{\partial r^{-1}}{\partial z}\bar{v}_1\left\{\cos\kappa\,(ct-r)-\kappa r\sin\kappa\,(ct-r)\right\}\right.$$

$$-\tfrac{1}{3}\frac{\partial^2 r^{-1}}{\partial x\partial y}\kappa^{-2}\bar{g}_1\left\{(3-\kappa^2 r^2)\cos\kappa\,(ct-r)-3\kappa r\sin\kappa\,(ct-r)\right\}$$

$$+\frac{\partial^2 r^{-1}}{\partial y^2}\kappa^{-2}\bar{f}_1\left\{\cos\kappa\,(ct-r)-\kappa r\sin\kappa\,(ct-r)\right\}$$

$$\left.+\frac{(z-z')^2+(x-x')^2}{r^4}\bar{f}_1\left\{\cos\kappa\,(ct-r)\right\}\right],\quad.\quad.\quad.\quad.\quad.\quad(64),$$

$$4\pi w_+ = \iint dx'dy'\left[\left(\bar{u}_1\frac{\partial r^{-1}}{\partial x}+\bar{v}_1\frac{\partial r^{-1}}{\partial y}\right)\left\{\cos\kappa\,(ct-r)-\kappa r\sin\kappa\,(ct-r)\right\}\right.$$

$$\left.-\tfrac{1}{3}\left(\frac{\partial^2 r^{-1}}{\partial x\partial z}\kappa^{-2}\bar{g}_1-\frac{\partial^2 r^{-1}}{\partial y\partial z}\kappa^{-2}\bar{f}_1\right)\left\{(3-\kappa^2 r^2)\cos\kappa\,(ct-r)\right.\right.$$

$$\left.\left.-3\kappa r\sin\kappa\,(ct-r)\right\}\right];\quad.\quad(65)\,;$$

$$4\pi f_+ = \iint dx'dy'\left[-\frac{\partial r^{-1}}{\partial z}\bar{f}_1\left\{\cos\kappa\,(ct-r)-\kappa r\sin\kappa\,(ct-r)\right\}\right.$$

$$-\frac{\partial^2 r^{-1}}{\partial t^2}\bar{v}_1\left\{\cos\kappa\,(ct-r)-\kappa r\sin\kappa\,(ct-r)\right\}$$

$$+\tfrac{1}{3}\frac{\partial^2 r^{-1}}{\partial x\partial y}\bar{n}_1\left\{(3-\kappa^2 r^2)\cos\kappa\,(ct-r)-3\kappa r\sin\kappa\,(ct-r)\right\}$$

$$\left.-\kappa^2\frac{(y-y')^2+(z-z')^2}{r^4}\bar{v}_1\left\{\cos\kappa\,(ct-r)\right\}\right],\quad.\quad.\quad.\quad.\quad(66),$$

$$4\pi g_+ = \iint dx'dy'\left[-\frac{\partial r^{-1}}{\partial z}\bar{g}_1\left\{\cos\kappa\,(ct-r)-\kappa r\sin\kappa\,(ct-r)\right\}\right.$$

$$-\tfrac{1}{3}\frac{\partial^2 r^{-1}}{\partial x\partial y}\bar{v}_1\left\{(3-\kappa^2 r^2)\cos\kappa\,(ct-r)-3\kappa r\sin\kappa\,(ct-r)\right\}$$

$$+\frac{\partial^2 r^{-1}}{\partial y^2}\bar{u}_1\left\{\cos\kappa\,(ct-r)-\kappa r\sin\kappa\,(ct-r)\right\}$$

$$\left.+\kappa^2\frac{(z-z')^2+(x-x')^2}{r^3}\bar{u}_1\left\{\cos\kappa\,(ct-r)\right\}\right],\quad.\quad.\quad.\quad.\quad(67),$$

$$4\pi h_+ = \iint dx'dy'\left[\left(\bar{f}_1\frac{\partial r^{-1}}{\partial x}+\bar{g}_1\frac{\partial r^{-1}}{\partial y}\right)\left\{\cos\kappa\,(ct-r)-\kappa r\sin\kappa\,(ct-r)\right\}\right.$$

$$\left.-\tfrac{1}{3}\left(\frac{\partial^2 r^{-1}}{\partial x\partial z}\bar{v}_1-\frac{\partial^2 r^{-1}}{\partial y\partial z}\bar{u}_1\right)\left\{(3-\kappa^2 r^2)\cos\kappa\,(ct-r)\right.\right.$$

$$\left.\left.-3\kappa r\sin\kappa\,(ct-r)\right\}\right]\quad.\quad.\quad(68).$$

Here the parts of $u_+,\,\ldots$ that arise from $\bar{u}_1\cos\kappa\,ct,\,\ldots$ are written down, the terms in $\sin\kappa\,ct$ being omitted.

[27. (*Re-written March*, 1901.)—The forms of the expressions for $u_+,\,\ldots$ have an important bearing on the determination of the functions $\bar{u}_1,\,\ldots$ The integrals,

which occur in these expressions, represent functions, which are continuous, and have definite values, at all points, that do not lie in the plane $z = z'$, and within the aperture. At points within the aperture, the functions, defined by these integrals, present discontinuities of one or other of three following kinds* :—(a.) The integral, obtained by replacing (x, y, z) by (x', y', z'), is convergent, and is different from the limit obtained by bringing (x, y, z) up to coincidence with (x', y', z') through values, for which $z > z'$, or through values, for which $z < z'$; these two limits are finite and definite, and they are not the same. The term of (63), containing $\partial r^{-1}/\partial z$, is an example of this peculiarity. (b.) The integral, obtained by replacing (x, y, z) by (x', y', z'), is not convergent; but the limit obtained by bringing (x, y, z) up to coincidence with (x', y', z'), on either side, is finite and definite, and these limits are the same. The term of (65), containing $\partial r^{-1}/\partial x$, is an example of this peculiarity. (c.) The integral obtained by replacing (x, y, z) by (x', y', z') is not convergent, nor is any definite limit obtained by bringing (x, y, z) up to coincidence with (x', y', z') on either side; but the difference of the two values, obtained by taking (x, y, z) at two points, near the aperture, and on opposite sides of it, can be made less than any assigned quantity by sufficiently diminishing the distance between the points. The term of (63), containing $\partial^2 r^{-1}/\partial x^2$, is an example of this peculiarity.

The discontinuity of the expressions for $u_+, \ldots$ arises from the representation or the disturbance on the further side $(z > z')$ as due to imagined sources in the aperture; there are not really any sources in the aperture, but the disturbance on the further side $(z > z')$ is continuous with the disturbance on the nearer side $(z < z')$. To restore continuity, it is most convenient to regard the disturbance on the nearer side as consisting of two superposed disturbances, denoted by A and B. The disturbance A is represented by functions, which are continuous in a region of space, containing all the points within the aperture, and within a finite distance on either side of it; these functions have no singularities on the nearer side, except the actual sources of the incident radiation. The disturbance B is represented by functions which have no singularities on the nearer side, but have discontinuities at the aperture, and these discontinuities may be of any of the kinds presented by the expressions for $u_+, \ldots$ We shall denote the magnetic and electric displacements, that belong to the disturbance B, by $u_-, \ldots, f_-, \ldots$, and those that belong to the disturbance A by $u', \ldots, f', \ldots$ The displacements, that belong to the disturbances A and B, satisfy the general equations of § 9. We shall take them to be given by the following equations :—

for A,
$$u = u' = u_1' \cos \kappa ct + u_2' \sin \kappa ct,$$
$$\cdots$$
$$f = f' = f_1' \cos \kappa ct + f_2' \sin \kappa ct,$$
$$\cdots$$
$$\left.\begin{array}{c}\end{array}\right\} \quad \cdots \cdots \cdots (69);$$

---

* *Cf.* POINCARÉ, 'Théorie du Potentiel Newtonien' (Paris 1899), ch. 3.

for B

$$u = u_- = u_1'' \cos \kappa \, \mathrm{c}t + u_2'' \sin \kappa \, \mathrm{c}t \,,$$
$$\cdots\cdots$$
$$f = f_- = f_1'' \cos \kappa \, \mathrm{c}t + f_2'' \sin \kappa \, \mathrm{c}t \,,$$
$$\left. \right\} \quad \cdots \cdots \quad (70).$$
$$\cdots\cdots$$

We have now to express the conditions of continuity of magnetic and electric displacement at the aperture. We suppose that u_+, ... are formed for a point P, $(z > z')$, and that u_-, ... are formed for a point P', $(z < z')$, and we take any
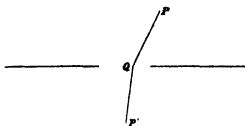
Fig. 4.

point Q in the aperture. The functions u_1', ... are continuous in the neighbourhood of Q, and have definite values at Q. We form the difference

$$u_+ (\mathrm{P}) - u_- (\mathrm{P}'),$$

and allow P and P' to approach Q by any paths,* the tangents to which at Q do not lie in the plane of the aperture. Then the conditions of continuity are

$$\lim \{ v_+ (\mathrm{P}) - u_- (\mathrm{P}') \} = u_1' (\mathrm{Q}) \cos \kappa \, \mathrm{c}t + u_2' (\mathrm{Q}) \sin \kappa \, \mathrm{c}t \,,$$
$$\cdots\cdots$$
$$\lim \{ f_+ (\mathrm{P}) - f_- (\mathrm{P}') \} = f_1' (\mathrm{Q}) \cos \kappa \, \mathrm{c}t + f_2' (\mathrm{Q}) \sin \kappa \, \mathrm{c}t \,,$$
$$\left. \right\} \quad \cdots \quad (71).$$

The functions u_-, ... satisfy the general equations of § 9 at all points on the nearer side $(z < z')$, and are free from singularities in this region; these conditions, with the conditions of continuity (71), suffice to determine the functions in question, in terms of the functions $\bar{u}_1$, ... introduced in § 25. One such determination will be worked out presently; here it is important to observe that it is effectively unique. The conditions (71) require, in fact, that the discontinuities of u_-, ... should be arranged so as to cancel exactly those of u_+, ... Now let us suppose that two sets of functions u_-, ..., and $u_- + \Delta u_-$, ... have been found, both of which obey all the conditions imposed upon the functions u_-, ... ; their differences Δu_-, ... have no discontinuities at the aperture, and no singularities on the nearer side $(z < z')$; thus the disturbance represented by Δu_-, ... belongs to the disturbance

* The path of P lies, of course, entirely in the region $z > z'$, and that of P' in the region $z < z'$.

A, and not to the disturbance B. The relation between A and B is similar to that between the "complementary function" and a "particular integral" of a linear differential equation, with a right-hand member; the difference between two particular integrals is part of the complementary function.*

Perhaps the simplest way of building up the functions u_-, ... is to act upon a hint, derived from a study of HELMHOLTZ'S theory of acoustical resonators.† We may, in fact, attempt to satisfy the conditions, by regarding the disturbance B as consisting of a system of standing waves; and we find that the method thus suggested is successful. We shall proceed on the assumption that the displacements, represented by u_-, ... , constitute a system of standing waves.]

28. Having regard to the proposed plan of passing to a limit when the point (x, y, z) is brought to coincidence with (x', y', z'), we see that especial importance attaches to the limiting values of such expressions as

$$\cos \kappa (ct - r) - \kappa r \sin \kappa (ct - r),$$

which, in § 26, have uniformly been placed in { }; and it appears that all these limiting values are numerical multiples of $\cos \kappa ct$. This remark indicates that the discontinuities of the terms $\bar{u}_2 \sin \kappa ct$, ... are independent of those of the terms $\bar{u}_1 \cos \kappa ct$, ... Again, when the expressions are, as above, replaced by their limiting values, it appears that every term in u_+, ... might be interpreted as a differential coefficient, either of the first order, or of the second order, of the potential of a distribution of surface density on the area within the aperture. Now it is known‡ that, the charged surface coinciding with the plane $z = $ const., first differential coefficients of the potential with respect to x and y are continuous in crossing the surface, and the first differential coefficient with respect to z has a definite discontinuity; further, it is known that all second differential coefficients are continuous in crossing the surface, except the two that are formed by differentiating with respect to z once and either x or y once. These considerations guide us to a proper choice of displacements in the standing waves represented by u_-, ... ; for example, in the second line of the expression for $4\pi u_+$, the factors

$$-\frac{\partial^2 r^{-1}}{\partial z^2} \kappa^{-2} \bar{g}_1$$

must be retained, and multiplied by a function of r and t, of which the limit at $r = 0$ is $\cos \kappa ct$. But we should not arrive at a proper choice by replacing the expressions in { } by their limiting values; for the system of displacements thus arrived at would not satisfy the fundamental differential equations. This consideration suggests that

---

* FORSYTH, 'Treatise on Differential Equations,' ch. 3.

† See the memoir already quoted, particularly equation (29c), HELMHOLTZ, 'Wiss. Abh.,' vol. 1, p. 377.

‡ *Cf.* POINCARÉ, 'Théorie du Potentiel Newtonien,' ch. 3.

the functions of r and t, that are to replace the expressions in { }, being factors in certain particular solutions of the differential equations, and even functions of t, could be arrived at by changing, in { }, t into $-t$, and taking half the sum; for example we should replace

$$\cos \kappa (ct - r) - \kappa r \sin \kappa (ct - r)$$

by

$$\tfrac{1}{2} \{\cos \kappa (ct - r) - \kappa r \sin \kappa (ct - r) + \cos \kappa (ct + r) + \kappa r \sin \kappa (ct + r)\},$$

or by

$$\cos \kappa \, ct \, (\cos \kappa r + \kappa r \sin \kappa r).$$

This comes to the same thing as picking out from each expression in { } the terms that contain $\cos \kappa \, ct$, and rejecting those that contain $\sin \kappa \, ct$.

We accordingly take for $u_-, \ldots$ forms given by such equations as

$$4\pi u_- = \cos \kappa \, ct \iint dx' dy' \left[-\frac{\partial r^{-1}}{\partial z} \, \bar{u}_1 \{\cos \kappa r + \kappa r \sin \kappa r\} \right.$$

$$- \frac{\partial^2 r^{-1}}{\partial c^2} \kappa^{-2} \bar{g}_1 \{\cos \kappa r + \kappa r \sin \kappa r\}$$

$$- \tfrac{1}{2} \frac{\partial^2 r^{-1}}{\partial x \partial y} \kappa^{-2} f_1 \{(3 - \kappa^2 r^2) \cos \kappa r + 3\kappa r \sin \kappa r\}$$

$$\left. - \frac{(y - y')^2 + (z - z')^2}{r^3} \bar{g}_1 \{\cos \kappa r\} \right], \quad \ldots \quad (72),$$

$$4\pi w_- = \cos \kappa \, ct \iint dx' dy' \left[\left(\bar{u}_1 \frac{\partial r^{-1}}{\partial x} + \bar{v}_1 \frac{\partial r^{-1}}{\partial y} \right) \{\cos \kappa r + \kappa r \sin \kappa r\} \right.$$

$$\left. - \tfrac{1}{2} \kappa^{-2} \left(g_1 \frac{\partial^2 r^{-1}}{\partial x \partial z} - f_1 \frac{\partial^2 r^{-1}}{\partial y \partial z} \right) \{(3 - \kappa^2 r^2) \cos \kappa r + 3\kappa r \sin \kappa r\} \right]. \quad (73).$$

29. According to explanations already given, we shall have

$$4\pi \lim \{u_+ (\mathrm{P}) - u_- (\mathrm{P}')\} = \cos \kappa \, ct \iint dx' dy' \, \bar{u}_1 \left[\left(-\frac{\partial r^{-1}}{\partial z} \right)_+ + \left(\frac{\partial r^{-1}}{\partial z} \right)_- \right]$$

$$= 4\pi \bar{u}_1 (\mathrm{Q}) \cos \kappa \, ct \, . \quad \ldots \quad \ldots \quad \ldots \quad (74).$$

Again

$4\pi \lim \{w_+ (\mathrm{P}) - w_- (\mathrm{P}')\}$ is the limit of

$$- \frac{1}{3\kappa^2} \iint dx' dy' \left[\left\{ g_1 \left(\frac{\partial^2 r^{-1}}{\partial x \partial z} \right)_+ - f_1 \left(\frac{\partial^2 r^{-1}}{\partial y \partial z} \right)_+ \right\} \{(3 - \kappa^2 r^2) \cos \kappa (ct - r) - 3\kappa r \sin \kappa (ct - r)\} \right.$$

$$\left. - \left\{ g_1 \left(\frac{\partial^2 r^{-1}}{\partial x \partial z} \right)_- - f_1 \left(\frac{\partial^2 r^{-1}}{\partial y \partial z} \right)_- \right\} \{(3 - \kappa^2 r^2) \cos \kappa r + 3\kappa r \sin \kappa r\} \cos \kappa \, ct \right].$$

We write $\frac{\partial_2 r^{-1}}{\partial x \partial z} = -\frac{\partial}{\partial x'} \frac{\partial r^{-1}}{\partial z}$, and similarly for y, and integrate by parts; the result contains a line integral round the boundary of the aperture and a surface integral, and the former contributes nothing to the limit we are seeking, unless the point Q is indefinitely close to the boundary of the aperture. Thus the limit we are seeking is that of

$$
\begin{aligned}
-\frac{1}{3\kappa^2} \iint dx' dy' \Bigg[&\left(\frac{\partial r^{-1}}{\partial z}\right)_+ \left(\frac{\partial \bar{g}_1}{\partial x'} - \frac{\partial \bar{f}_1}{\partial y'}\right) \{(8 - \kappa^2 r^2) \cos \kappa (ct - r) - 3\kappa r \sin \kappa (ct - r)\} \\
&- \left(\frac{\partial r^{-1}}{\partial z}\right)_- \left(\frac{\partial \bar{g}_1}{\partial x'} - \frac{\partial \bar{f}_1}{\partial y'}\right) \{(3 - \kappa^2 r^2) \cos \kappa r + 3\kappa r \sin \kappa r\} \cos \kappa ct \\
&- \left(\frac{\partial r^{-1}}{\partial z}\right)_+ \left(\bar{g}_1 \frac{x' - x}{r} - \bar{f}_1 \frac{y' - y}{r}\right) \{-\kappa^2 r^2 \sin \kappa (ct - r) + \kappa^2 r \cos \kappa (ct - r)\} \\
&+ \left(\frac{\partial r^{-1}}{\partial z}\right)_- \left(\bar{g}_1 \frac{x' - x}{r} - \bar{f}_1 \frac{y' - y}{r}\right) \{\kappa^2 r^2 \sin \kappa r + \kappa^2 r \cos \kappa r\} \cos \kappa ct ;
\end{aligned}
$$

in this expression, the two last lines vanish in the limit, and the others yield the value at Q of

$$
4\pi \kappa^{-2} \left(\frac{\partial \bar{g}_1}{\partial x'} - \frac{\partial \bar{f}_1}{\partial y'}\right) \cos \kappa ct .
$$

Thus we have

$$
4\pi \lim \{w_+(P) - w_-(P')\} = 4\pi \bar{w}_1(Q) \cos \kappa ct ; \quad . \quad . \quad . \quad (75),
$$

and it follows that the conditions of continuity are satisfied by putting

$$
\left.
\begin{aligned}
\bar{u}_1 &= u_1', \quad \bar{u}_2 = u_2', \\
\bar{f}_1 &= f_1', \quad \bar{f}_2 = f_2', \\
&\quad \dots
\end{aligned}
\right\} \quad . \quad . \quad . \quad . \quad (76).
$$

By this result, the transmitted waves on the further side are connected with the waves on the nearer side; and it is manifest that the result would not be disturbed if the surface were not plane, provided that all the linear dimensions of the aperture are small compared with the radii of curvature of the surface.

[30. (*Partly re-written March,* 1901.)—We return now to the general question propounded in § 24, and seek to estimate the character of the answer that we have found. In § 25 the question is made more precise by showing that the distribution over the aperture of sources to which the transmitted radiation can be regarded as due, depends upon the values, at certain times and at points within the aperture, of a certain system of magnetic and electric displacements; these values are the quantities denoted by $\bar{u}_1, \ldots$ In § 26 the transmitted radiation is expressed in terms of these

quantities; the functions by which it is expressed are those denoted by u_+, ... These functions are defined for points on the further side by expressions which are not continuous up to and across the aperture; but they represent the transmitted radiation at any finite distance from the aperture. The actual disturbance is continuous up to and across the aperture. We seek accordingly to represent the disturbance on the nearer side by means of functions which are defined for the nearer side, but are not continuous up to and across the aperture, the discontinuities being so arranged that the displacements on the nearer side shall be continuous with those on the further side. In § 27 we separate the expressions of these functions into two parts, thus regarding the disturbance on the nearer side as consisting of two super-posed disturbances, there called A and B. The functions representing the disturbance A are continuous up to and across the aperture; those representing the disturbance B are not; but their discontinuities cancel exactly those of the functions u_+, ... The determination of B is in a certain sense unique. In § 28 we verify the supposition that B may be regarded as a system of standing waves, by actually determining, in accordance with this supposition, the functions involved in B, viz., u_-, . . . , in terms of the functions $\bar{u}_1$, ... In § 29 we show that the displacements, of which the func-tions $\bar{u}_1$, . . . are the values, at certain times, and at points within the aperture, are the displacements belonging to the disturbance A. The disturbance B and the trans-mitted radiation are thus determined in terms of A, and the general question of § 24 is reduced to the determination of A.

The components u', . . . f', . . . of A are subject to the following conditions :—

(1.) On the nearer side they satisfy the equations of § 9 everywhere, except possibly at certain singular points.

(2.) These singular points are the actual sources of the incident radiation.

(3.) The functions u', . . . f', . . . are continuous up to, and across, the aperture.

(4.) At all points of the screen, not points of the aperture, they satisfy certain boundary conditions.

The boundary conditions depend, to some extent, on the material of the screen; and they will usually take the form that some components of electric or magnetic displacement vanish. The components, affected by the condition, are those of the displacement on the nearer side compounded of A and B, i.e., such quantities as $u' + u_-$; but, as B falls off rapidly, with increasing distance from the aperture, it will generally be sufficient to impose the boundary condition on the components of A only.

We may now give the following interpretation of the analysis :—The disturbance B, consisting of a system of standing waves, which are important in the neighbour-hood of the aperture only, can be described as the "effect of the aperture." The disturbance A can be described as "the incident radiation, as modified by the action of the screen." The result of § 29 can be stated in the form :—The transmitted radiation is to be calculated from the incident radiation, as modified by the action

of the screen, in the same way as if this radiation passed freely through the aperture.

This result differs from the ordinary optical rule, that the transmitted radiation is to be calculated from the incident radiation, unmodified, as if this radiation passed freely through the aperture. In the application of this rule no attention is paid to the boundary conditions at the screen. If we could assume that the disturbance at points of the *aperture* when the incident radiation is modified by the action of the screen, differs very little from the unmodified incident disturbance, then the result and the optical rule would be in practical agreement. The success of the optical rule seems to show that the modification of the incident radiation by the screen is unimportant, at points within the aperture, when the wave-length is short.]

The result obtained may be applied with greater certainty when the disturbance on the nearer side of the screen has been calculated in accordance with a known boundary condition, holding over all the unperforated portion. This is the case when, instead of an incident train of waves, we have, on the nearer side, standing vibrations, for which the boundary condition is satisfied. In such a case, the values to be assigned to the components   ,

$$u_1' \cos \kappa \, ct + u_2' \sin \kappa \, ct ,$$

$$\cdots \cdots$$

of the disturbance A, . . . at points of the aperture, are the values that $u, . . .$ would have if the screen were unperforated. This remark applies to the problem of the communication of vibrations from a condensing system to the surrounding æther. We shall now take up this problem, having regard especially to the example of concentric spherical conducting surfaces, with a very thin dielectric plate between them, the outer surface being perforated by a small circular aperture.

Electrical Oscillations between Concentric Spheres.

31. It has been pointed out by LARMOR[*] that the most important modes of electrical oscillation in a condenser, with a thin dielectric plate, are those in which the charge surges over the conducting surfaces, the lines of electric force being always normal to these surfaces, and the lines of magnetic force tangential to them. In a condenser with concentric spherical conducting surfaces such modes of oscillation exist, whatever the thickness of the dielectric plate may be; and the analysis requisite for dealing with them has been developed by LAMB.[†] The required solutions of the

* 'London Math. Soc. Proc.,' vol. 26 (1895), p. 119.

† 'London Math. Soc. Proc.,' vol. 13 (1882), p. 51; or 'Hydrodynamics,' pp. 555, *et seq.* The notation here used will be that of the 'Hydrodynamics.' It is worth while to recall some of the properties of the functions ψ, defined in equations (77): they satisfy the equations

fundamental equations are included among those obtained in § 10, by proper choice of the function ϕ_0.

Taking the centre of the spheres as origin, let ω_n denote any spherical solid harmonic, and write

$$\psi_n(\zeta) = (-)^n \left(\frac{1}{\zeta}\frac{\partial}{\partial\zeta}\right)^n \frac{\sin\zeta}{\zeta},$$

$$\Psi_n(\zeta) = (-)^n \left(\frac{1}{\zeta}\frac{\partial}{\partial\zeta}\right)^n \frac{\cos\zeta}{\zeta}, \qquad \ldots \ldots \quad (77),$$

$$A\psi_n(\zeta) + B\Psi_n(\zeta) = \chi_n(\zeta),$$

A and B being constants. Then, in the modes of oscillation under discussion, we have

$$(u, v, w) = \cos\kappa\, ct \cdot \chi_n(\kappa r) \cdot \left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \; z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}, \; x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\omega_n, \quad (78),$$

$$f = -\frac{1}{2n+1}\cos\kappa\, ct\left\{(n+1)\chi_{n-1}(\kappa r)\frac{\partial\omega_n}{\partial x} - n\chi_{n+1}(\kappa r)\,\kappa^2 r^{2n+3}\frac{\partial}{\partial x}\left(\frac{\omega_n}{r^{2n+1}}\right)\right\}, \quad (79),$$

with similar formulæ for g and h; the value of κ and the ratio $A : B$ are to be found from the boundary conditions, which are that the electric force is normal to the bounding surfaces.

The vector (f, g, h) has a radial component of amount

$$- n(n+1)\cos\kappa\, ct\, r^{-1}\chi_n(\kappa r)\,\omega_n \quad \ldots \ldots \ldots \quad (80).$$

If then we form a new vector from (f, g, h), by resolving this radial component in the directions of the axes, and subtracting its resolved parts from f, g, h, the new vector will have the same tangential components at any sphere as (f, g, h) has; the x-component of the new vector is

$$\frac{n(n+1)}{2n+1}\cos\kappa\, ct\left[\left\{\chi_n(\kappa r) - \frac{1}{n}\chi_{n-1}(\kappa r)\right\}\frac{\partial\omega_n}{\partial x}\right.$$
$$\left. - \left\{\chi_n(\kappa r) - \frac{\kappa^2 r^2}{n+1}\chi_{n+1}(\kappa r)\right\}r^{2n+1}\frac{\partial}{\partial x}\left(\frac{\omega_n}{r^{2n+1}}\right)\right],$$

$$\left(\frac{d^2}{d\zeta^2} + 1 - \frac{n(n+1)}{\zeta^2}\right)\{\zeta^{n+1}\psi_n(\zeta)\} = 0,$$

$$\psi_n{}'(\zeta) = -\zeta\psi_{n+1}(\zeta),$$

$$(2n+1)\psi_n(\zeta) = \psi_{n-1}(\zeta) + \zeta^2\psi_{n+1}(\zeta),$$

and these equations are also satisfied by Ψ, and by χ, provided the constants A and B are supposed not to change with n. Reductions made by using these equations will be introduced without remark.

or, what comes to the same thing, the new vector is

$$-\frac{\cos\kappa\,ct}{(2n+1)r^n}\frac{\partial}{\partial r}\{r^{n+1}\chi_n(\kappa r)\}\left[(n+1)\left(\frac{\partial\omega_n}{\partial x},\ \frac{\partial\omega_n}{\partial y},\ \frac{\partial\omega_n}{\partial z}\right)+nr^{2n+1}\left(\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}\right)\left(\frac{\omega_n}{r^{2n+1}}\right)\right];$$

and the condition that (f, g, h) should be radial, at the conducting surfaces $r = r_0$ and $r = r_1$, is that

$$\frac{\partial}{\partial r}\{r^{n+1}\chi_n(nr)\} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (81)$$

at $r = r_0$ and $r = r_1$. These two equations determine the ratio $A : B$ and the value of κ.

When the conducting surfaces are very near together, we have approximately

$$\frac{\partial^2}{\partial r^2}\{r^{n+1}\chi_n(\kappa r)\} = 0,$$

or, in virtue of the differential equation, satisfied by χ_n,

$$\kappa^2 r_0^2 = n(n+1), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (82),$$

a result otherwise obtained by LARMOR.[*] The ratio $A : B$ is determined by the equation

$$A\frac{\partial}{\partial r}\{r^{n+1}\psi_n(\kappa r)\} + B\frac{\partial}{\partial r}\{r^{n+1}\Psi_n(\kappa r)\} = 0, \quad . \quad . \quad . \quad . \quad (83),$$

which holds for $r = r_0$.

32. We consider, in particular, modes of oscillation, for which the axis z is an axis of symmetry. We take

$$z/r = \mu, \quad \omega_n = r^n P_n(\mu), \quad . \quad . \quad . \quad . \quad . \quad . \quad (84),$$

where $P_n(\mu)$ is the zonal surface harmonic (LEGENDRE'S coefficient) of order n. We find

$$\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\omega_n = r^{2n+1}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\{r^{-(n+1)}P_n(\mu)\},$$

$$= \frac{(-)^n}{n\,!}r^{2n+1}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\frac{\partial^n r^{-1}}{\partial z^n}$$

$$= \frac{(-)^n}{n\,!}r^{2n+1}\left\{y\frac{\partial^{n+1}r^{-1}}{\partial z^{n+1}} + yz\frac{\partial^n r^{-3}}{\partial z^n}\right\}.$$

Now it is easy to show[†] that

$$\frac{(-)^n}{n\,!}\frac{\partial^n r^{-3}}{\partial z^n} = \frac{1}{r^{n+3}}\left\{(n+1)P_n + \mu\frac{dP_n}{d\mu}\right\}, \quad . \quad . \quad . \quad . \quad (85),$$

---

* 'London Math. Soc. Proc.,' vol. 26.

† One way of doing this is as follows :—

$$\frac{(-)^n}{n\,!}\frac{\partial^n r^{-3}}{\partial z^n} = \text{the coefficient of } h^n \text{ in the expansion of } \{x^2 + y^2 + (z - h)^2\}^{-\frac{3}{2}};$$

also

$$\{x^2 + y^2 + (z - h)^2\}^{-\frac{3}{2}} = r^{-3}\left(1 - 2\mu\frac{h}{r} + \frac{h^2}{r^2}\right)^{-\frac{3}{2}},$$

and we deduce without difficulty that, when $\omega_n = r^n P_n(\mu)$,

$$(u, v, w) = \cos \kappa \, ct \, r^{n-1} \chi_n (\kappa r) \frac{dP_n}{d\mu} \cdot (y, -x, 0) \quad . \quad . \quad . \quad . \quad (86).$$

Forms for f, g, h could be obtained by a similar analysis; but the values that they would take near the boundaries r_0 and r_1 can be written down immediately from the formula (80), viz.: we have, at the boundaries,

$$(f, g, h) = -n(n+1) \cos \kappa \, ct \, r^{n-2} \chi_n (\kappa r) P_n(\mu) \cdot (x, y, z) \quad . \quad . \quad . \quad (87).$$

When r_0 and r_1 are nearly equal, this formula holds approximately for all values of r that are involved.

33. The kinetic energy of the mode of oscillation here discussed can be calculated without difficulty. We require the value of

$$\frac{1}{8\pi} \iiint (u^2 + v^2 + w^2) \, d\tau,$$

$$= \frac{\kappa^2 c^2}{8\pi} \sin^2 \kappa \, ct \int_{r_1}^{r_0} dr \int_{-1}^{1} d\mu \, r^{2n+2} \{\chi_n (\kappa r)\}^2 \, 2\pi (1 - \mu^2) \left(\frac{dP_n}{d\mu}\right)^2$$

$$= \frac{1}{2} \frac{n(n+1)}{2n+1} \kappa^2 c^2 \sin^2 \kappa \, ct \int_{r_1}^{r_0} \{r^{n+1} \chi_n (\kappa r)\}^2 \, dr \quad . \quad . \quad . \quad . \quad (88).$$

To calculate the integral in this expression, we have recourse to the differential equation

$$\left\{ \frac{d^2}{dr^2} + \kappa^2 - \frac{n(n+1)}{r^2} \right\} \{r^{n+1} \chi_n (\kappa r)\} = 0, \quad . \quad . \quad . \quad . \quad (89),$$

and the condition that

$$\frac{d}{dr} \{r^{n+1} \chi_n (\kappa r)\} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (81)$$

and
$$(1 - 2\mu k + k^2)^{-\frac{1}{2}} = \frac{1}{(1 - 2\mu k + k^2)^{\frac{1}{2}}} + (2\mu - k) \frac{k}{(1 - 2\mu k + k^2)^{\frac{3}{2}}}$$

$$= \left\{ 1 + (2\mu - k) \frac{d}{d\mu} \right\} \frac{1}{(1 - 2\mu k + k^2)^{\frac{1}{2}}}$$

$$\cdot 1 + \sum_{1}^{\infty} k^n \left(P_n + 2\mu \frac{dP_n}{d\mu} - \frac{dP_{n-1}}{d\mu} \right);$$

and we also have the known relation

$$\frac{dP_{n-1}}{d\mu} = \mu \frac{dP_n}{d\mu} - nP_n.$$

at both boundaries. Taking κ and κ' to be two possible values of κ, and λ and λ' the corresponding forms of $r^{n+1} \chi_n(\kappa r)$, we find

$$(\kappa^2 - \kappa'^2) \int_{r_1}^{r_0} \lambda \lambda' dr = \left[\lambda \frac{d\lambda'}{dr} - \lambda' \frac{d\lambda}{dr} \right]_{r_1}^{r_0} ;$$

and then, by the usual limiting process,*

$$\int_{r_1}^{r_0} \lambda^2 dr = \frac{1}{2\kappa} \left[\frac{d\lambda}{d\kappa} \frac{d\lambda}{dr} - \lambda \frac{d}{d\kappa} \frac{d\lambda}{dr} \right]_{r_1}^{r_0} ,$$

or

$$\int_{r_1}^{r_0} r^{2n+2} \{\chi_n(\kappa r)\}^2 dr = \frac{1}{2} \left[\left\{ 1 - \frac{n(n+1)}{\kappa^2 r_0^2} \right\} r_0^{2n+2} \{\chi_n(\kappa r_0)\}^2 \right.$$
$$\left. - \left\{ 1 - \frac{n(n+1)}{\kappa^2 r_1^2} \right\} r_1^{2n+2} \{\chi_n(\kappa r_1)\}^2 \right]. \quad . \ (90)$$

When r_1 is very nearly equal to r_0, this becomes, approximately,

$$\int_{r_1}^{r_0} r^{2n+2} \{\chi_n(\kappa r)\}^2 dr = (r_0 - r_1) \kappa^{-2} n(n+1) r_0^{2n} \{\chi_n(\kappa r_0)\}^2 \quad . \quad . \ (91)$$

After the appropriate expression (90) or (91) has been substituted in the expression (88), the total energy, kinetic and potential, is to be obtained by suppressing the factor $\sin^2 \kappa ct$. Thus, when r_1 is very nearly equal to r_0, the total energy of the oscillating charges on the condenser is

$$\frac{1}{2} (r_0 - r_1) \frac{n^2 (n+1)^2}{2n+1} c^2 r_0^{2n} \{\chi_n(\kappa r_0)\}^2 \quad . \quad . \quad . \quad . \ (92)$$

Communication of Electrical Oscillations to External Medium.

34. When there is no aperture in the outer conductor, the oscillations considered in §§ 31–33 would, in the absence of dissipation due to imperfect conduction, continue indefinitely; but they would not produce any effect in any external electrical system. When there is an aperture, we may take account of it by supposing that the displacements (magnetic and electric), in space external to the condenser, are of the character corresponding to waves diverging from sources distributed over the aperture only, and that the displacements within the dielectric plate of the condenser differ from those, which would be found in a normal mode of oscillation, by the superposition of displacements corresponding to a system of standing waves, which are insensible except in the immediate neighbourhood of the aperture. We may suppose the

---

* *Cf.* Lord RAYLEIGH, 'Theory of Sound,' vol. 1, p. 325.

diverging waves, and the standing waves, thus introduced, to have the same period as the oscillations in the normal mode; and then the displacements corresponding to them will be determined, as in § 29, by the conditions that the electric and magnetic displacements must be continuous across the aperture. As we are concerned rather with the general features of the transmission of disturbances across an aperture than with special details, we may select any normal mode of oscillation for examination. We shall suppose the aperture to be a circle of radius a, small compared with the distance $r_0 - r_1$ between the conducting surfaces; and we shall consider particularly modes of oscillation symmetrical about the axis of the circle, taken as axis of z.

35. For the calculation of the energy dissipated we shall take, in the notation of § 25,

$$(\bar{u}_1, \bar{v}_1, \bar{w}_1) = \frac{n(n+1)}{2} r_0{}^{n-1} \chi_n (\kappa r_0) \cdot (y', -x', 0)$$

$$= \xi \cdot (y', -x', 0), \text{ say}, \quad \ldots \ldots \ldots \ldots (93),$$

where $\dfrac{n(n+1)}{2}$ has been written for $\left(\dfrac{d\mathrm{P}_n}{d\mu}\right)_{n=1}$; and we shall take

$$(\bar{f}_1, \bar{g}_1, \bar{h}_1) = -n(n+1) r_0{}^{n-2} \chi_n (\kappa r_0) \cdot (x', y', r_0)$$

$$= \eta \cdot (x', y', r_0), \text{ say}, \quad \ldots \ldots \ldots \ldots (94),$$

these being with sufficient approximation the values obtained in § 32: the normal mode of oscillation here discussed will accordingly be one for which the axis z is an axis of symmetry.

We have now to find the most important terms in $4\pi u_+, \ldots$ at a great distance from the aperture, the values above written being substituted for $\bar{u}_1, \ldots$. We shall take R for the distance of the point (x, y, z) from the centre of the aperture, and, whenever we wish to do so, we shall expand r in the form

$$r = \mathrm{R} + x' \frac{-x}{\mathrm{R}} + y' \frac{-y}{\mathrm{R}} + \ldots$$

Now, taking $4\pi u_1$, the first line of equation (63) is

$$\xi \iint \frac{z - r_0}{\mathrm{R}^3} y' \left\{ 1 + 3 \frac{xx' + yy'}{\mathrm{R}^2} \right\} [\sin \kappa (ct - \mathrm{R}) (-\kappa \mathrm{R})$$
$$+ \cos \kappa (ct - \mathrm{R}) \{ 1 - \kappa^2 (xx' + yy') \}] \, dx' dy'$$

approximately, where the integration is taken over the area within the circle $x'^2 + y'^2 = a^2$, and terms of order higher than x'^2 have been neglected. The most important part of this is

$$-\frac{\pi a}{4} \xi \frac{z - r_0}{\mathrm{R}} \frac{\kappa^2 y}{\mathrm{R}} \frac{\cos \kappa (ct - \mathrm{R})}{\mathrm{R}}$$

which is of order $R^{-1} \cos \kappa (ct - R)$, the right order when R is great. We treat every term exactly as this term has been treated. The second line of $4\pi u_+$ is of order $R^{-2} \cos \kappa (ct - R)$, and is to be rejected; the third line gives

$$\frac{\pi a^4}{4} \eta \kappa \frac{x^2 y}{R^3} \frac{\sin \kappa (ct - R)}{R},$$

and the fourth line gives

$$\frac{\pi a^4}{4} \eta \kappa \frac{y}{R} \frac{y^2 + (z - r_0)^2}{R^3} \frac{\sin \kappa (ct - R)}{R}.$$

Hence the most important part of u_+ at a distance from the aperture is

$$- \frac{a^4}{16} \left\{ \xi \frac{\kappa^2 y (z - r_0)}{R^3} \cos \kappa (ct - R) - \eta \frac{\kappa y}{R^3} \sin \kappa (ct - R) \right\}.$$

In like manner the most important part of v_+ is

$$\frac{a^4}{16} \left\{ \xi \frac{\kappa^2 x (z - r_0)}{R^3} \cos \kappa (ct - R) - \eta \frac{\kappa x}{R^3} \sin \kappa (ct - R) \right\},$$

and all the terms of w_+ are of a higher order than these.

The results just obtained can be written

$$(u_+, v_+, w_+) = \frac{\kappa a^4}{16R} \left\{ \kappa \xi \frac{z - r_0}{R} \cos \kappa (ct - R) - \eta \sin \kappa (ct - R) \right\} \cdot \left(- \frac{y}{R}, \frac{x}{R}, 0 \right) \quad (95).$$

By a similar process it may be shown that the approximate forms for f_+, g_+, h_+ at a great distance from the aperture are given by the equation

$$(f_+, g_+, h_+) = \frac{\kappa^2 a^4}{16R} \left\{ \kappa \xi \frac{z - r_0}{R} \sin \kappa (ct - R) + \eta \cos \kappa (ct - R) \right\}$$
$$\cdot \left(- \frac{x (z - r_0)}{R^3}, - \frac{y (z - r_0)}{R^3}, \frac{x^2 + y^2}{R^3} \right) \quad . \quad . \quad (96).$$

We observe that the value of the magnetic force $(\dot{u}_+, \dot{v}_+, \dot{w}_+)$ at a great distance is

$$(\dot{u}_+, \dot{v}_+, \dot{w}_+) = c \frac{\kappa^2 a^4}{16R} \left\{ \kappa \xi \frac{z - r_0}{R} \sin \kappa (ct - R) + \eta \cos \kappa (ct - R) \right\} \cdot \left(\frac{y}{R}, - \frac{x}{R}, 0 \right)$$
$$\quad . \quad . \quad . \quad (97),$$

so that the factors, that contain t, are the same for the electric and magnetic forces at a great distance from the aperture.

36. Now let l, m, n be the direction cosines of the normal to a closed surface S drawn in a specified direction (inwards or outwards); then the rate at which energy

is transmitted across the surface, in the sense of the normal (l, m, n), is, by POYNTING's Theorem,

$$\frac{c^2}{4\pi} \iint \{l (g\dot{w} - h\dot{v}) + m (h\dot{u} - f\dot{w}) + n (f\dot{v} - g\dot{u})\}\, dS \quad . \quad . \quad (98).$$

We can, therefore, find the rate of dissipation of energy from the condenser, by forming this integral, for a sphere of large radius, R, and for the functions $\dot{u}_+, f_+, \ldots,$ the normal being drawn outwards.

If we write, for brevity,

$$x = l\text{R}, \qquad y = m\text{R}, \qquad z - r_0 = n\text{R}, \qquad \kappa (ct - \text{R}) = \psi,$$

the rate of dissipation is

$$\frac{c^3}{4\pi} \left(\frac{a}{2}\right)^3 \frac{\kappa^4}{\text{R}^2} \iint (\kappa \xi n \sin \psi + \eta \cos \psi)^2 \{l^2 (l^2 + m^2) + m^2 (l^2 + m^2) + n^2 (l^2 + m^2)\}\, dS$$

$$. \quad . \quad . \quad (99),$$

taken over the sphere; and the amount of energy transferred across the sphere in a period, $2\pi/(\kappa c)$, is given by the expression[*]

$$\frac{c^3}{4\pi} \left(\frac{a}{2}\right)^3 \frac{\kappa^4}{\text{R}^2} \frac{\pi}{\kappa c} \iint (\kappa^2 \xi^2 n^2 + \eta^2) (l^2 + m^2)\, dS, \quad . \quad . \quad . \quad . \quad (100),$$

or it is

$$\frac{c^3}{4} \left(\frac{a}{2}\right)^3 2\pi \int_0^\pi (\kappa^2 \xi^2 \cos^2 \theta + \kappa^2 \eta^2) \sin^3 \theta\, d\theta \,,$$

which is

$$\left(\frac{a}{2}\right)^3 \left(\frac{\kappa^2 \xi^2}{15} + \frac{\kappa^2 \eta^2}{3}\right). \quad . \quad . \quad . \quad . \quad . \quad (101).$$

Restoring the values of ξ and η, this expression becomes

$$\tfrac{2}{3} \pi c^2 \left(\frac{a}{2}\right)^3 \kappa^2 n^2 (n + 1)^2 r_0^{2n-4} \{\chi_n (\kappa r_0)\}^2 (1 + \tfrac{1}{10} \kappa^2 r_0^2), \quad . \quad . \quad . \quad (102),$$

where n is now the order of the spherical harmonic involved in the oscillations.

When $r_0 - r_1$ is small compared with r_0, the fraction of the total energy, which is dissipated in one period, is obtained by dividing the expression here written by the expression (92); it is

$$\tfrac{4}{3} \pi (2n + 1) (\kappa r_0)^3 (1 + \tfrac{1}{10} \kappa^2 r_0^2) \frac{a^3}{2\kappa r_0^7 (r_0 - r_1)}; \quad . \quad . \quad . \quad . \quad (103);$$

---

[*] The expression shows that equal amounts of energy are transmitted across the hemisphere in front of the aperture and that behind. This arises from the circumstance that the wave-length is of the same order of magnitude as the radius of the outer conducting surface, so that the waves bend completely round that surface.

and, at any rate when n is not too great, this is a very small fraction, if $r_0 - r_1$ is large compared with a, as has been supposed throughout the investigation.

37. The form of the result shows that the number of vibrations of the higher modes, that are executed before the disturbance sinks into insignificance, is much less than that of the lower ones. The occurrence of $r_0 - r_1$ in the denominator of (103) suggests that the principal factor in securing permanence of the vibrations is not the capacity of the vibrating system, but the screening action of the external conductor. The latter point might be illustrated further by considering the example of a spherical condenser, in the case where r_1 is small compared with r_0. The boundary condition at the inner surface can be satisfied approximately by putting, in equation (83), $B = 0$; and the frequency is determined by the equation

$$\frac{d}{dr}\{r^{n+1}\psi_n(\kappa r)\} = 0 . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (104),$$

when $r = r_0$. The total energy, for a mode of oscillation given by $\omega_n = r^n \mathrm{P}_n(\mu)$, is

$$\tfrac{1}{4}\frac{n(n+1)}{2n+1}\kappa^2 \mathrm{C}^2 \left\{1 - \frac{n(n+1)}{\kappa^2 r_0^2}\right\} r_0^{2n+3} \{\psi_n(\kappa r_0)\}^2, \quad . \quad . \quad . \quad (105),$$

and the energy dissipated in a period is

$$\tfrac{2}{3}\pi \mathrm{C}^2\left(\frac{a}{2}\right)^3 \kappa^3 n^2 (n+1)^2 \, r_0^{2n-4} \{\psi_n(\kappa r_0)\}^2 (1 + \tfrac{1}{30}\kappa^2 r_0^2) ; \quad . \quad . \quad (106),$$

and it is clear that the fraction of the total energy dissipated in one period is of the same order of magnitude as before, except that $(a/r_0)^3$ is substituted for the product of $(a/r_0)^7$ and $\{a/(r_0 - r_1)\}$. For the mode of least frequency $n = 1$, and we have $\kappa r_0 = 2\cdot75$ nearly,[*] instead of $1\cdot41$, its value when r_1 is nearly equal to r_0; and thus the fraction in question becomes approximately

$$82\pi\left(\frac{a}{2r_0}\right)^3, \quad \text{instead of being approximately} \quad 12\pi\left(\frac{a}{2r_0}\right)^3 \frac{r_0}{r_0-r_1} ;$$

or the rate of dissipation of energy, for the spherical condenser, is less when the capacity is very small than when it is very large.

* See J. J. THOMSON, 'Recent Researches,' p. 373.

INDEX SLIP.

II. *Argon and its Companions.*

By WILLIAM RAMSAY, *F.R.S.*, and MORRIS W. TRAVERS.

[PLATES 1–3.]

"*Natura nihil agit frustra*, is the only indisputed Axiome in Philosophy. There are no Grotesques in Nature; not anything framed to fill up empty Cantons, and unnecessary spaces."—SIR THOMAS BROWNE'S ' *Religio Medici*.'

CONTENTS.

INTRODUCTION.

ON June 9, 1898, we had the honour of giving the Society a preliminary account of " Krypton, a new Constituent of Atmospheric Air" ; and on June 16, we brought to the notice of the Society the existence of two other gases in air, which we named " Neon " and " Metargon." And at the meeting of the British Association at Bristol, we gave a further short account of these gases, and also announced the discovery of yet another gas, still heavier than the former, which we named " Xenon." In a note to 'Nature' in October, 1898, Mr. BALY called attention to the presence of helium lines in the spectrum of neon; the D_3 line had been previously remarked in atmospheric air by Professor KAYSER, of Bonn ('Chemical News,' August 23, 1895), and by Dr. FRIEDLÄNDER, of Berlin (*ibid.*, December 9, 1895).

Since publishing the preliminary notes referred to, we have been engaged in a study of these gases; and we may say at the outset that we have failed to isolate

"Metargon." It will be remembered that Professor Schuster, in a letter to 'Nature' (vol. 58, p, 199, June 30, 1898) pointed out that the spectrum of metargon presents a close analogy with that of carbon in the "Swan" spectrum. This we verified, and subjected the gas to prolonged sparking with oxygen in presence of caustic soda, attempting thus to oxidise and remove the carbon compound. The carbon spectrum was, however, still visible in a "Plücker" tube at a pressure of several millimetres of mercury. We were inclined to ascribe this spectrum to the presence of a compound of carbon with one of the usually inert gases. But we have since failed to detect this spectrum ; and we think it probable that in removing oxygen from the sparked gas some carbon was introduced with the phosphorus we then used. Remsen ('American Chemical Journal,' V, No. 6) has drawn attention to the fact that some specimens of yellow phosphorus contain carbon, probably in combination, and that on burning the phosphorus, carbon monoxide and dioxide are formed. After the supposed purification from carbon compounds, we must have introduced carbon monoxide into the argon of which the bulk of the sample of gas consisted, and still recognising a spectrum similar to that of carbon, we ascribed it to the presence of "metargon." The phosphorus we at present employ does not produce such a spectrum, and we must therefore withdraw our statements regarding this supposed gas.

On making a mixture of pure yellow phosphorus with powdered charcoal, and using it to remove oxygen from argon, the spectrum of the remaining gas was identical with that which we described as "metargon," even to the bright bands in the violet, which are not present in the spectrum of pure carbon monoxide, but which are visible in that of cyanogen. Some crude phosphorus, kindly given us by Mr. George Albright, gave the same spectrum, when used to remove oxygen from argon.

We can only express our regret that such a simple cause has led us into an erroneous statement, which we here retract.

It will be seen from what follows that the inert gases of the atmosphere form a well-defined series parallel to that of the halogens, and of the metals of the alkalies. Thus we have :—

?	Fluorine.	Chlorine.	Bromine.	Iodine.
Helium.	Neon.	Argon.	Krypton.	Xenon.
Lithium.	Sodium.	Potassium.	Rubidium.	Cæsium.

It is unlikely that any gas possessing an atomic weight higher than that of xenon is to be found in atmospheric air. The question as to the precursor of helium will be discussed later. The atomic weights of these gases are taken as twice their densities, on the apparently well-founded assumption that the ratio 1·66 between the specific heat at constant volume and that at constant pressure implies that the molecule and the atom are identical. If this assumption, which is fully justified on physical grounds, is denied, then these elements can find no place in the periodic

table. This conclusion, as will be shown when the question of their atomic weights is discussed, is wholly untenable.

The separation of these gases from air, and from one another, is a very trouble-some one, and demands much time and patience. Moreover, the tests of purity are not easily applied. The pocket spectroscope is, of course, a useful instrument, when it is required to decide as to the nature of the main constituent of a gaseous mixture. It is convenient, in using the instrument, to remove the lenses and the slit, and to examine the capillary portion of the Plücker tube at a distance of 6 or 8 feet. The brightly illuminated capillary then itself forms a sufficiently good slit, and the spectrum is a very luminous one. This plan, however, fails when one gas is con-taminated by traces of others. By altering the pressure in the Plücker tube, and by interposing a jar and a spark-gap, it is often possible to reveal the presence of a small proportion of one of these gases in a large amount of another. But at the best it is only a rough guide. The final criterion of purity was always the balance. If two portions of gas were found to possess nearly the same density, they were fractionated and their density again compared. If the density was not altered by fractionation, they were regarded as homogeneous and identical.

Apparatus employed in these Researches.

It will conduce to brevity and clearness, if we begin with an account of the apparatus and methods which we found advantageous in preparing the liquid air, in fractionating the gases, in determining their density, their refractivity, their molecular volumes, their compressibilities, and in measuring their vapour-pressures. In some cases, the methods have been employed before, and where this has been the case, we will confine our description to any modifications which we have found useful for the particular purpose in view, giving references to the papers previously pub-lished on the subject.

The Liquefaction of Air* was carried out by means of an apparatus built from the design of the inventor, Dr. HAMPSON, by the " Brin" Oxygen Company. Its principle, which is common to Dr. LINDE's machine, consists in the expansion of air compressed under a pressure of 150 to 175 atmospheres, through a valve L regulated by the spindle M; the "Thomson-Joule" effect is thus utilised, and the expanded air passes upwards over the surface of the coils KK, through which it has passed downwards in a state of compression, thereby cooling the coils pro-gressively, and absorbing more and more heat in expanding, until the temperature is ultimately reduced to below the temperature at which the air liquefies. The liquid collects in the vacuum-vessel N and can be drawn off at the stop-cock O. The

Fig. 1.
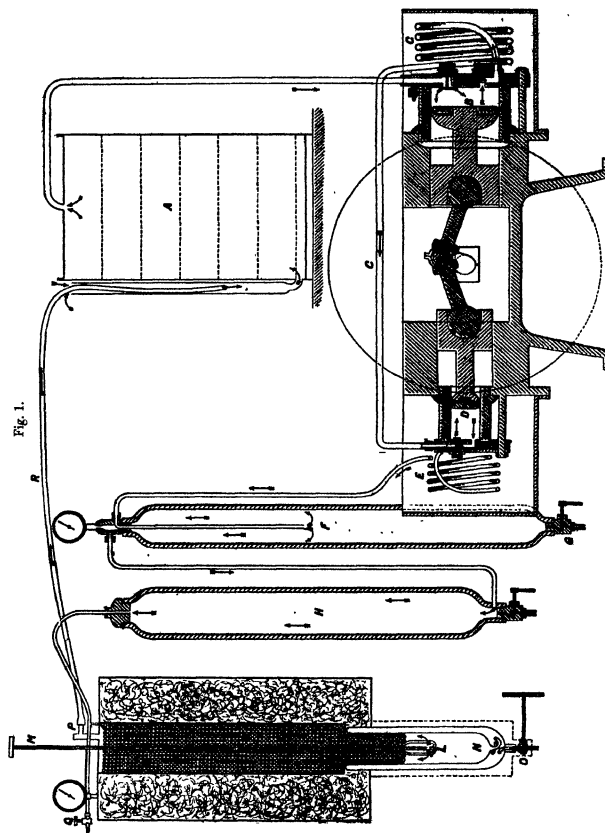
air is compressed by a "torpedo air-compressor" made by the Whitehead Company of Fiume, Austria; it enters the compressor through a low-pressure purifier A, consisting of a drum fitted with trays, each covered with slaked lime, to effect a preliminary absorption of carbon dioxide. In the first cylinder, B, it is compressed with a pressure of about 60 atmospheres, and passes through a tube C to the intake valve of the second cylinder D, where the pressure is raised as high as is desirable; the air then passes through the cooling coil E to the separator F, where the water which enters the cylinders for the purpose of lubrication, is separated and discharged at the cock G; it passes through a high-pressure purifier H, charged with caustic potash in lumps, which removes water-vapour, and any carbon dioxide which may have escaped absorption in the low-pressure purifier. From the high-pressure purifier, it enters the liquefier. The power required to drive the compressor is about five horse-power; and when working at a pressure of 175 atmospheres, it delivers a litre-and-a-quarter of liquid air per hour. The air begins to run liquid in eight minutes after starting the compressor; and the amount of attention it requires is no more than that of an ordinary steam-engine, in oiling, cleaning, &c.

The apparatus employed in fractionating the gases was similar to that described in the 'Proceedings,' vol. 64, p. 188. To that description there is nothing to add, save that the bulb b in the figure is relatively much smaller, when only small amounts of gas are available; and the tube-reservoir a, had a capacity of 35 cub. centims. It was also possible to compress the gas in a by raising the attached reservoir,

Fig. 2.
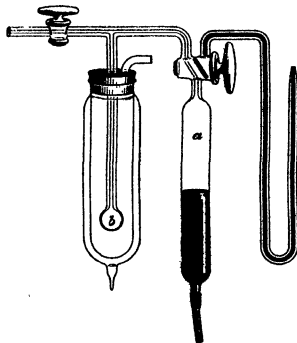
under a pressure of over an additional atmosphere, in order to facilitate liquefaction. This device was particularly useful in the separation of neon from argon, and in the liquefaction of argon, mixed with the heavier gases, which could then be accomplished without the necessity of causing the air to boil at reduced pressure.

By enclosing the vacuum-vessel shown in the diagram in a wider outer vessel, filled with liquid air, and by reducing pressure in the inner vessel, the temperature could be reduced as low as 205° and maintained with hardly any labour, owing to the very slow ingress of heat. This device was particularly useful in the separation of neon from argon ; but it was unnecessary in the separation of argon from the heavier gases.

Determination of Density.—As the densities of the gases obtained by the process of fractional distillation gave us the surest indication of the manner in which the fractionation was progressing, very many determinations were made during the course of the investigation. Except in the case of argon itself, the quantities of gas at our disposal were exceedingly small ; and as we were consequently obliged to reduce the dimensions of the weighing globes to the smallest capacity compatible with the degree of accuracy which we desired to attain, and to design a combined gasholder and manometer for introducing as much of the gas as possible into the globe.

Some four years ago, in conjunction with Dr. NORMAN COLLIE, in a paper read before the Chemical Society, we pointed to the possibility of determining the density of helium to within 1 per cent. above or below its actual value by weighing the gas in a globe of about 30 cub. centims. capacity. The results obtained for the much heavier gas, argon, naturally showed much closer concordance ; and we have found that the accuracy of such experiments is conditioned only by the sensibility of the balance. Since neon, the lightest of the new gases, is five times as heavy as helium, we resolved to employ a globe of 7 cub. centims. capacity. Determinations of the density of the same sample of gas, carried out by each of us independently, showed that it was possible to obtain results of sufficient accuracy and concordance, even when the pressure on the gas did not exceed 250 millims. of mercury ; this statement is substantiated by the figures in the following table :—

The gas chosen is neon containing helium ; the capacity of the bulb is 7·152 cub. centims.

	Weight of gas.	Pressure.	Temperature.	Density.
	grammes.	millims.	°	
W. R. . . .	0·00531	788·6	16·20	8·48
M. W. T. . .	0·00539	507·1	16·95	8·44
W. R. . . .	0·00331	422·5	18·40	9·94
M. W. T. . .	0·00320	417·2	20·15	9·80

In these and all other determinations, the density has been referred to O = 16.

The Weighing Globes.—Three globes were employed of capacities corresponding to

163·170 cub. centims., 32·138 cub. centims., and 7·152 cub. centims. ; the largest was used only to determine the density of pure argon. Sealed glass globes of the same external volume as the weighing globes were used as counterpoises.

The Weights.—The values of the weights were carefully referred to that of a centigramme rider. It was not necessary to know their absolute values, for since the volumes of the globes were obtained by weighing them filled with water at a known temperature with the same weights as were used in weighing the gases, the weights of the gases were referred through the weight of an equal volume of water to the weight of an equal volume of oxygen. The weight of a litre of oxygen was taken as 1·4296 gram. ('Phil. Trans.,' A, 1895, Part 1, p. 201)*.

The Balance.—The globes were weighed in a long-beam Oertling balance to 0·01 milligram. They were usually allowed to hang for about half-an-hour in the balance-case, before the final weighing was made.

The Preparation of the Gases.—The gases were sparked with oxygen in presence of caustic soda to remove nitrogen, &c. ; the oxygen was subsequently removed by means of phosphorus, the water-vapour being at the same time absorbed by the products of its combustion. As this process differs somewhat from any previously described, it may be advisable to enter into it in some detail. After the gas had been sparked, it was withdrawn into a burette, and separated from the soda. A tube was then completely filled with mercury, and a piece of phosphorus, as large as a small pea, was introduced, so that it floated up above the mercury. The top of the tube was then warmed, so as to melt the phosphorus, and the gas was introduced from the gas-burette, through the mercury, bubble by bubble ; as each bubble entered it caused the phosphorus to inflame. When all the gas had been introduced, the top of the tube was again warmed, and a rain of phosphorus fell through the tube. After cooling, the gas could be withdrawn if desired ; but it was often kept in the same tube until required. It appears to be necessary to have a sufficiency of oxygen in the gas which is admitted to the phosphorus ; for if the phosphorus does not inflame, we have found that the spectrum shows lines belonging to phosphorus ; if, however, the phosphorus actually catches fire, there is no sign of any volatile compound being present in the gas.

The apparatus employed for introducing the gas into the globe is shown in fig. 3. The gas can be introduced through the siphon into the burette A, which can be placed in communication with the globe E. The side-tube B, which is of the same diameter as the upper part of the burette, facilitates the measurement of the pressure on the gas, after it has been allowed to fill the globe, thus eliminating any correction for capillarity. The pressure was read off on a silvered glass scale, to which the burette was fixed. The tube c, which was about 2·5 millims. in-diameter, contained a roll of

---

* In previous determinations with the same bulbs different weights were used, and the capacity is given as a slightly different number. As these were also referred to water, no correction as regards the densities previously determined is necessary.

amalgamated silver-foil, which effectually retained any small globule of mercury which might have been carried over into the globe E by the stream of gas. The apparatus communicated with the pump through the stop-cock c.

Fig. 3.
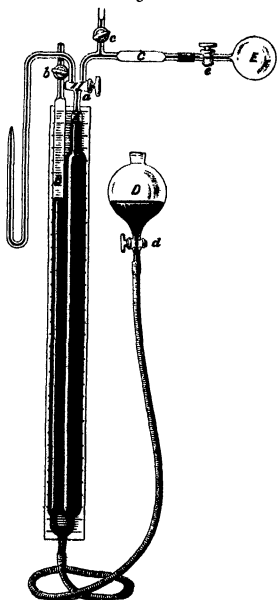
The method of manipulation was briefly as follows. The apparatus was first filled with mercury to the level of the stop-cocks a and b, which were closed; the reservoir D was then lowered, so as to leave a barometric vacuum in the tubes A and B; the stop-cock d was then closed. The globe E was now attached to C by a short length of thick-walled india-rubber tubing, which was wired at both ends; the globe itself was enclosed in a small cardboard box, which also contained the bulb of the thermometer. The stop-cocks a, c, and e were now turned, so as to place the whole of the apparatus in communication with the pump. It was particularly important to remove the air in the bore of the stop-cock a, as its volume was not inconsiderable in comparison with that of the gas weighed. After exhausting the apparatus, the stop-cock a was closed, and by turning the stop-cock d, the mercury was allowed to rise in the burette. The gas was then introduced through the siphon into the burette, and the stop-cock c was closed. On turning the stop-cock a, the gas entered the globe, while the reservoir D was held in such a position that the mercury did not rise above a. The stop-cock b was then opened, the stop-cock a being momentarily closed, in order to avoid any danger of mercury being forced up into C; and the position of the reservoir D was adjusted, so that the surface of the mercury in A lay as near as possible to the top of the tube. After about a quarter of an hour, the temperature and pressure were observed, the stop-cock e was closed, and the barometer was read. The gas contained in the tube C and in the burette was removed by exhaustion, and collected through the pump, and the globe was detached from the apparatus and suspended in the balance.

After weighing the globe, it was again attached to the apparatus, and when the tube C had been exhausted, the stop-cock e was opened, and the gas was removed from the globe through the pump. The globe was then weighed empty, and from the difference of the two weighings the weight of the gas was determined. With a globe of the small size of those employed it was unnecessary to apply any correction for shrinkage under atmospheric pressure.

The Preparation of Neon.

The process which led to the discovery of neon was the fractionation of a large quantity of liquid argon. A partial description of this has already been given in the 'Proceedings,' vol. 64, p. 188, where we have described the preparation and some of the properties of pure argon; and also in a note "On the Companions of Argon," vol. 63, p. 437. But for the sake of completeness, we shall briefly recapitulate the main points.

The whole of the argon which we had made, probably about 15 litres, was liquefied in a bulb immersed in liquid air boiling under reduced pressure. The bulb was then placed in communication with a series of mercury reservoirs, and successive fractions of gas were collected; the remaining liquid was then allowed to evaporate, and the gas was returned to the gasometer in which it had originally been stored. The first fraction of gas turned out to be rich in neon, and was afterwards found to contain helium. After repeated fractionation its density was reduced to 14.

The main quantity of the argon was submitted to a second fractionation, and a further quantity of neon was obtained. A third liquefaction of the argon yielded a first distillate, of which the density was only slightly lower than that of argon, viz., 19·05. After one-sixth of the argon had evaporated, the remaining fractions manifested no further change in density; and it was therefore assumed that the middle fractions consisted of pure argon.

The gas of density 14 was refractionated by means of the apparatus shown in fig. 2. The gas was introduced from the gasholder a into the bulb b, which was immersed in liquid air, boiling under reduced pressure. The apparatus communicated with a Töpler pump through the stop-cock; before admitting the gas, all air had been removed. On raising the reservoir attached to a, and applying an additional atmosphere's pressure, the mixture of neon and argon condensed in the bulb b. The stop-cock on the gasholder a was then closed, and the liquid was allowed to stand for a few minutes, so that it might become homogeneous.

On lowering the mercury reservoir, and opening the stop-cock, a portion of the gas, about one-fifth of the whole, was allowed to boil off into the gasholder, the temperature of the bulb being kept as low as possible. This fraction of the gas, which contained the greater quantity of the lower-boiling constituents, was transferred to a tube, and a second, third, and fourth fraction were taken off in the same manner. The last fraction was removed through the pump, and collected as usual. These five fractions, consisting of approximately equal quantities of gas, formed

a series of ascending densities; they are represented by the numbers 2 to 6 in the annexed table. The method employed for their further fractionation was as follows:—

TABLE I.
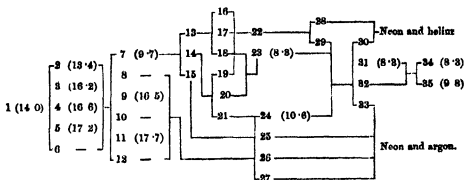
The heaviest portion 6 was placed in the gasholder, and liquefied in the bulb; a portion of it was allowed to boil off into the gasholder; and the remainder (the heavier portion 12) was transferred to a tube. The next heaviest portion 5 was then placed in the gasholder, liquefied together with the gas already in the bulb, and subsequently divided by evaporation into two portions. This operation was repeated in the case of each fraction, the proportion of gas taken back into the gasholder increasing at each operation so as to produce six approximately equal fractions, represented in the table by numbers 7 to 12.

This process, as is clearly shown in the table, resulted in the separation of the heavy and light constituents of the mixture with a certain degree of sharpness. Fraction 7, when examined spectroscopically, was found to contain only a trace of argon, while fraction 12 consisted almost entirely of that gas; beyond this, it was found that the gas of density 9·7 could not be liquefied at −200° under a pressure of two atmospheres.

In order to determine whether fraction 7 consisted of homogeneous gas, mixed with only a trace of argon, it was necessary to fractionate it further, and to submit each fraction to a rigid spectroscopic examination. Since the gas would not liquefy, it was mixed with about eight times its volume of oxygen; the mixture was liquefied in the apparatus, and the liquid was allowed to evaporate in three fractions, Nos. 13, 14, and 15. After the oxygen had been removed from the inactive residue, it was found that in the third fraction (15) alone, could argon be detected by visual observation of the spectrum. When, however, the spectrum of the lighter fraction was photographed, it was found that the lines represented by the wave-lengths 5876 and 5016 were present. The gas contained helium, which is accordingly present in the atmosphere; KAYSER's and FRIEDLANDER's observations were confirmed. After knowing that they were present, it was possible to see the principal helium lines with a direct-vision spectroscope.

This observation introduced a new feature into the work; for, although no great difficulty had been experienced in the separation of the light gas from argon, it appeared likely that the separation of two gases so nearly akin as helium and neon would be considerably more troublesome. And this proved to be the case.

The next stage in the fractionation consisted in the separation of fractions 13 and 14 each into three parts by mixing them with oxygen, and distilling the mixture. The oxygen was removed from each fraction separately; and after spectroscopic examination, the fractions 16, 17, and 19, which contained the greatest quantity of helium, were mixed, as were also fractions 18 and 20, which consisted of nearly pure neon. The density of the latter, 23, was 8·3. The fractions 8 to 12, with 15 and 21, were also submitted to a further fractionation yielding about 20 cub. centims. of neon of density 10·6, containing only a little argon.

Up to this point, the gas had been divided into five portions :—

      i. (22), consisting of neon with much helium ;
     ii. (23)   „   „   trace of helium ; (D = 8·3)
   iii. (24)   „   „   trace of argon ; (D = 10·6)
   iv. (25)   „   „   much argon ;
    v. (26) and (27) residues.

Following this scheme, fraction 22 was divided into two parts, and the fraction containing least helium (29) was mixed with fractions 23 and 24. This gas, mixed with oxygen, was liquefied and divided into four parts, the lightest containing helium, and the heaviest, argon. The two middle fractions, 31 and 32, were mixed, and after addition of excess of oxygen, were again fractionated. The lightest fraction, amounting to about 10 cub. centims., had a density of 8·3, and still contained helium ; fraction 35, amounting to 5 cub. centims., had the density 9·8 ; and a third fraction not numbered, for it consisted only of a single small bubble of gas, showed distinctly the spectrum of argon. We regarded the sample of density 9·8 as the purest ; but there is no doubt that it still contained both helium and argon.

It appeared, therefore, an almost impossible task to separate these gases by fractionation. If it had been possible to liquefy the mixture without addition of oxygen, separation might have been thus accomplished; but fractionation under these circumstances can only be only an incomplete operation ; and we therefore resolved to attempt another plan. But for this, a larger quantity of neon was required, and a method of separation from the oxygen and nitrogen of the air, which we shall now describe, was devised.

In describing the compressor (fig. 1), it was mentioned that the air drawn in had passed through a low-pressure purifier, filled with trays charged with slaked lime, in order to remove carbon dioxide. As a rule, the air which had escaped liquefaction was allowed to blow off through the nozzle Q and mix with the atmosphere. But by causing it to blow through an india-rubber tube R, the further end of which

delivered into the intake of the low-pressure purifier A, the same air was returned
into the compressor; the amount which had been liquefied was replaced by ordinary
air, drawn into the purifier, and from it to the compressor, at the same time. In
this manner the air which passed through the liquefier was fractionated and
refractionated, for the heavier and more condensable portions were continually
removed, while the lighter portions passed again through the machine. The escape
valve, Q, at the top of the liquefier (fig. 1) was connected by thick-walled india-rubber
tubing with the bulb a (fig. 4) immersed in a vacuum-jacket which was filled with
the liquid air continually being produced during the process; that is, with the
portion which had condensed from the total quantity entering the machine. The
boiling point of this liquid air was unusually low, for it contained, owing to the
process of circulation, a larger proportion of nitrogen than usual. The air entering
the bulb a was under an additional pressure of about an atmosphere and a half, as
shown by a gauge. When the bulb a was filled with liquid to near the top, the

Fig. 4.
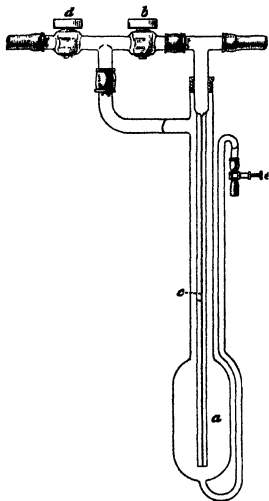
stop-cock b was shut, so that the compressed
air no longer freely entered the bulb,
but passed through the narrow tube c, and
bubbled through the liquid in the bulb.
On opening the stop-cock d, which commu-
nicated with a tube leading to a large gas-
holder, the liquid in the bulb evaporated;
when about one-sixth had boiled off into
the gasholder, this part of the operation
was stopped; and by shutting the stop-
cocks b and d, and opening the clip e, the
liquid in the bulb was forced out, and flowed
into the outside jacket, increasing the
volume of air employed for the purpose of
condensation. In this manner, the gas-
holder, of about 120 litres capacity, was
filled with the lightest and lowest-boiling
portion of liquid air. In case others may
desire to repeat this operation, a word
of caution may be given: it is absolutely
necessary to blow air through the contents
of the liquefying bulb, while evaporation is
proceeding; if this is not done, evaporation
takes place chiefly from the surface, and
the gas collected in the gasholder will con-
tain very little neon; it was only after
several unsuccessful trials that we dis-

covered the cause of our failure to obtain gas rich in neon. Moreover, liquid nitrogen is very apt to become superheated, and to boil from the surface.*

The gas contained in the gasholder was freed from oxygen, of which there was very little present, by burning with hydrogen, as described later; and from nitrogen, of which it chiefly consists, by a red-hot mixture of lime and magnesium. From about 120 litres of the light air, about 800 cub. centims. of argon comparatively rich in neon was obtained; it was fractionated, in a manner similar to that described on p. 55, mixing it with nitrogen instead of oxygen; for it was supposed that neon might have a somewhat greater solubility in the lower-boiling liquid. No improvement was noticed; hence the further fractional solution was carried out with oxygen as a solvent, the temperature being lowered to −205°, by causing the oxygen to boil under reduced pressure (see p. 52). The gas obtained by the last process, together with the lighter fractions from previous operations (34 and 35), were condensed with 200 cub. centims. of oxygen, and the liquid was evaporated in six fractions (36 to 41); of these, the last three (39, 40, and 41) were discarded, as consisting mainly of argon; the gas was examined spectroscopically after removal of the oxygen with phosphorus. The first three portions were divided into five (42 to 46), and the last fraction was again discarded after examination. In each case, the last fraction consisted of the main bulk of the oxygen. The operation was repeated with the first three fractions (42, 43, and 44), giving four portions (47 to 50); the last fraction was mixed with No. 45. These fractions contained argon, and it was hoped that they would contain but little helium.

The operations described are shown diagrammatically on the table which follows.—
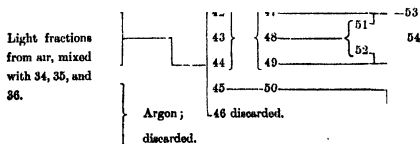
It was supposed that Nos. 53, 54, and 55 contained helium and no argon, and that 56 contained no helium, but a fair amount of argon. The first three of these were again fractionated, and a small quantity of argon was found in the oxygen remaining after the chief part of the gas had evaporated; this operation was repeated, and a trace of argon was again discarded, thus :—

* We owe our thanks to Messrs. GRAY HEHNER and LIDBURY for kindly superintending this operation.

Density.

$$
\begin{array}{l}
57 \; \rbrace \\
58 \; \rbrace - \lbrace \\
59 \; \rbrace
\end{array}
\quad
\begin{array}{l}
61 \; . \; . \; . \; . \; 7{\cdot}34 \\
62 \; . \; . \; . \; . \; 7{\cdot}52 \\
63 \; . \; . \; . \; . \; 8{\cdot}27
\end{array}
$$

55   64

56 . . . . 8·57   Argon; discarded.

The density of 63 was not greatly different from that of 56, which contained argon, hence 63 also obviously contains that gas; these two were accordingly mixed, and liquefied along with 500 cub. centims. of oxygen; the mixture was collected in two portions, 65 and 66; the first fraction, 65, had the density 7·24; the last, 66, contained nearly all the argon in these later samples.

It became a question whether it was possible to separate helium from neon by this process. It was obviously possible to remove practically all the argon; and this is not surprising, since the boiling-points of argon and oxygen are nearly the same; but an experiment to determine the relative solubilities of helium and of the impure neon which we possessed showed that there was no very great difference in favour of the latter. The solubility in oxygen at −205° of both gases was approximately 80 volumes of gas in 100 of liquid oxygen under normal pressure. The presence of argon greatly increases the solubility, and, indeed, these gases appear to be freely soluble in liquid argon. The results obtained by fractional solution in liquid oxygen were not sufficiently promising to induce us to continue the process.

The next attempt to free the neon from helium was made by means of diffusion. The density of helium being in round numbers 2, and that of neon presumably 10, the rate of diffusion of helium should be about twice as great as that of neon. A small diffusion apparatus similar to the one described in the 'Proceedings' (vol. 62, p. 318, A in figure) was employed, and a preliminary experiment with oxygen and hydrogen showed that it was possible to reduce the quantity of hydrogen considerably in one operation. Collecting the diffusate in three fractions of relative volumes, 59, 45, and 16 per cent. of the original, the last fraction contained only 16·5 per cent. of hydrogen; the percentage of hydrogen in the original mixture had been 54.

All the available neon, therefore, which had been freed from argon by fractional solution in liquid oxygen, was united; the density of the mixed samples was 7·3. It was divided by diffusion into three equal amounts, 1, 2, and 3, and these were further fractionated in the same manner as in conducting a fractional distillation; that is, fraction 1 was introduced into the diffusion apparatus, and the first portion which passed over was collected separately; portion 2 was then introduced, and the first portion of the diffusate was added to 1; the main bulk was collected in 2; and fraction 3 was added to the gas remaining in the diffusion apparatus, the first portion of the diffusate being collected in tube 2, and the remainder in tube 3. This

operation was repeated three times; the lightest portion showed the helium spectrum with brilliancy. During the third fractionation, the diffusate was collected in four fractions. The density of the heaviest was 8·90, and of the lightest, 6·24. The heaviest fraction was again diffused, and the first quarter was rejected; the density of the remainder had risen to 9·18. In each case the gas was sparked with oxygen before it was weighed, so as to remove any accidental contamination. It was thus evident that 9·18 was a minimum density for neon.

As it was evident that no complete separation of neon from helium could be effected by means of any process of diffusion or solution, these attempts were given up. The separation was finally accomplished by means of liquid hydrogen, the operation being conducted in precisely the same manner as the separation of argon from krypton and xenon, and the latter two gases from each other by condensing them with liquid air.

The liquid hydrogen was made in an apparatus constructed by one of us (M. W. T.) with the valuable assistance of the laboratory mechanician, Mr. J. HOLDING. As it is described in detail elsewhere,* suffice it to say that hydrogen, made from zinc and sulphuric acid, and submitted to no further purification than by a rapid passage through a wash-bottle charged with a solution of potassium permanganate, was compressed by the Whitehead compressor into a set of coils cooled by liquid air boiling under low pressure; the hydrogen was allowed to expand through a jet to atmospheric pressure, at the end of a regenerator-coil, similar to that in the HAMPSON's apparatus. The operation presents no special difficulty to those used to the Hampson liquefier, and the yield is rapid and ample. Any impurities can be easily separated by filtration. For complete separation of neon and helium, 100 cub. centims. is sufficient; for liquid hydrogen is fairly permanent, and does not evaporate much more quickly than liquid air.

On admitting the mixed gases into a bulb cooled by liquid hydrogen, all the gas at once entered, even under reduced pressure, there being sufficient neon present to dissolve and liquefy the helium. The pressure in the mercury gasholder was made very low, by depressing the reservoir, and the first fraction of gas re-entered the gasholder. It consisted of helium, mixed with a trace of neon. It was set aside as not further needed. The residue was removed through the pump, and re-liquefied. The stop-cock to the pump being opened, a few bubbles were removed. The second fraction, which had a vapour-pressure of only a few millimetres at the boiling-point of liquid hydrogen, was removed by pumping; the residue, amounting to a few bubbles only, being kept separate, for it might possibly have contained a little argon, the vapour-pressure of which was practically zero at that temperature. It was pumped off after the removal of the liquid hydrogen. It is evident from the very low vapour-pressure of neon at the boiling-point of hydrogen that helium must have been completely removed by this process. It is true that a little neon was lost by

* 'Phil. Mag.,' 1901, vol. 1, p. 411.

neglecting to refractionate the first distillate: but as our object was to procure pure neon, and not pure helium, this was of no consequence.

The density of this neon was determined independently by each of us.

	W. R.	M. W. T.
Volume of bulb	32·038 cub. centims.	32·038 cub. centims.
Weight of gas	0·01100 gramme.	0·01120 gramme.
Pressure	306·5 millims.	313·8 millims
Temperature.	292·05° abs.	292·2° abs.
Density of neon	9·99	9·94

The Preparation of Krypton and Xenon.

These gases are left behind after the evaporation of liquid air, along with a large quantity of oxygen and some nitrogen. It has been necessary to use considerable quantities of liquid air for refrigerating purposes, and in almost all cases, the residue of a few cub. centims., instead of being allowed to evaporate away, was collected in a gasholder. When a sufficient quantity had accumulated, the oxygen was removed by combustion with hydrogen in the following manner :—

An iron tube, about half an inch in internal diameter, filled with platinised asbestos, was heated to redness in a long-flame gas burner. At one end of the tube an arrangement like a blowpipe was attached by screwing. and through one of the tubes a current of hydrogen entered from a KIPP's apparatus, while through the other the mixture containing the rare gases was admitted. These gases burned, and by regulating the stop-cocks, it was possible to adjust the proportions fairly well. But to make it possible to leave the operation to itself, and to render it in a sense automatic, a tell-tale tube of hard glass, containing copper oxide in one half, and metallic copper in the other, was connected with the other end of the iron tube : so that if excess of oxygen passed it oxidised the copper, and if excess of hydrogen, it reduced the copper oxide. The further end of the glass tube was connected with a reservoir, and with a gasholder. The water produced condensed in the reservoir, which was so arranged that it could be run off from time to time ; and the gases, now free from oxygen, passed into the gasholder. The nitrogen was next removed with a mixture of lime and magnesium. This mixture, as already stated by MAQUENNE, absorbs nitrogen at a much lower temperature than magnesium itself, and consequently it is not difficult to manipulate it in hard-glass tubes. The only precaution required is that the lime must be free from hydroxide and carbonate, for, if not, the magnesium is apt to act so quickly on these impurities as to produce a rush of gas,—hydrogen and carbon monoxide—so rapid as practically to amount to an explosion. The lime, therefore, should be made directly before use, by heating precipitated carbonate in a muffle for several hours; and it is conveniently mixed with magnesium powder immediately before using, in about equal proportions. Care must be taken to pass the current of

gas containing nitrogen slowly over this mixture, heated to dull redness, else the temperature may rise so high as to cause the tube to fuse. There is no limit to the rapidity of the absorption, which is conditioned merely by the melting-point of the glass, for great heat is liberated by the reaction. The nitrogen is completely removed by this method, no trace being left behind; but as it is impossible to exclude all traces of carbonate or hydroxide, the gas collected always contains some hydrogen and carbon monoxide. These must be subsequently removed by passing the gas over copper oxide at a red heat, and absorbing in soda-lime the carbon dioxide produced.

The first preparation of krypton was described in the 'Proceedings,' vol. 63, p. 405; the density of the sample of argon showing the spectrum of krypton brilliantly was 22·5. But we were surprised to find that on refractionating, the density of the gas, instead of rising, fell. This we found subsequently due to the fact that the gas was pumped out of the fractionating bulb without removing the jacket of liquid air; and under these circumstances, the xenon, of which the density is 64, remained in the bulb. On one occasion, the jacket was removed; and the small remaining quantity was collected separately; it turned out to possess a spectrum totally differing from that of krypton, and was subsequently named "xenon." The spectrum of krypton is one which is remarkable for its brilliancy, and for the fact that a small amount of gas, present along with argon, is easily recognised by the spectroscope; and this led us to believe that krypton possessed a density not differing greatly from that of argon. Repeated fractionation, moreover, did not appear to produce any marked change in its character or brilliancy.

It would be tedious to relate in detail all the unsuccessful experiments which were made before the truth was discovered, that krypton is a gas present in very small quantity, but possessing a density of 40·8. Suffice it to state that the gas was fractionated more than twelve times, and that 24 determinations of density were made. We will, therefore, at once proceed to describe the method by which krypton and xenon were prepared in a pure state.

The gas remaining after the evaporation of the air used in various experiments, which we estimate as having been not less than 30 litres in all, was purified in the manner already described from oxygen and nitrogen. The remaining gas, of which, of course, argon constituted the main part, was liquefied and fractionated, so as to separate the heavier portion. The process employed in fractionation was as follows :— The heavy gas was liquefied in the apparatus described on page 51, at the temperature of liquid air. When it was rich in argon, some external pressure was necessary for liquefaction; but later, the gases liquefied, and when nearly pure, solidified, even under reduced pressure. The lighter portion was allowed to boil back into the gas-holder, and the remainder collected in one portion; this operation was repeated, and the remaining gas is alluded to as fraction 1.

The whole of this fraction, amounting to about 300 cub. centims., was introduced into the gasholder of the liquefaction apparatus, and was condensed at the

temperature of liquid air by applying a slight pressure. The liquid was then allowed to evaporate back into the gasholder in six successive fractions, which were stored in separate tubes, Nos. 2 to 7. Fraction 2 was discarded, as the spectrum showed that but little krypton was present.

The remaining five fractions, Nos. 3 to 7, were submitted to a further fractionation in the following manner. The fraction containing the greatest quantity of argon (3) was introduced into the gasholder and condensed in the bulb. About two-thirds of this was then allowed to evaporate, and the gas transferred to a tube (8); this gas was afterwards discarded. The contents of the next tube (4) were then condensed, along with the residue from 3, and again partially evaporated; this gas formed the second fraction (9) of the next series. This process was repeated with each of the fractions from 3 to 7, the quantities of gas taken off at each fractionation being so graduated that the six resulting fractions (8 to 13) were approximately equal. The last fraction was taken off through the pump, while the liquid air in the jacket was removed so as to make sure that no traces of xenon were lost. By treating the fractions 9 to 13 in the same way, six other fractions (14 to 19), were obtained; of these the first two (14 and 15) were rejected after spectroscopic examination. The numbers indicating the rejected fractions are enclosed in square brackets.

As the four remaining fractions (16 to 19) solidified at the temperature of liquid air, we thought that by mixing the gases with oxygen it might perhaps be possible to obtain a more rapid and complete separation. A single experiment sufficed to convince us that the gain would have to be considerable to compensate for the additional trouble and difficulty in manipulation; the method was therefore abandoned.

```
 [2]    [8]    [14]
        9     [15]
 3            16 ─┐
 4     10          ├─[20]
 5 ─┐       17 ────┴─[22]          24 (D = 32·07)
     ├─ 11          ├─23        ┌─
 6   │ 12    18 ────┘            25 (lost)          ·27─┐
     │                  ·       └─26 ──────────────      ├─(D = 47·55)
 7 ─┘ └─13   19 ──────────────────────            ·28─┘

                                                  ·29  (D = 56·04)
```

Proceeding by the original method, fraction 16 was condensed, and evaporated in two portions, 20 and 21. The lightest portion (20) amounting to two-thirds of the whole was discarded; the heavier (21) was liquefied together with fraction 17, and divided into a light portion (22), amounting to about one-fifth of the volume of the mixed gases, and a heavier portion (23). In the same way, fractions 24, 25, and 26 were obtained from fractions 18 and 23; and fractions 27, 28, and 29, from fractions 19 and 26.

In the early stages of the fractionation the spectroscope had been the sole guide in

dealing with these gases; but later on, it became evident that we had in our hands gases possessing very different vapour-pressures at the temperature of liquid air; for while argon required additional pressure for its liquefaction, the vapour-pressure of krypton did not exceed 160 millims., and that of xenon appeared to be less than could be measured.

On account of the ease with which these gases solidified, we were obliged to take certain precautions usually unnecessary. If the bulb and stem of the liquefaction bulb were immersed in the liquid air before the introduction of the gas, the gas did not solidify in the bulb but in the stem; and as the liquid air evaporated, it began to volatilise without proper fractionation. To overcome this difficulty, the vacuum vessel was held in a clamp, so that the surface of the liquid air just touched the bottom of the liquefaction bulb, and when the gas was admitted, it condensed in the bulb and not in the tube; the vacuum vessel was then raised so that the liquid air completely covered the bulb. In fractionating krypton and xenon, the fractions of gas were taken off through the pump, as their vapour-pressure was too low to allow of them being collected into the gasholder.

To return to the fractionation of the gases from which the argon was now nearly removed, we had at this point five fractions, viz., Nos. 24, 25, 27, 28, and 29. Fraction 25 was unfortunately lost through an accident; but as the remaining fractions consisted of fairly pure krypton and xenon, and amounted to nearly 50 cub. centims., the loss was not of vital importance. The following table illustrates the method employed in the next series of fractionations.

$$
\begin{array}{l}
24 \quad (D = 32\cdot07) \quad
\begin{bmatrix} 30 \\ 31 \end{bmatrix} - (D = 36\cdot6) \\[2ex]
\left.\begin{array}{l} 27 \\ 28 \end{array}\right] - (D = 47\cdot55) -
\begin{bmatrix} 30 \\ 31 \end{bmatrix} - (D = 41\cdot44) \\[2ex]
29 \quad (D = 56\cdot01) -
\begin{bmatrix} 31 \\ 32 \end{bmatrix} - (D = 61\cdot7)
\end{array}
$$

Fractions 27 and 28 were mixed together, and each of the three quantities of gas was in turn solidified in the fractionation bulb, and separated into a lighter and a heavier part. These fractions were mixed as is shown in the table, so as to obtain three samples of gas, two of which appeared from their spectra to be nearly pure krypton; while the third, of density 61·7, consisted chiefly of xenon. The two lighter fractions were separately solidified, and separated into two portions :—

$$ (D = 36\cdot6) \begin{cases} 33 \text{ (one-third).} \\ 34 \text{ (two-thirds)} \quad (D = 40\cdot82). \end{cases} $$

$$ (D = 41\cdot44) \begin{cases} 35 \text{ (two-thirds)} \quad (D = 40\cdot73). \\ 36 \text{ (one-third).} \end{cases} $$

The heavier two-thirds of the light gas, and the lighter two-thirds of the heavy gas had the same density.

As the atomic weight of krypton depends upon these determinations, the experimental data are given below :—

	Fraction 34.	Fraction 35
Volume of bulb . . .	7·152 cub. centims.	7·152 cub. centims.
Weight of gas . . .	0·01515 gramme.	0·01521 gramme.
Pressure	470·6 millims.	474 millims.
Temperature	18·1° C.	18·4° C.
Density	49·82	40·73

The refractivities of the two samples with regard to air were next determined ; they were found to agree within the limits of experimental accuracy :—

Refractivity of fraction 34	1·450
„ „ 35	1·449

We considered, therefore, that since we had succeeded in separating by fractional distillation from a lighter and from a heavier impurity respectively two samples of gas which agreed in two distinct physical properties, we were justified in assuming that we had isolated a definite chemical substance.

The samples 34 and 35 were mixed, and will from this point be referred to as pure krypton.

The separation of the xenon from traces of krypton proved to be more tedious than difficult ; for though their vapour-pressures at the temperature of liquid air differed considerably, the krypton appeared to remain dissolved in the xenon, and could only be removed by repeated fractionation. Fraction 36, the heavier portion from the last experiment, was introduced into the fractionation bulb, where it of course became solid, and the krypton was allowed to volatilise into the pump, while the exhaustion was continued as long as any gas could be removed. The vessel containing liquid air was then lowered, and the contents of the bulb was pumped off and mixed with fraction 32, the gas of density 61·7 obtained in the previous experiment (p. 65). This operation was repeated several times, with the result that the greater part of the xenon was removed from fraction 36.

Fraction 32 was then submitted several times to the inverse operation. The gas was solidified in the fractionation bulb, which was then placed in connection with the pump and exhausted as far as possible. The liquid air was then removed, and the heavy gas was pumped off and collected ; the operation was repeated, and the gas, after sparking with oxygen to remove traces of nitrogen and carbon compounds, was weighed. Its density was found to be 62·9. After repeating the same operation two or three times the density was again determined :—

Volume of bulb . . . 7·152 cub. centims.
Weight of gas . . . 0·01224 gramme.
Pressure . . . . . 245·4 millims.
Temperature . . . . 21·6° C.

The density calculated from these numbers is 64·0.

Unfortunately, after this experiment, a small quantity of air became mixed accidentally with the gas, and lowered the density to 62·96; but after refractionation its density was redetermined with the following results :—

Volume of bulb . . . 7·152 cub. centims.
Weight of gas . . . 0·01114 gramme.
Pressure . . . . . 225·3 millims.
Temperature . . . . 22·45° C.
Density of xenon. . . 63·64

The Vapour-pressures of Argon, Krypton, and Xenon.

We have not found it possible to contrive an arrangement whereby the temperature of liquid argon, krypton, and xenon should be gradually altered, as is generally done in measuring the vapour-pressures of substances liquid at the ordinary temperature, by means of a bath of some liquid, the temperature of which can be raised at will, or by jacketing the vapour-pressure tube with vapour from some pure liquid, boiling under different pressures. All substances liquid at the ordinary temperature freeze at the lower temperatures required for these gases. But it is possible to employ liquid ethylene, cooled either by external application of liquid air, or by causing it to boil under reduced pressure; both these means of obtaining a known low temperature have been used, but only a small range of temperature is thus covered. The results, moreover, were not so trustworthy as those obtained when a liquid of known boiling-point, boiling under atmospheric pressure, was used as a refrigerant.

The plan adopted, therefore, was to determine the pressure corresponding to certain known temperatures, and then by means of the formula suggested by RAMSAY and YOUNG,[*] the accuracy of which has been demonstrated in a great number of instances, to ascertain the vapour-pressures at intermediate temperatures. This method depends on the fact that the ratios of the absolute temperatures of two liquids corresponding to a series of pressures are a linear function for one of them corresponding to the same pressures. It therefore becomes possible to calculate a whole series of vapour-pressures of any substance, if the pressures are corresponding to two or more temperatures accurately known; the application is briefly as follows :—

The vapour-pressures of the substances are determined at absolute temperatures

* ‘Phil. Mag.,’ vol. 21, p. 33, 1886; vol. 22, p. 37, 1886.

T_a, T_a', &c. ; and by reference to tables, the absolute temperatures T_b, T_b', &c., for some standard substance, such as water or methyl alcohol, corresponding to the same pressures, are found. The ratios T_a/T_b, T_a'/T_b', &c., are then calculated, and are plotted on curve-paper against the temperatures T_b, T_b', &c., as ordinates. The points so obtained should lie on a straight line, and the temperature corresponding to any particular vapour-pressure can be discovered by multiplying the absolute temperature corresponding to that pressure for the standard substance by the value of the ratio corresponding to that temperature which is read from the line.

For example :– the following result was obtained with krypton—

Pressure.	T_a (krypton)	T_b (CH³OH).	T_a/T_b.
millims	Abs	Abs	
385·6	112·7°	321·8°	0·3502
898·7	123·1	342·3	0·3596
11970·0	170·9	429·3	0·3981
28800·0	197·9	471·4	0·4193

If, now, it is required to calculate the boiling-point of krypton at 760 millims., the ratio is plotted against the values of T_b, and the value of the ratio corresponding to the temperature 338° (the boiling-point of methyl alcohol at normal pressure), is read from the line. It is 0·3578 ; and the product 0·3578 × 338° is the boiling-point of krypton on the absolute scale, 120·9° abs.

This relation is expressed by the formula given by RAMSAY and YOUNG :—

$$T_a/T_b = T_a'/T_b' - c(T_a - T_a').$$

The value of c, it will be noticed, is constant for each pair of substances ; and if referred to a standard such as methyl alcohol, the ratio between the values of c for various gases has a physical meaning ; this will be considered later.

The temperatures were based on measurements made with a CALLENDAR's compensated constant-pressure hydrogen thermometer ('Roy. Soc. Proc.,' vol. 50, p. 247) ; in certain cases, a platinum-resistance thermometer, worked in conjunction with a CALLENDAR's recorder, was used ; it had been standardised by help of the hydrogen thermometer ; in others, the hydrogen thermometer was directly employed.

For low pressures, an apparatus was used, the arrangements of which are shown in the annexed figure (fig. 5).

The gas was introduced by means of the inverted siphon into the tube-reservoir a ; this reservoir was in communication with the small bulb b, which in its turn was connected with the pressure-gauge c, and with a TÖPLER's pump through the stop-cock d ; the bulb could be cooled by means of one of the devices previously alluded to. After evacuating the bulb, gauge, and connecting-tubes, the gas was introduced through

the stop-cock at the top of the reservoir; the bulb was cooled; and the pressures were read on the gauge on a mirror-scale, *e*, parallax being thus avoided.

For high pressures, a pressure-apparatus was made use of, similar to the one depicted in a memoir on Ethyl Oxide, published in the 'Transactions' for 1887 (A, vol. 178, p. 59). The tube containing the gas, however, was bent twice at right-angles, and was a small-bore capillary; for the amounts of gas at our disposal for compression were small (about 3 cub. centims.) because the total volume of the screw displacing the mercury did not admit of the compression of a greater amount; and, moreover, the amount of xenon in our possession was not greater than 3 cub. centims. The end of the capillary tube dipped into the cooling liquid; it was, of course, impossible to measure the volume of the liquefied gas, for care had to be taken to keep the mercury in the compression-tube sufficiently far away from the cold region, in case of its freezing.

Details of Results. –*Helium and Neon.*—These gases have been liquefied only at the temperature of liquid hydrogen. DEWAR states that helium was liquefied at that temperature under a pressure of eight atmospheres; and our experiments show that even at that temperature, neon possesses a vapour-pressure of a few millimetres. Attempts were made to liquefy neon by compressing it to about 100 atmospheres at the temperature of air boiling *in vacuo* (about − 205°). The pressure rose continuously on decrease of volume, and no liquid could be detected in the tube. The critical temperature of neon, therefore, lies in the unbridged interval between the boiling-point of liquid hydrogen, and that of air under reduced pressure.

Fig. 5.

Argon.—The lowest temperatures were obtained by the use of liquid nitrogen, boiling in a vacuum-jacket under atmospheric pressure; and the low-pressure apparatus was employed. Liquid air was added from time to time, and the rise of pressure consequent on the rise of temperature was noted. The range was from 77·3° to 87·8° abs.; the pressure from 215 to 821 millims. At the lower tempera-

ture, the argon was solid. The temperatures were measured by the resistance of a platinum coil, immersed in the liquid.
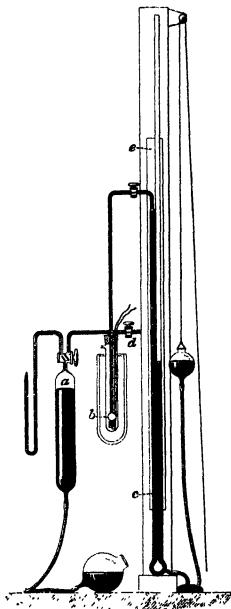
For the remaining experiments, the pressure-apparatus was used. The tube was first surrounded by liquid nitric oxide, boiling at 123·1° abs. under atmospheric pressure ; the temperature was measured simultaneously by help of the platinum-resistance thermometer ; the boiling-point of the nitric oxide, a large quantity of which had been prepared and purified by fractional distillation, was previously measured by the hydrogen thermometer. The next temperature was that of ethylene, boiling under reduced pressure ; readings were obtained at 136·8° and at temperatures up to 149° abs. As with nitric oxide, a large quantity of ethylene was prepared, and purified by fractional distillation. The temperatures were here determined with a hydrogen thermometer. Owing to the length of tube immersed in the ethylene, we did not succeed in reading directly the critical temperature, for the mercury froze in the capillary tube. But on using an air-bath (a vacuum-jacket, previously cooled with liquid air), containing the experimental tube, together with the hydrogen thermometer, all being cooled externally by surrounding the inner vacuum-tube with liquid air, the critical change was twice observed, at 155·5° and 155·7° abs. The critical pressure was afterwards determined graphically from the vapour-pressure curve plotted by means of the straight line indicating ratios.

Krypton.—Two observations were made at the temperature of liquid air ; at these temperatures the krypton was solid. The next observation was at the temperature of boiling methane (112·7° abs.), determined as the mean of five experiments with the platinum-resistance thermometer. The boiling-point of nitric oxide (125·1°) gave the next reading ; a further observation was made at the boiling-point of ethylene under atmospheric pressure (170·9°) ; this temperature had previously been determined by the hydrogen thermometer. Lastly, a series of observations was made, using pure dry ether, contained in a vacuum-vessel of half a litre capacity ; the rise of temperature of the ether was very slow, owing to its considerable mass, and to the slow transmission of heat through the walls of the vessel ; and close to the tube were placed the bulb of the hydrogen thermometer and a mechanical stirrer, so that the temperature was nearly uniform throughout. To hasten the rise of temperature, small quantities of ether were introduced from time to time, and two observations were taken at two consecutive temperatures, close together, before adding a fresh dose of warm ether, so as to raise the temperature for the next set of observations. The critical temperature and pressure were also observed.

Xenon.—Observations were made, as before, at the temperature of liquid air, but the vapour-pressure was hardly appreciable. When the temperature of liquid air is alluded to, it must be understood that the thermometer was always used to determine the actual temperature ; and that as the temperature rose owing to the evaporation of nitrogen, fresh observations were made. An attempt was also made

to measure the vapour-pressures of xenon at the boiling-point of nitric oxide and ethylene in the low-pressure apparatus; but the amount of gas at our disposal was insufficient. We had therefore to content ourselves with measurements in boiling nitrous oxide (183·2°; see 'Trans. Chem. Soc.,' vol. 63, p. 836), in cooled ether, and in water. The critical temperature is 14·8° Cent.; and xenon is thus liquefiable at atmospheric temperature.

The results obtained by measurement are as follows:—

VAPOUR-PRESSURES of Argon, Krypton, and Xenon.

	Argon.			Krypton.			Xenon.		
	T. abs.	P.	Ratio.	T. abs.	P	Ratio	T abs.	P.	Ratio.
Temperatures read with platinum thermometer and recorder	77·3	215 0	0·2504	84·2	9·0	0 3314	183 2	2020	0·5008
	81·7	410 1	0·2529	88·6	17 4	0·3359	(B P of		
	82 1	423 9	0·2536	112 7	386·6	0 3502	N₂O).		
	82·4	440 1	0 2537	(B. P. of	—	—	231 4	11134	0·5431
	82 6	465·0	0 2533	CH₄).			237 4	13505	0 5460
Liquid nitrogen mixed with liquid air as a refrigerant.	83 0	485 8	0 2537	—	—	—	239·7	13971	0 5495
	83 8	524 3	0 2548	—			241 2	14696	0 5503
	85 5	633 4	0·2565				241·2	15868	0·5525
	85 7	644·6	0·2567			—	244 6	16301	0·5512
	86·5	704·5	0 2575	—			248 7	18153	0·5546
	86 6	716 5	0·2574	—	·	—	250 5	18929	0·5548
	86·9	757·3	0·2572	—	—	—	252·6	19984	0·5576
	87·3	787·6	0·2575	—	—	—	254·0	21512	0·5592
	87 8	821·2	0 2582	—	—	—	255·6	21967	0 5586
	—	—	—	—	—	—	257 4	22310	0·5615
	123·1	10306	—	—	—	·	In ether bath, slowly warming, hydrogen thermometer.		
NO, atmospheric press	,,	10322		—	—	·			
	,,	10312	0 2910	—	—	—			
Liquid ethylene, boiling under reduced pressure.	136 8	20700	0·3009	170 9	11970	0 3981	273 35	31501	0·5740
	137 8	21334	0 3021	C²H⁴, boiling at atmospheric pressure			In water containing ice, stirred		
	139 0	23251	0·3019						
				197 9	28808	0·4198	287 75	44112	0 5814
				201 0	30837	0·4229	(Critical temperature and pressure)		
				201·5	31621	0·4230			
Hydrogen thermometer in air space; pressure read from curve.	155 5	40200	—	204 1	34693	0 4240			
	155·7	,,	—	206·4	37006	0 4256	In water, stirred		
	(Critical temperature and pressure)			290·5	41245	0 4287			
				(Critical temperature and pressure)					
				Ether bath, slowly warming; hydrogen thermometer					

On constructing the diagram in which these ratios are mapped against the absolute temperature of methyl alcohol, it was seen that the points representing ratios between the temperatures of the liquid and that of methyl alcohol at the same

pressure bore a linear relation to the absolute temperatures of methyl alcohol. From these results, plotted on a much more open scale, the temperatures of the liquefied gases corresponding to definite pressures were obtained, by finding the temperatures of methyl alcohol corresponding to the same pressures, and reading the ratio corresponding to those temperatures. By multiplying the temperatures of methyl alcohol by the ratios, the temperatures of the liquids at the corresponding pressures were calculated. The following table of vapour-pressures was then constructed (compare Plate 1):—

Pressure	Temperature			Pressure	Temperature		
	Argon	Krypton	Xenon		Argon	Krypton	Xenon
millims				millims			
300	78·94	110·43	148·9	18000	134·88	182·79	248·3
400	81·31	113·84	153·2	20000	137·36	186·00	252·6
500	83·32	116·14	156·8	22000	139·63	188·88	256·6
600	84·89	118·35	159·7	24000	141·80	191·68	260·2
700	86·38	120·19	162·0	26000	143·82	194·32	263·8
760	86·90	121·33	163·9	28000	145·80	196·72	267·2
800	87·77	122·01	164·7	30000	147·58	199·02	270·2
1000	89·90	124·84	168·8	32000	149·28	201·21	273·7
1500	93·52	129·55	176·8	34000	150·88	203·32	276·0
2000	97·95	135·23	182·9	36000	152·52	205·40	278·9
3000	103·33	142·17	192·4	38000	154·11	207·25	281·9
4000	107·27	147·34	199·6	40200	155·60	-	---
6000	113·58	155·47	210·8		Critical		
8000	118·60	161·95	219·6	41240	---	210·5	-
10000	122·72	167·31	226·8			Critical	
12000	126·31	171·87	233·1	43500	---	---	287·7
14000	129·47	175·88	238·7				Critical
16000	132·73	179·45	243·7				

The vapour-pressures of argon were measured for certain temperatures by Professor Olszewski; the data which he gives are ('Phil. Trans.,' A, 1895, p. 258):—

Pressures	T. argon.	T. water.	Ratios	Olszewski. Ratios smoothed.	R. and T. ratios.	Diff.
740·5	86·1	372·27	0·2313	0·2311	0·2337	0·0026
18010	132·9	495·00	0·2705	0·2699	0·2729	30
19230	134·7	498·4	0·2703	0·2710	0·2738	28
20750	136·8	503·4	0·2718	0·2727	0·2754	27
22040	137·9	505·8	0·2726	0·2733	0·2762	29
22650	138·6	507·4	0·2732	0·2738	0·2767	29
27210	143·5	527·8	0·2771	0·2771	0·2800	29
28880	144·4	521·2	0·2771	0·2782	0·2811	29
38460	152·0	538·6	0·2822	0·2834	0·2860	26

On mapping the ratios between the absolute temperatures of water and argon against the absolute temperature of water, a straight line is obtained, which in its slope coincides with that obtained by us. In the table above, we see that the difference between the values of OLSZEWSKI's and our ratios is practically a constant one; and the change of value between 372·27° and 538·6° for water is in each case 0·0523. The position of OLSZEWSKI's line is, however, different from ours; and the differences of temperature are shown in the following short table :—

Pressure.	Temperatures.		
	Olszewski.		Ramsay and Travers.
	Observed.	Re-calculated.	
740·5	86°·1	86°·03	87°·00
18010	132·9	133·6	135·1
19230	134·7	135·1	136 5
20750	136·8	137·3	138·6
22040	137·9	138·2	139 7
22650	138·6	138·9	140·4
27210	143·5	143 5	145 0
28880	144·4	145·0	146·5
38460	152·0	152·6	154·0

It is evident that OLSZEWSKI's temperatures are uniformly below ours; and we imagine that it may be attributed to his having used a constant-volume thermometer, whereas ours was a constant-pressure one. Mr. BALY, who has made measurements of the boiling-points of oxygen and nitrogen with the same thermometer, finds results that also differ from OLSZEWSKI's by 0·36° for O^2, and by 0·15° for N^2 in the same direction (higher). On the whole, however, we may take OLSZEWSKI's and our vapour-pressure curves as practically identical, save in the fact that their positions are not quite coincident.

The Molecular Volumes of Argon, Krypton, and Xenon.

The apparatus by means of which the volume of 1 cub. centim. of each of these gases at their boiling-points under atmospheric pressure was measured is shown in the figure (fig. 6).

After the apparatus had been completely exhausted of air through a, which was connected with the pump, the gas was introduced through an inverted siphon, and dried in transit with phosphoric anhydride; when a sufficient quantity had entered, the tube a was sealed. On raising the reservoir b, the volume of the gas

was read at a known temperature and pressure in the tube c, which had previously been graduated and calibrated. The capillary tube d, which was a piece of even-bore thermometer tubing, was surrounded with a suitable cooling agent, so that the gas was near its boiling-point under normal pressure. By altering the level of b,

Fig. 6.
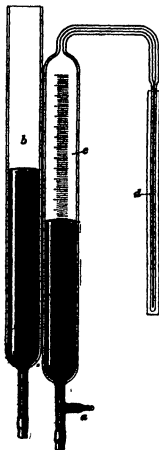
and reading the difference in level of b and c, the actual pressure of the gas was measured. A portion liquefied in d; its volume was read, and also the volume of gas remaining in c, after a portion had been liquefied. As the density of the gas was known, it was easy to calculate the weight of the portion remaining in c, and consequently of the portion liquefied in d.

The volume of one division of the tube c was equivalent to 0·5893 gramme of mercury; that of the capillary d, to 0·000678 gramme. The volume in grammes of mercury from the highest mark on c to the end of the capillary was found by determining the change in volume of the gas in c, consequent on known changes of pressure, and as a mean of five concordant determinations, it was found to be equal to the volume of 6·55 grammes of mercury.

LIQUEFACTION of Argon in Liquid Air.

Argon D = 19·95.

	Volume of gas uncondensed.	Temp.	Pressure.	Volume at N.P.T.	Volume condensed.	Volume of liquid.	Weight of 1 cub. centim.
1	12·44	15·5	840·2	13·01	23·08	0·03390	1·211
2	10·62	15·2	831·8	11·01	25·08	0·03674	1·214
3	5·43	15·3	816·0	5·52	30·57	0·04474	1·215
4	5·37	15·4	815·7	5·46	30·63	0·04508	1·208
5	5·08	15·6	815·4	5·16	30·93	0·04542	1·211
6	5·61	15·6	818·5	5·72	30·37	0·04461	1·211

Mean weight of 1 cub. centim. · 1·212 gramme.

The temperature was about 88° abs., the boiling-point of argon at 760 millims. being 86·9° abs.

LIQUEFACTION of Krypton in Ethylene, cooled below its Boiling-point by being surrounded by Liquid Air.

Krypton D = 40·78.

	Volume of gas uncondensed.	Temp.	Pressure	Volume at N.P.T.	Volume condensed.	Volume of liquid	Weight of 1 cub. centim.
1	20·58	12·8	1298·0	33·57	16·46	0·02820	2·126
2	20·99	,,	1299·7	34·29	15·74	0·02724	2·115
3	21·93	,,	1302·3	35·90	14·13	0·02434	2·115
4	19·46	,,	1295·1	31·68	18·35	0·03044	2·196
5	20·22	,,	1296·4	32·95	17·08	0·02841	2·190
6	20·52	,,	1297·9	33·48	16·55	0·02773	2·174
7	20·75	,,	1298·3	33·88	16·17	0·02705	2·177
8	21·05	,,	1299·8	34·39	15·64	0·02625	2·170
9	21·28	,,	1300·2	34·74	15·25	0·02569	2·162
10	21·78	,,	1302·2	35·80	14·23	0·02434	2·130

Mean weight of 1 cub. centim. 2·155 grammes

The temperature was about 127° abs., the boiling-point at 760 millims. being 121·3° abs.

LIQUEFACTION of Xenon condensed in Ethylene, boiling under Atmospheric Pressure.

Xenon D = 64·0.

	Volume of gas uncondensed.	Temp	Pressure.	Volume at N.T.P.	Volume condensed.	Volume of liquid.	Weight of 1 cub. centim.
1	4·95	12°·2	1268·5	15·13	3·96	0·00630	3·59
2	4·85	,,	1268·5	15·03	4·06	0·00644	3·60
3	3·95	,,	1268·5	14·50	4·51	0·00730	3·50
4	3·20	11°·7	1279·4	13·62	5·47	0·00915	3·42

Mean weight of 1 cub. centim. 3·52 grammes.

The temperature of the xenon was 171° abs.; its boiling-point at 760 millims. is 163·9° abs.

Although these determinations of density have not been made at the actual boiling-points under normal pressure, still they are sufficiently near to show correspondence. The atomic volumes may be obtained by dividing the atomic weights of the elements by their densities. They are :—

$$\text{Argon . } 39·9/1·212 = 32·92.$$
$$\text{Krypton : } 81·55/2·155 = 37·84.$$
$$\text{Xenon : } 128·0/3·52 = 36·40.$$

The Relations between the Volume, Temperature, and Pressure of the Inactive Gases.

The experiments bearing on this question were carried out in an apparatus similar to that of which a figure is given in the 'Phil. Trans.,' A, 1887, p. 59. The volume-tube, carefully calibrated, was charged with a known volume of gas, accurately measured under known conditions of temperature and pressure, and introduced without contamination. The helium was a portion of what had been fractionated by diffusion from any admixture of argon, and it had the density 1·98; the argon was free from helium, neon, krypton, and xenon; they had been removed by fractional distillation; the neon, krypton, and xenon were pure samples of these gases, with densities, 9·97, 40·8, and 64, respectively. Before use, the krypton and xenon were solidified, and freed by the pump from any admixture of gas with higher vapour-pressure.

The method of filling the volume-tube requires special description. It will be understood by reference to the figure (fig. 7).

The volume tube A was sealed with a constriction at B to the apparatus C,

communicating through the stop-cock D with the
pump, on the one hand, and through E with the
inverted siphon on the other. The point of the
siphon was sealed at F, and drawn out to a fine
point which was scratched, so that the end could be
easily broken by pressing down on it the inverted
gas-tube containing the gas under experiment. By
raising the reservoir G, mercury entered the appa-
ratus to above the level of the india-rubber tube.
The clip K on the india-rubber connecting-tube was
then closed. The whole apparatus was then evacu-
ated with the pump, until no bubbles were carried
over, an operation lasting half-an-hour. The stop-
cocks D, E, and L being shut, the point of the capil-
lary siphon was broken by pressing on it with the
inverted test-tube at F, and gas entered the space
comprised between the stop-cocks D, E, and L. The
stop-cock L was then opened; the gas entered the
apparatus, and "washed it out," diluting the resi-
dual air (of which there was only the minutest trace,
seeing that an almost complete vacuum had been
established by the TOPLER's pump); and this gas was
pumped out, and in the case of the rarer gases,
collected and preserved. The main body of gas was
then admitted by closing D and opening E; and the
clip K on the rubber tube was then opened, the
mercury reservoir G being temporarily lowered. On
raising it again, mercury fell in the reservoir until
it took its level in the capillary part opposite B,
while the mercury in the volume-tube stood at H.
This capillary part of the volume-tube was made of
tubing of the same bore as the lower capillary of
the reservoir G, so as to avoid errors due to the
capillarity of mercury. The temperature and the
relative positions on the scale of the levels of the
mercury in the capillary tubes and the barometer
were then read. The reservoir M was next filled
with mercury, and pushed up, until the capillary
constriction of the volume-tube at B was well
covered. The volume-tube was then bent over, and
broken at the capillary B; mercury entered, and

Fig. 7.
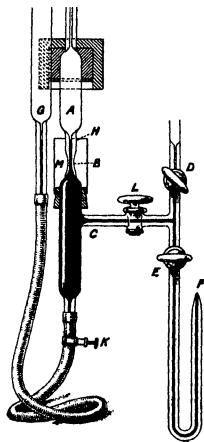
confined a known volume of gas. The volume-tube was transferred to the pressure apparatus, and measurements were made.

The high-pressure manometer had previously been compared with the low-pressure manometer, and the latter, with an open tube manometer. The volume-tube and the high-pressure gauge were jacketed in the first instance with water at atmospheric temperature, which, of course, was accurately known; and the heights of mercury in the air-gauge and in the volume-tube were read, both during ascending and during descending pressure. The results were quite concordant. The readings of the air-gauge were corrected for deviations from BOYLE's law, according to AMAGAT's results for air, in comparison with directly read pressures; and the readings of the volume-tube were translated into real volumes, and, by assuming GAY-LUSSAC's law, corrected to what they would have been at 11·2° C., which happened to be the temperature at which readings were taken with the gas first investigated, helium. As the highest temperature recorded did not exceed 15°, it was assumed that no actual variation from GAY-LUSSAC's law would influence the results, within such a small interval of temperature as that between 15° and 11·2°.

For a higher temperature, the boiling-point of quinoline under atmospheric pressure, about 237·3°, was chosen. The pressure of the atmosphere was not always quite normal; but the barometer was always read, and the necessary change in volume was calculated according to GAY-LUSSAC's law for the small interval of temperature required—only 0·1°.

The pressures, volumes, and their products are given in the accompanying tables. The volumes are stated in terms of cubic centimetres occupied by the molecular weight of the gas taken in grammes at the pressures and temperatures specified. They are thus all comparable with each other, and with the corresponding constants for 28 grammes of atmospheric nitrogen, as measured by AMAGAT with a direct-reading manometer. The basis of calculation has been taken as the volume occupied by 32 grammes of oxygen. Data on this constant vary slightly among themselves. The mean of REGNAULT's, RAYLEIGH's, JOLLY's and LEDUC's determinations for the weight of a litre of oxygen is 1·42961 gramme; that of RAYLEIGH is 1·42952; and of LEDUC, 1·42920. Taking RAYLEIGH's number as occupying an intermediate position between the other two, 32 grammes of oxygen would occupy at 0° and 760 millims. 22,395 cub. centims.; multiplying by 0·760, the normal pressure as a fraction of a metre, the value of P.V. is 17,012 metre-cubic-centims. At 11·2°, from GAY-LUSSAC's law, this value is increased to 17,710; and at 237·3°, the other temperature under consideration, to 31,800. On the hypothesis that the products of pressure and volume for a perfect gas remain constant, these values should represent P.V. at all pressures and volumes. We shall now see how far this condition is fulfilled by the gases in question. For the sake of convenient comparison, AMAGAT's results for atmospheric nitrogen have been inserted, both in the tables, and in the diagram (Plate 2) representing the results.

PRESSURE, VOLUME, and P.V. at Atmospheric Temperature, 112° C.

Nitrogen, 28 grammes						Helium, 4 grammes			Neon, 20 grammes		
Amagat			R and T								
P.	V.	P.V.	P.	V.	P.V	P.	V.	P.V.	P.	V.	P.V.
metres	cub. centim.		metres	cub. centim.		metres	cub. centim.		metres	cub. centime.	
20	—	17595	—	—	—	21·120	835·1	17496	29·870	590·5	17638
25	—	17580	—	—	—	21·962	793·4	17425	31·609	557·6	17626
30	—	17560	—	—	—	22·918	761·6	17455	33·574	523·8	17587
35	—	17548	—	—	—	23·924	729·3	17448	35·819	491·9	17618
40	—	17539	41·573	423·4	17625	25·070	695·3	17445	38·370	458·4	17588
45	—	17539	45·037	390·9	17607	26·312	663·3	17454	41·322	426·1	17609
50	—	17540	49·146	357·0	17548	27·684	629·7	17431	44·791	392·8	17603
55	—	17549	54·138	324·2	17554	29·317	596·8	17437	48·876	359·4	17564
60	—	17557	60·307	291·5	17577	30·918	564·6	17460	53·868	327·7	17652
65	—	17572	—	—	—	32·833	530·9	17431	57·214	309·6	17714
70	—	17594	68·096	257·4	17530	35·013	498·2	17443	59·980	294·2	17643
75	—	17624	—	—	—	37·499	465·3	17449	64·407	275·4	17739
80	—	17651	79·353	224·8	17436	40·377	432·2	17452	67·782	260·7	17673
						43·749	399·7	17485	71·458	248·8	17777
						47·724	364·8	17408			
						52·557	332·3	17466			
						58·535	299·3	17523			
						66·119	266·9	17647			
						77·012	231·9	17859			
At 237·3°											
—	—	—	39·010	826·0	32232	40·956	755·7	30950	33·810	925·9	31304
—	—	—	42·100	766·4	32286	44·329	697·1	30903	36·041	868·0	31285
—	—	—	49·720	648·4	32283	48·347	638·1	30848	38·592	809·1	31226
—	—	—	54·798	589·9	32302	53·306	578·7	30790	41·530	751·1	31193
—	—	—	65·493	491·4	32338	59·938	519·9	30796	44·987	691·0	31086
—	—	—	68·919	471·8	32517	65·929	461·5	30892	49·070	633·7	31094
—	—	—	74·669	434·1	32615	77·884	401·7	31285	54·157	574·0	31036
—	—	—	76·852	421·3	32414				57·381	541·0	31119
—	—	—			32378				60·169	517·2	31190
									64·562	483·0	31187
									68·112	457·9	31184
									71·840	433·2	31124

PRESSURE, Volume, and P.V. at Atmospheric Temperature, 11·2° C.

Argon, 39·9 grammes.			Krypton, 81·5 grammes.			Xenon, 164 grammes.		
P	V	P.V.	P.	V.	P V.	P.	V.	P.V.
metres	cub centims		metres	cub centims		metres	cub centims	
—	—	—	19·669	879·2	17166	19·492	771·6	15042
—	—	—	20·405	839·0	17120	20·230	737·2	14913
—	—	—	21·211	811 8	17092	21·040	702·3	14776
22·945	774·6	17771	22·065	770·5	17000	21·884	667 6	14609
23 947	740·5	17732	23·033	736·5	16965	22·851	632·6	14456
25·097	705 9	17717	24·046	700·8	16851	23·868	595·6	14217
26·316	672·3	17692	25·197	668·4	16841	25·018	560·5	14022
27 686	637 7	17655	26·438	632·7	16728	26·270	525 5	13805
29·214	602·4	17598	27·819	597·7	16629	27·656	490·0	13552
30 935	568 4	17583	29 359	562 8	16523	29·199	452·2	13203
32·854	532 9	17509	31 073	528 3	16416	30·923	417·0	12894
35 012	497·0	17400	32 991	494 1	16302	32·863	378 2	12430
37·497	463·2	17369	35·180	459·2	16157	35·125	339·4	11917
40·358	428·0	17274	37 682	422 8	15934	37·537	300·1	11266
43·722	403·1	17221	40·828	389·3	15895	40·430	234·3	9420
47·698	357·9	17069	43 933	354 3	15564	—	—	—
52·509	324·4	17033	47 933	318 8	15280	- -	—	—
58·479	289 3	16921	52·792	283·9	14990	—	—	—
66 021	255·3	16853	54 798	248·9	14633	—	—	- -
76·852	219·7	16889	66·420	214·8	14264	—	-	
			77·322	180·1	13924			

At 237·3°.

—		—	38·707	824 8	31924	—		—
		- -	41·664	765·3	31886	40·681	736·4	30114
44·420	721·8	32060	45·113	704·5	31783	44·061	680·1	29965
48·421	659·8	31949	49·488	644·0	31680	48·042	618 5	29714
53·303	599·2	31940	54 109	583·7	31583	52·915	560·4	29656
59·340	539 3	32001	60·229	523·3	31521	58·927	497·4	29313
66 988	479·9	32150	68·044	462·5	31473	66·997	436 2	29223
77·954	412·7	32174	79·108	402·4	31830	77·035	375·8	29288

ARGON. Repetition at 237·3°.

P.	V	P.V.
32·844	959·1	32460
32·370	899·3	32370
38·511	827·7	32267
44·450	777·6	32233
44·900	715·8	32142
48·880	654·7	32001
53·927	595·3	32103
57·267	560·0	32069
60·047	534·0	32067
64·482	499·7	32222
67·849	474·1	32170
71·542	450·1	32203

These results are graphically reproduced on Plate 2.

Determination of Refractivity.—The apparatus which we employed is essentially the same as that described by Lord RAYLEIGH, in the ' Proceedings,' vol. 59, p. 202, and vol. 64, p. 67, and differed from it only in dimensions. The parallel tubes of the refractivity apparatus were of glass, and had an internal diameter of 2 millims.; their length was 32 centims.; the gauge-tubes were 4 millims. in diameter, and the connecting tubes were of capillary bore; and the total capacity at full pressure was 5 cub. centims., but it was possible to work with 3 cub. centims. of gas. The focal length of the lenses was 30 centims.

The refractivity was determined in the manner previously described.

Helium	0·1238
Neon . . . . (I.) (0·235; II.) 0·234	0·2345
Argon	0·968
Krypton . . . . . . (I.) 1·450; (II.) 1·449	1·450
Xenon . . . (I.) 2·361; (II.) 2·365; (III.) 2·379; (IV.) 2·352 .	2·364

The refractivity of helium and argon were previously determined (' Proc. Roy. Soc.,' vol. 64, p. 190); that of neon was compared with hydrogen; and the refractivity of hydrogen compared with air was found equal to 0·4733 (' Proc. Roy. Soc.,' vol. 62, p. 227).

Two samples of krypton were obtained, one by fractionating from argon (I.);

M

the other after fractionating it from xenon (II.). The ratio between the refractivities of (I.) and (II.) were 1·003 to 1 ; the mean of all observations is 1·450.

Spectra of the Inactive Gases.

It is not intended to give a complete account of the spectra of the inactive gases; exceedingly accurate measurements have been made by Mr. E. C. C. BALY, with a ROWLAND's grating, which will shortly be published. But to the general reader, it is of interest to gain some idea of the visual spectra, and these have been reproduced in Plate 3 from drawings made under the supervision of Professor ERDMANN, from tubes furnished to him by us.

The spectra of neon, krypton, and xenon cannot be detected in that of crude argon, even by photographic methods.

The change of spectrum produced in argon by the introduction of a jar and spark-gap is well known ; but less attention has been paid to the fact that helium, too, alters its spectrum, though much less markedly. The effect of the jar and spark-gap with krypton is almost to obliterate the spectrum ; the brilliant yellow and green lines disappear. The spectrum of neon is hardly changed ; but that of xenon is enormously intensified ; indeed the plate shows the jar and spark-gap spectra of xenon. With the ordinary discharge, it is far less luminous, and the blue lines become only faintly visible ; the blue and green are much intensified by the interposition.

The colour of these tubes is very striking ; the yellow of helium and the red-purple of argon are now well known ; neon has a brilliant flame-colour, as intense as the yellow of helium ; krypton has a pale violet tint, and is not striking ; and xenon is sky-blue, and with the jar acquires a greener tint.

The question of the identity of the spectrum of krypton and that of the aurora borealis can be settled after the publication of Mr. BALY's work.

Conclusions, and Résumé of Results.

The densities of the inactive gases are respectively :—

Helium.	Neon.	Argon.	Krypton.	Xenon.
1·98	9·96	19·96	40·78	64·0

There are two lines of argument which show that these numbers, multiplied by 2, give approximately the atomic weights of the elements.

(1.) The ratio between their specific heats at constant pressure and at constant volume is 1·66. It is true that no measurements have been made with pure neon,

krypton, and xenon, yet many determinations by the Kundt method were carried out with the approximately pure gases. It is hardly necessary to point out that if the theoretical ratio of 1.66 is attained with a mixture of monatomic gases, it applies also to the constituents of that mixture. This reasoning supports the above assertion.

(2.) Supposing that the atomic weights, as in the case of oxygen, nitrogen, &c., be taken as identical with the numbers expressing density, there is no place for these elements in the periodic table. This argument, of course, does not necessarily apply to helium, for it occupies the gap between hydrogen (1) and lithium (7). But neon would occupy a place between beryllium (9) and boron (11) ; argon between fluorine (19) and sodium (23), taking the place we have assigned to neon ; krypton between calcium (40) and scandium (44) ; and xenon, between copper (63.6) and zinc (65.4). But if it be granted that they are monatomic, they form a group by themselves. Their atomic weights would then be :—

Helium.	Neon.	Argon.	Krypton.	Xenon.
4	20	40	82	128

The group of elements, the atomic weights of which come next in order, is :—

Hydrogen.	Helium.	Lithium.	Beryllium.
1	4	7	9

Fluorine.	Neon.	Sodium.	Magnesium.
19	20	23	24

Chlorine.	Argon.	Potassium.	Calcium.
35·5	40	39	40

Bromine.	Krypton.	Rubidium.	Strontium.
80	82	85	87

Iodine.	Xenon.	Cæsium.	Barium.
127	128	133	137

It may appear remarkable that hydrogen is placed as the first member of the chlorine-group ; but arguments may be adduced in favour of this position, which need not be here recapitulated, and which have been well summed-up by Professor ORME MASSON ('Chem. News,' vol. 73, 1896, p. 283). It is also to be noticed that the group ends here ; for no elements of the fluorine, sodium, and magnesium series with higher atomic weights than iodine, cæsium and barium have been discovered. We have no absolute proof that "xenon" does not contain some element of still higher weight ; the same could have been said of argon at the time of publication of the 'Phil. Trans.' memoir ; but it may be confidently stated that if xenon contains elements of higher atomic weight, they must be present in an extremely minute amount ; inasmuch as the pressure does not rise during the process of its condensation

to liquid; and the critical phenomena are sharp. This is known not to be the case with a mixture. Our opinion is that this portion of the table is complete.

We have next to show that the properties of these elements vary periodically, like those of the cognate elements.

Refractivities.—The found refractivities of the inactive gases compared with air as unity are:—

Helium.	Neon.	Argon.	Krypton.	Xenon.
0·124	0·235	0·968	1·450	2·368

We see here a progressive increase with atomic weight.

The following table gives a résumé of the refraction-equivalents of the elements, taken from the summary in OSTWALD'S 'Allgemeine Chemie,' vol. 1, p. 447; the refractivities of the inactive gases have been calculated to the same basis as those of the other elements:—

Hydrogen.	Helium.	Lithium.	Beryllium	Boron	Carbon.	Nitrogen.	Oxygen.
1·47	0·47	3·8	4·0	4·0	5·0	4·1	3·5

Fluorine.	Neon.	Sodium.	Magnesium.	Aluminium.	Silicon.	Phosphorus.	Sulphur
1·4	0·89	4·2	7·0	8·4	7·5	18·3	16·0

Chlorine.	Argon.	Potassium.	Calcium.	Scandium.	Titanium.	Vanadium.	Chromium.
10·0	3·68	8·1	10·4	—	25·5	25·3	19·9

	Iron.						
	12·0						

Manganese.	Nickel.	Copper.	Zinc.	Gallium.	Germanium.	Arsenic.	Selenium.
12·2	10·4	11·6	10·2	—	—	15·4	30·5

	Cobalt.						
	10·8						

Bromine.	Krypton.	Rubidium.	Strontium.	Yttrium.	Zirconium.	Niobium.
15·3	5·51	14·0	13·6	—	21·0	—

	Rhodium.					
	24·2					

Silver.	Cadmium	Indium	Tin.	Antimony.	Tellurium.
13·0	13·6	—	27·0	24·5	21·6

	Palladium				
	22·4				

Iodine.	Xenon.	Cæsium.	Barium.
24·5	9·0	19·5	15·8

Of these, hydrogen, chlorine, bromine, nitrogen, and oxygen, are, with the exception

of the inactive gases, the only elements of which the refractivities have been
measured directly ; the other refraction-equivalents are deduced from measurements
with compounds of the elements.

  These figures have been plotted on the accompanying curve; the ordinates are
atomic weights, and the abscissæ, the refraction-equivalents. It is noticeable that those
of helium, neon, argon, krypton, and xenon form the lower apices of each series.   It
must of course be understood that many of the numbers for the other elements are
far from accurate; still the diagram is interesting inasmuch as it shows clearly the

Fig. 8.

Atomic Weights.
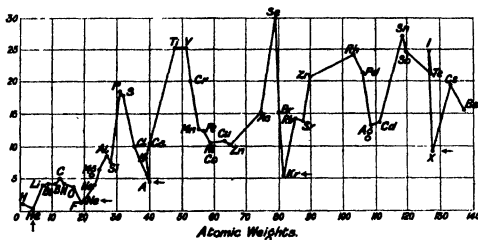
periodicity of the property, and carries with it the conviction that the inactive
gases find a fitting place among the other elements.

  The comparative retardation of light caused by equal numbers of molecules of
the inactive gases compared with that produced with air is given by the figures on
p. 84 ; on dividing by the atomic weights the retardation for equal quantities of
matter is arrived at :—

Helium.	Neon.	Argon.	Krypton.	Xenon.
0·0309	0·0118	0·0242	0·0177	0·0184

  It is remarkable that if, for example, krypton be submitted to such a pressure
that its density is equal to that of xenon at atmospheric pressure, it offers nearly
equal opposition to the passage of light, a fact which is highly suggestive, when
the ultimate nature of these gases is considered.

  Somewhat similar relations are found with the neighbouring series of elements if
their values are reduced to the scale of air as unity. The refractivities are far less
certain, and the results must be accepted with caution ; but the figures are :—

Hydrogen.	Fluorine.	Chlorine.	Bromine.	Iodine.
0·473	0·05 ?	0·073	0·048	0·048

Lithium.	Sodium.	Potassium.	Rubidium.	Cæsium.
0·13	0·048	0·051	0·036	0·038

Compressibilities.—On referring to the plate (Plate 2) it will be seen that at the lower temperature, 11·2°, the value of P.V. for a perfect gas is 17,710 metre-cubic-centims. The values for nitrogen are taken from AMAGAT's results, and they deviate very little from the theoretical. The compressibility of helium decreases with rise of pressure, or in other words the value of P.V. increases. This is also the case with hydrogen, though in a less degree. Neon is a more perfect gas than nitrogen, though at high pressures, it, too, becomes too little compressible. Argon, on the other hand, is more compressible than the former, and krypton still more ; though the shape of the curve indicates that at still higher pressures their P.V. would become normal at some point. Xenon is below its critical temperature, and its compressibility is therefore very great.

At higher temperatures the results are difficult to interpret. Measurements made with nitrogen show that it retains nearly a uniform P.V. as pressure rises ; but of all the other gases the P.V. diminishes and then increases. And the order of position of the curves is not in accordance with the densities of the gases. That the measurements are fairly correct may be inferred from the two sets for argon being nearly identical. Although these observations at a high temperature cannot be taken as of great importance in themselves, yet they show the necessity of a careful comparison of the behaviour of diatomic gases as regards their compressibility.

The results of the measurements are reproduced on Plate 2.

Vapour-pressures.—As already remarked, it was found impossible to liquefy neon, except by the aid of liquid hydrogen ; and as the constants for hydrogen are unknown, it was useless to attempt a measurement of its vapour-pressure. The same remarks apply to helium. The diagram shows that the vapour-pressure curves of the remaining three elements exhibit a gradual progression.

Using the formula of RAMSAY and YOUNG, as described on p. 68, the ratios between the absolute temperatures of these liquids, and that of methyl alcohol at equal pressures for all are expressible by straight lines, when mapped against the absolute temperature of the alcohol. The slope of such a line affords a physical constant, which is peculiar to each substance. The values of $\Delta R/\Delta t$ for the three elements, argon, krypton and xenon, are :—

Argon.	Krypton.	Xenon.
0·0350	0·0467	0·0675

Here again a progression is noticeable ; it does not, however, appear to be possible to

extrapolate the results, and so to obtain means of calculating the vapour-pressure curves of neon and helium.

Melting-points, Boiling-points and Critical Temperatures.—During the determination of the vapour-pressures and the atomic volumes of argon, krypton, and xenon, we have frequently solidified these elements. The melting-points are as follows :—

Argon.	Krypton.	Xenon.
−187·9° C.	−169° C.	−140° C. (approx.)
85·1° abs.	104° abs.	133° abs.

The melting-point of argon was determined by aid of a platinum thermometer immersed in liquid air ; that of krypton in liquid methane ; and that of xenon in ethylene cooled below its boiling-point. As already mentioned, the temperatures are reduced to the hydrogen standard ; a portion of the vapour-pressure curve of solid krypton was measured, but only two points were obtained, viz., at 9·00 and at 17·4 millims. pressure. The ratios calculated from these two points and plotted on curve-paper, cuts the line representing the ratios for the vapour-pressures of the liquid at a point representing the ratio for the melting-point ; it corresponds to the temperature −166·6° C., a fairly near concordance with the found number −169° C. (See RAMSAY and YOUNG, 'Trans. Chem. Soc.,' vol. 49, p. 460.)

The boiling-points at 760 millims., calculated by the method of ratios, are :—

Argon.	Krypton.	Xenon.
−186·1° C.	−151·67° C.	−109·1° C.
86·9° abs.	121·33° abs.	163·9° abs.

The critical temperatures and pressures are :—

	Argon.	Krypton	Xenon.
Temperatures . .	−117·4° C.	−62·5° C.	+14·75° C.
	155·6° abs.	210·5° abs.	287·75° abs.
Pressures. . . .	40,200 millims.	41,240 millims.	43,500 millims.

The boiling-point and melting-points of the inactive elements also show periodicity ; it has not been deemed necessary, however, to insert diagrams illustrative of this. Suffice it to remark that these constants place the elements at the bottom of the descending point of such a curve as accompanies LOTHAR MEYER's 'Modern Theories of Chemistry ;' and that here, too, they fall into place.

Atomic Volumes.—So far as is known, the following table gives the atomic volumes at their boiling-points of the elements of interest in connection with those of the argon group.

		Hydrogen.	Helium.	Lithium.
		14·3	?	?(11·9)
Nitrogen.*	Oxygen.*	Fluorine.*	Neon.	Sodium.†
17·7	14·1	17·15	?	31·0 (23·7)
Phosphorus.†	Sulphur.†	Chlorine.*	Argon.	Potassium.
20·9 (17·0)	21·6 (15·7)	23·5	32·9	?(45·4)
		Bromine.†	Krypton.	
		27·1 (26·9)	37·8	
Antimony.	Tellurium.	Iodine.*	Xenon.	Cæsium.
— (17·9)	— (20·3)	34·2 (25·7)	36·4	?(70·6)

The figures in brackets refer to atmospheric temperature, and are taken from C. Schmidt's work. It is to be noticed, if these figures are mapped as ordinates against atomic weights as abscissæ, as shown by Lothar Meyer, that argon follows chlorine on the ascending branch, of which the highest member is potassium; that krypton occupies a regular position on the curve of which the members with lowest atomic volume are cobalt and nickel and the highest rubidium; while the position of xenon is near that of iodine, and lies between it and cæsium. In short, the atomic volumes of these elements exhibit that partial regularity characteristic of all the others; and their place on the atomic volume curve is a normal one.

If it is permissible to speculate on the atomic volumes of helium and of neon by interpolation, they are respectively 13·5 and 20·2. This would make the densities of the liquids 0·30 and 1·0, approximately. These numbers would then complete the following series of densities :—

Helium.	Neon.	Argon.	Krypton.	Xenon.
0·3 ?	1·0 ?	1·212	2·155	3·52

The whole table of atomic volumes, as far as that of cæsium, is reproduced in the accompanying diagram. The periodicity of the inactive elements is obvious. A word may be said regarding the apparently abnormal position of argon; with the atomic weight 40, there is a reversal of direction in passing from argon to potassium, and from potassium to calcium; but the same peculiarity is observable with iodine and tellurium. While the difference between the atomic weight of any element and that preceding it or succeeding it on the table is generally positive, although very irregular, it appears that a few instances of reversal cannot be ignored, and argon forms one of these. The peculiarity of the diagram, it must be noted, is due to the atomic weight of argon and not to its atomic volume.

* Drugman and Ramsay, 'Trans. Chem. Soc.,' vol. 67, p. 1228.

† Ramsay, ibid., vol. 39, p. 472; Ramsay and Masson, ibid., vol. 39, p. 49; Thorpe, ibid., vol. 37, p. 391.

General Remarks.—The great value of NEWLAND's, MENDELÉEF's, and LOTHAR MEYER's generalisation, known as the periodic arrangement of the elements, is universally acknowledged. But a study of this arrangement, it must be allowed, is a somewhat tantalising pleasure; for, although the properties of elements do undoubtedly vary qualitatively, and, indeed, show approximate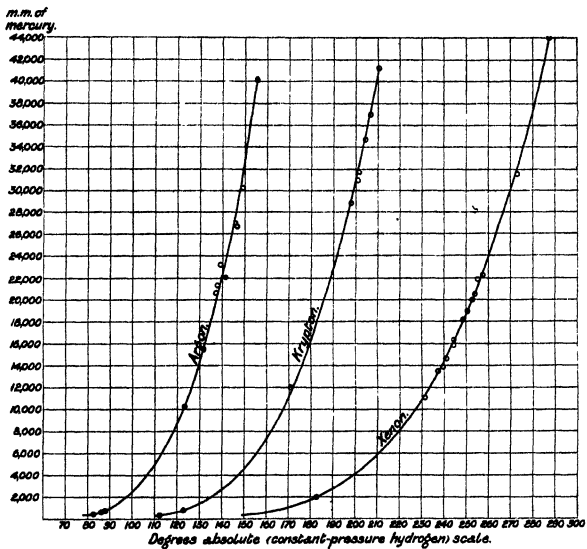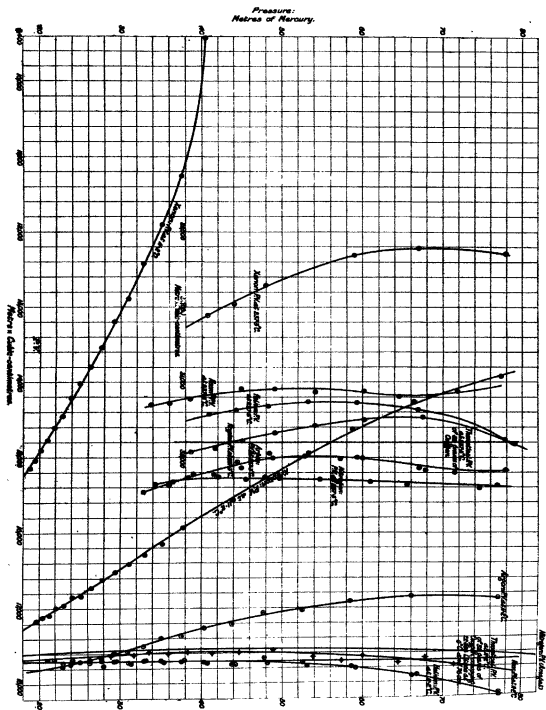 quantitative relations to their position in the periodic table, yet there are inexplicable deviations from regularity, which hold forth hopes of the discovery of a still more far-reaching generalisation. What that generalisation may be is not yet to be divined; but that it must underlie what is known, and must furnish a clue to the explanation of irregularities, cannot be disputed.

Fig. 9.

Atomic Weights.
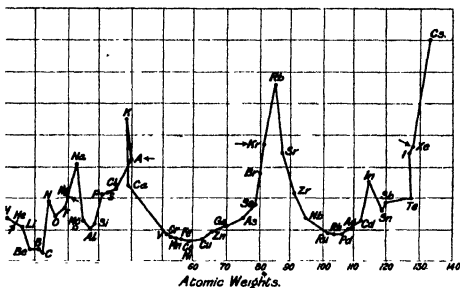
When we began the search for the elements of which the physical properties are described in the foregoing pages, we were not without a strong hope that their discovery would solve the problem. For there can be little doubt that these inactive elements constitute simple, if not the simplest forms of matter. They have no tendency to form compounds, and are monatomic; their physical relations, we may therefore presume, are not subject to interference by the simultaneous exercise of chemical functions. But our hope has been fruitless. While the same rough quantitative correspondence between the order in the periodic table and the physical properties is manifest, as with other similar series of elements, we have failed to trace any simple mathematical expressions which would make it possible to predict with accuracy the physical properties of any one of these elements, from a knowledge of those of its congeners. It is possible that such expressions exist; we venture to hope that others, more mathematically gifted than we are, may succeed where we have failed.

m.m. of mercury.

44,000
42,000
40,000
38,000
36,000
34,000
32,000
30,000
28,000
26,000
24,000
22,000
20,000
18,000
16,000
14,000
12,000
10,000
8,000
6,000
4,000
2,000

Argon   Krypton   Xenon

70 80 90 100 110 120 130 140 150 160 170 180 190 200 210 220 230 240 250 260 270 280 290 300

Degrees absolute (constant-pressure hydrogen) scale.

Pressure:
Metres of Mercury.

III. *On the Theory of Consistence of Logical Class-frequencies, and its Geometrical Representation.*

By G. UDNY YULE, *formerly Assistant Professor of Applied Mathematics in University College, London.*

Communicated by Professor K. PEARSON, F.R.S.

Received February 9,—Read February 28, 1901.

CONTENTS.

§ 1. IN the ordinary treatment of logic the field of discussion is strictly limited to premises of a non-numerical character, numerically definite data being rigidly excluded. The statistician obtains no help from ordinary logic towards solving even the simplest problems, *e.g.*, the deduction of inferences from data of the type "*x* per cent. of A's are B, *y* per cent. of A's are C," or the inferring of association between B and C from known associations of A with B and with C.

It is now more than half a century since DE MORGAN, in the chapter "On the Numerically Definite Syllogism," of his 'Formal Logic' (1847), laid the foundations of a theory of strictly quantitative logic. Substituting the modified notation of JEVONS, employed by me in a recent paper,* for DE MORGAN's own notation, his Theorem may be expressed in the form "if (AB) + (AC) > (A), (BC) must be *at least* equal to the difference (AB) + (AC) − (A)." For if, *e.g.*, we imagine (A) boxes, into which

---

* "On the Association of Attributes in Statistics," &c., 'Phil. Trans.,' A, 1900, vol. 194, p. 257. The notation is essentially that of JEVONS, save that small Greek letters have been substituted for his italics.

we deal a certain number (viz., (AB)) of cards marked (B), one into each box, leaving only (A) — (AB) unoccupied boxes, and then proceed to deal into the remaining boxes cards marked C, one into each ; some of these must fall into boxes already occupied by a B if their number exceed (A) — (AB). From similar simple reasoning DE MORGAN derived the complete conditions of consistence for (AB), (AC), and (BC).* He does not, however, consider the case of more than three attributes, and the whole discussion is rendered very lengthy owing to his standpoint being still that of the older logicians.

The theory of numerical logic was carried somewhat further by BOOLE in Chapter 19 of the 'Laws of Thought' (1854), entitled "Of Statistical Conditions." After taking a series of propositions for finding the major and minor limits to class-frequencies or sums of frequencies of any order, in terms of the first-order frequencies and the total frequency only, he proceeds to the general problem "given the respective numbers of individuals comprised in any classes, s, t, &c., logically defined, to deduce a system of numerical limits of any other class w also logically defined." I must confess myself unable to follow the physical meaning of the processes symbolically developed in BOOLE's general theorem, and this chapter has not, to my knowledge, been discussed by subsequent logicians. One naturally turns to the "Symbolic Logic" of Dr. VENN, whose lucid treatment clears up many difficulties of the "Laws of Thought," but he does not appear to deal with the problems "of statistical conditions."

In the following memoir I have endeavoured to deal with the general theory of logical consistence, as I prefer to term it, from a standpoint slightly different to that of BOOLE.†

§ 2. Let (U) be the total frequency in some defined universe, and let

$$(A)\ (B)\ (C)\ \ldots\ (AB)\ (AC)\ \ldots\ (ABC)\ \ldots\ \&c.,$$

be the frequencies of the positive groups (classes) up to, say, groups of the nth order, the number of attributes specified being m ($m > n$). It will be remembered that all other group-frequencies can be expressed in terms of those of the positive groups, so that no others need be considered. BOOLE, in his general theorem, quoted above, supposes certain of these frequencies to be known, and requires to find the resulting limits to some one other. I propose instead to make, at the outset, no supposition as to the frequencies that are known, but simply to discuss what conditions must hold if the whole set of frequencies is to be self-consistent. By proceeding in this way symmetrical systems of conditions are obtained of great interest and generality. They may be applied at once to such cases of limit-inference as are dealt with by

---

* See § 5 below.

† I may perhaps state that this work was not directly suggested by DE MORGAN'S or by BOOLE'S writings. Difficulties had arisen in the invention of numerical examples to illustrate certain points of theory, and I was driven to working out the theory of consistence in order to clear up these difficulties.

BOOLE, while exhibiting in a clearer light the data that are necessary for any inference, and the limitations of inference caused by assigned limitations of data.

The whole of the conditions of consistence are derived from one source, or one condition only, viz., that *no frequency can be less than zero.* But as it is evident that if *all* the frequencies of any order be greater than zero, the frequencies of all lower orders must, *à fortiori*, be greater than zero, we may limit the above statement by saying that all the conditions of consistence are covered by the dictum that *no ultimate frequency* can be less than zero.* If, however, we are dealing with groups of the *n*th and lower orders only in *m* specified attributes, it is convenient to divide the conditions into two classes—(1) the " inferior conditions of consistence," which may be derived from the fact that no *n*th order frequency can be less than zero; (2) the " superior conditions of consistence" which can only be derived from the consideration that frequencies of order greater than *n* cannot be less than zero.

A distinction of this sort is, it may be noted, made by DE MORGAN. Inferences drawn from the *inferior* conditions of consistence for second order groups he terms *spurious* inferences;† they do not really follow from given premises (*i.e.*, given values of (AB) and (AC) or (AB) and (BC), &c.), but are " true by the constitution of the universe."‡

§ 3. It will be convenient to use the following terms in addition to those defined in my previous memoir.

A set of frequencies formed by taking the frequency of any positive group ABCD . . . N, together with the frequencies of all possible groups of the same order formed by substituting the contraries $a \beta \gamma \delta$ . . . ν for one or more of the attributes ABCD . . . N, will be called an " aggregate " of frequencies. Any one aggregate contains only one positive group which may be used to denote the aggregate, so that one may speak of the AB aggregate or the ABCD aggregate. The *order* of an aggregate may be defined as the order of the groups contained in it. An aggregate of order *n* contains 2^n groups. The sum of the frequencies of these 2^n groups of the aggregate, is evidently equal to the total frequency or number of observations (U).

If *m* attributes be specified, the number of positive groups of the *n*th order that can be formed from them is

$$m (m - 1) (m - 2) \ldots (m - n + 1)$$

The complete set of *consistent* aggregates corresponding to these positive groups will be termed a congruence of aggregates or simply a congruence. The number

---

* The frequency of a group specified by all the *m* attributes noted. " On the Association of Attributes," &c., *loc. cit.*, p. 259.

† 'Formal Logic,' p. 159.   ‡ Note on same page.

of attributes specified, m, will be termed the *degree* of the congruence, its *order* being the order of the component aggregates.*

A congruence of degree m and order n may be regarded as built up of a series of congruences of degree lower than m. Thus the congruence of the fourth degree and second order containing the aggregates AB, AC, AD, BC, BD, CD, may be split up into four congruences of the third degree only, viz., those containing the aggregates (AB, AC, BC) (AB, AD, BD) (AC, AD, CD) and (BC, BD, CD) respectively. In general, any congruence of the mth degree may be divided into

$$\frac{m(m-1)\ \ldots\ (m-r+1)}{r!}$$

congruences of the rth degree, one to each positive group of the rth order that can be formed from the m attributes. These component congruences evidently, as in the above example, overlap : any one aggregate occurs in two or more congruences.

In the following sections (§ 4–§ 10) I proceed to the discussion of congruences of the third, fourth, and fifth degrees ; §§ 11–13 deal with the general case, and the remainder of the paper consists of a discussion of the geometrical representation, by means of polyhedra, of the conditions of consistence, together with a few numerical illustrations.

Congruence of the Third Degree.

§ 4. In terms of the definition § 2, p. 93, the inferior conditions of consistence are given at once by expanding all second-order frequencies in terms of the positive groups only, and putting the resulting expression $\not< 0$. Thus, retaining for convenience the second-order terms only on the left of the inequality, we must have

$$
\left.
\begin{aligned}
(AB) &\not< 0 & &\text{or} & (AB) \text{ will be negative} \\
&\not< (A) + (B) - (U) & &\text{or} & (\alpha\beta) \quad ,, \qquad ,, \\
&\not> (A) & &\text{or} & (A\beta) \quad ,, \qquad ,, \\
&\not> (B) & &\text{or} & (\alpha B) \quad ,, \qquad ,,
\end{aligned}
\right\} \quad . \quad . \quad . \text{ I.}
$$

Similar conditions hold of course for (AC) and (BC).

§ 5. To find the superior conditions of consistence (*cf.* again the definition), write down the *inferior* conditions for the (ABC) aggregate. These are of course given similarly by expanding the third-order frequencies in terms of the positive groups, and putting the resulting expansion $\not< 0$. Thus—

---

* In any systematic tabulation of frequencies I think the grouping should be made by aggregates. Such an arrangement would be distinctly better than that adopted by me in the sample table on p. 318 of my paper on "Association," *loc. cit.*

Or the frequency
given below
will be negative.

$(ABC) \not< 0$ . . . . (ABC) [1]

     $\not< (AB) + (AC) - (A)$. . . . . $(A\beta\gamma)$ [2]

     $\not< (AB) + (BC) - (B)$. . . . . $(\alpha B\gamma)$ [3]

     $\not< (AC) + (BC) - (C)$ . . . . $(\alpha\beta C)$ [4]   II.

     $\not> (AB)$ . . . . $(AB\gamma)$ [5]

     $\not> (AC)$ . . . . $(A\beta C)$ [6]

     $\not> (BC)$ . . . . (αBC) [7]

     $\not> (AB) + (AC) + (BC) - (A) - (B) - (C) + (U)$. $(\alpha\beta\gamma)$ [8]

Now evidently, any third-order aggregate whatever is impossible *if any one of the minor limits* [1]–[4] *be greater than any one of the major limits* [5]–[8]. If the second-order frequencies be such as to create this condition they must be impossible within one and the same universe, *i.e.*, they are inconsistent or incongruent. There are sixteen comparisons to be made, taking each of the major limits in turn with each of the other minor limits, but the majority of these comparisons, viz., 12, only lead back to the inferior conditions. The four comparisons of expansions due to contrary frequencies alone lead to new conditions—

II. [1][8] $(AB) + (AC) + (BC) \not< (A) + (B) + (C) - (U)$ . . . [1]

II. [2][7] $- (AB) + (AC) + (BC) \not> (C)$ . . . . [2]   III.

II. [3][6] $(AB) - (AC) + (BC) \not> (B)$ . . . . [3]

II. [4][5] $(AB) + (AC) - (BC) \not> (A)$ . . . . [4]

These are the four superior conditions of consistence for the congruence of the third degree. In order that the second-order frequencies may be consistent with each other and with the given frequencies of the first order, they must fulfil all the conditions of type I. and the conditions of type III. These inequalities are highly interesting; but discussion is best deferred till after we have obtained the similar conditions for the congruences of higher degree.

Congruence of the Fourth Degree.

§ 6. Two congruences of this degree are possible, viz., those of the second and third orders. The third-order congruence will be taken first, as the conditions of consistence for the second-order congruence may be obtained directly from the third-order conditions.

The inferior conditions need not be written down as they have been given already (II. of § 5); similar conditions hold of course for all four groups (ABC), (ABD), (ACD), (BCD).

§ 7. To obtain the superior conditions, the fourth-order groups must be expanded (*cf.* § 5), and the expansions again put ≮ 0, thus giving the systems of limits for (ABCD) as below.

Or the frequency
given below
will be negative.

(ABCD) ≮ 0 . . . . (ABCD)   [1]

≮ (ACD) + (BCD) − (CD) . . . . (αβCD)   [2]

≮ (ABD) + (BCD) − (BD) . . . . . (αByD)   [3]

≮ (ABC) + (BCD) − (BC) . . . . . (αBCδ)   [4]

≮ (ABD) + (ACD) − (AD) . . . . (AβyD)   [5]

≮ (ABC) + (ACD) − (AC) . . . . (AβCδ)   [6]

≮ (ABC) + (ABD) − (AB) . . . . (ABγδ)   [7]

≮ (ABC) + (ABD) + (ACD) + (BCD) − (AB)
− (AC) − (AD) − (BC) − (BD) − (CD) + (A) } (αβγδ)   [8]
+ (B) + (C) + (D) − (U) . . . . .

≯ (ABC) . . . . (ABCδ)   [9]

≯ (ABD) . . . . (ABγD)   [10]   **IV.**

≯ (ACD) . . . . (AβCD)   [11]

≯ (BCD) . . . . (αBCD)   [12]

≯ (ABD) + (ACD) + (BCD) − (AD) − (BD)
− (CD) + (D) . . . . } (αβγD)   [13]

≯ (ABC) + (ACD) + (BCD) − (AC) − (BC)
− (CD) + (C) . . . . } (αβCδ)   [14]

≯ (ABC) + (ABD) + (BCD) − (AB) − (BC)
− (BD) + (B) . . . . } (αBγδ)   [15]

≯ (ABC) + (ABD) + (ACD) − (AB) − (AC)
− (AD) + (A) . . . . } (Aβγδ)   [16]

Any fourth-order aggregate will be impossible if any one of the minor limits to (ABCD) be greater than any one of the major limits. There are eight of each, or sixty-four comparisons to be made ; but thirty-two of these lead only to the already known inferior conditions of consistence. The remaining thirty-two, involving comparisons of pairs of groups that are contrary as regards *three* attributes (as in § 5), give new conditions all obtainable by cyclic substitution from the eight given in V. below.

There are four third-order positive groups to be formed from four attributes, and therefore four possible sets of three, giving four sets of the eight inequalities V. The remaining three sets may be at once written down by substituting A, B, or C for D (and conversely) in the set given below.

$$\left.\begin{array}{l}
(ABD) + (ACD) + (BCD) \not\gtrless (AB) + (AC) + (AD) + (BC) + (BD) \\
\qquad . \quad + (CD) - (A) - (B) - (C) - (D) \\
\qquad + (U) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\
\qquad\qquad \not\lessgtr (AD) + (BD) + (CD) - (D) \quad . \quad . \\
(ABD) + (ACD) - (BCD) \not\gtrless (AD) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\
\qquad\qquad \not\lessgtr (AB) + (AC) + (AD) - (BC) - (A) \\
(ABD) - (ACD) + (BCD) \not\gtrless (BD) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\
\qquad\qquad \not\lessgtr (AB) - (AC) + (BC) + (BD) - (B) \\
-(ABD) + (ACD) + (BCD) \not\gtrless (CD) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\
\qquad\qquad \not\lessgtr - (AB) + (AC) + (BC) + (CD) - (C)
\end{array}\right\} V.$$

with bracket labels [1], [2], [3], [4], [5], [6], [7], [8]

§ 8. The superior conditions of consistence for the *second*-order congruence of the fourth degree may be at once obtained from V. For in order that the congruence of the third order may be self-consistent, it is clearly necessary that the congruence of the second order should be so; if this condition do not hold the conditions of ·consistence V. will prove themselves impossible. But the limits V. only become impossible if, in either of the four pairs of limits ([1] [2], [3] [4] &c.), the minor limit be greater than the major. If we express the condition that each minor limit must be the less we have,

from V [1]|[2] $(AB) + (AC) + (BC) \not\gtrless (A) + (B) + (C) - (U)$

[3]|[4] $(AB) + (AC) - (BC) \not\lessgtr (A).$

[5]|[6] $(AB) - (AC) + (BC) \not\lessgtr (B).$

[7]|[8] $-(AB) + (AC) + (BC) \not\lessgtr (C)$

But these are simply the conditions III. of § 5, the superior conditions of consistence for the second-order congruence of the *third* degree. Similar conditions for the aggregates ABD, ACD, BCD, will of course be derived from the unwritten sets of inequalities corresponding to (V.). The theorem therefore holds—" A congruence of the fourth degree and second order is self-consistent, if each of the three congruences of the third degree, into which it can be resolved, is self-consistent."

Congruence of the Fifth Degree.

§ 9. Congruences of the fifth degree may be of either the second, third or fourth order. The congruence of the fourth order may be taken first and the others derived from it.

The inferior conditions of consistence for fourth-order groups have been already given in § 7, IV. The similar conditions for fifth-order groups, obtained of course in precisely the same way, are as follows :—

Or the frequency
given below is
negative.

$(ABCDE) \not< 0$ . . . . $(ABCDE)$ [1]

,, $\not< (ACDE) + (BCDE) - (CDE)$ . . . $(\alpha\beta CDE)$ [2]

,, $\not< (ABDE) + (BCDE) - (BDE)$ . . . $(\alpha B\gamma DE)$ [3]

,, $\not< (ABCE) + (BCDE) - (BCE)$ . . . $(\alpha BC\delta E)$ [4]

,, $\not< (ABCD) + (BCDE) - (BCD)$ . . . $(\alpha BCD\epsilon)$ [5]

, $\not< (ABDE) + (ACDE) - (ADE)$ . . . $(A\beta\gamma DE)$ [6]

,, $\not< (ABCE) + (ACDE) - (ACE)$ . . . $(A\beta C\delta E)$ [7]

,, $\not< ABCD) + (ACDE) - (ACD)$ . . . $(A\beta CD\epsilon)$ [8]

,, $\not< (ABCE) + (ABDE) - (ABE)$ . . . $(AB\gamma\delta E)$ [9]

,, $\not< (ABCD) + (ABDE) - (ABD)$ . . . $(AB\gamma D\epsilon)$ [10]

,, $\not< (ABCD) + (ABCE) - (ABC)$ . . . $(ABC\delta\epsilon)$ [11]

,, $\not< (ABCE) + (ABDE) + (ACDE) + (BCDE)$
$- (ABE) - (ACE) - (ADE) - (BCE)$
$- (BDE) - (CDE) + (AE) + (BE)$
$+ (CE) + (DE) - (E)$ . . . . . $\left.\right\}(\alpha\beta\gamma\delta E)$ [12]

,, $\not< (ABCD) + (ABDE) + (ACDE)$
$+ (BCDE) - (ABD) - (ACD) - (ADE)$
$- (BCD) - (BDE) - (CDE) + (AD)$
$+ (BD) + (CD) + (DE) - (D)$ . . $\left.\right\}(\alpha\beta\gamma D\epsilon)$ [13]

, $\not< (ABCD) + ABCE) + (ACDE)$
$+ (BCDE) - (ABC) - (ACD) - (ACE)$
$- (BCD) - (BCE) - (CDE) + (AC)$
$+ (BC) + (CD) + (CE) - (C)$. . . $\left.\right\}(\alpha\beta C\delta\epsilon)$ [14]

,, $\not< (ABCD) + (ABCE) + (ABDE)$
$+ (BCDE) - (ABC) - (ABD) - (ABE)$
$- (BCD) - (BCE) - (BDE) + (AB)$
$+ (BC) + (BD) + (BE) - (B)$ . . . $\left.\right\}(\alpha B\gamma\delta\epsilon)$ [15]

,, $\not< (ABCD) + (ABCE) + (ABDE)$
$+ (ACDE) - (ABC) - (ABD) - (ABE)$
$- (ACD) - (ACE) - (ADE) + (AB)$
$+ (AC) + (AD) + (AE) - (A)$ . . . $\left.\right\}(A\beta\gamma\delta\epsilon)$ [16]

,, $\not> (BCDE)$ . . . . $(\alpha BCDE)$ [17]

, $\not> (ACDE)$ . . . . $(A\beta CDE)$ [18]

,, $\not> (ABDE)$ . . . . $(AB\gamma DE)$ [19]

,, $\not> (ABCE)$ . . . . $(ABC\delta E)$ [20]

, $\not> (ABCD)$ . . . . $(ABCD\epsilon)$ [21]

,, $\not> (ABDE) + (ACDE) + (BCDE) - (ADE)$
$- (BDE) - (CDE) + (DE)$ . . . . . $\left.\right\}(\alpha\beta\gamma DE)$ [22]

VI

Or the frequency
given below is
negative.

$$(ABCDE) \not> (ABCE) + (ACDE) + (BCDE) - (ACE) \\ - (BCE) - (CDE) + (CE) \quad . \quad . \quad .\} (\alpha\beta C\delta E) \quad [23]$$

$$,, \quad \not> (ABCD) + (ACDE) + (BCDE) - (ACD) \\ - (BCD) - (CDE) + (CD) . \quad . \quad . \quad .\} (\alpha\beta CD\epsilon) \quad [24]$$

$$,, \quad \not> (ABCE) + (ABDE) + (BCDE) - (ABE) \\ - (BCE) - (BDE) + (BE) \quad . \quad . \quad .\} (\alpha B\gamma\delta E) \quad [25]$$

$$,, \quad \not> (ABCD) + (ABDE) + (BCDE) - (ABD) \\ - (BCD) - (BDE) + (BD) . \quad . \quad .\} (\alpha B\gamma D\epsilon) \quad [26]$$

$$,, \quad \not> (ABCD) + (ABCE) + (BCDE) - (ABC) \\ - (BCD) - (BCE) + (BC) . \quad . \quad . \quad .\} (\alpha BC\delta\epsilon) \quad [27]$$

$$,, \quad \not> (ABCE) + (ABDE) + (ACDE) - (ABE) \\ - (ACE) - (ADE) + (AE) \quad . \quad . \quad .\} (A\beta\gamma\delta E) \quad [28]$$

$$, \quad \not> (ABCD) + (ABDE) + (ACDE) - (ABD) \\ - (ACD) - (ADE) + (AD) \quad . \quad .\} (A\beta\gamma D\epsilon) \quad [29]$$

$$,, \quad \not> (ABCD) + (ABCE) + (ACDE) - (ABC) \\ - (ACD) - (ACE) + (AC) \quad . \quad .\} (A\beta C\delta\epsilon) \quad [30]$$

$$,, \quad \not> (ABCD) + (ABCE) + (ABDE) - (ABC) \\ - (ABD) - (ABE) + (AB) \quad . \quad . \quad .\} (AB\gamma\delta\epsilon) \quad [31]$$

$$,, \quad \not> (ABCD) + (ABCE) + (ABDE) + (ACDE) \\ + (BCDE) - (ABC) - (ABD) - (ABE) \\ - (ACD) - (ACE) - (ADE) - (BCD) \\ - (BCE) - (BDE) - (CDE) + (AB) \\ + (AC) + (AD) + (AE) + (BC) + (BD) \\ + (BE) + (CD) + (CE) + (DE) - (A) \\ - (B) - (C) - (D) - (E) + (U) \quad . \quad .\} (\alpha\beta\gamma\delta\epsilon) \quad [32]$$

$\} VI.$

Any fifth-order aggregate will be impossible if any one of the minor limits to (ABCDE) be greater than any one of the major limits. Since there are sixteen of each, there are 256 comparisons to be made. Eighty of these lead only to the already known inferior conditions. Of the remaining 176, 160 may be obtained by cyclic substitution from the sixteen conditions involving only three fourth-order groups each, as given in VII. below, the remaining sixteen, each involving all five fourth-order positive groups, being given in VIII.

VII

$$
\begin{aligned}
(ABDE) + (ACDE) + (BCDE) \gtrless\ & (ABE) + (ACE) + (ADE) + (BCE) + (BDE) + (CDE) - (AE) \\
& \quad - (BE) - (CE) - (DE) + (E) \tag{1} \\
\lessgtr\ & (ABD) + (ACD) + (ADE) + (BCD) + (BDE) + (CDE) - (AD) \\
& \quad - (BD) - (CD) - (DE) + (D) \tag{2} \\
\lessgtr\ & (ADE) + (BDE) + (CDE) - (DE) \tag{3} \\
\lessgtr\ & (ABD) + (ABE) + (ACD) + (ACE) + (ADE) + (BCD) + (BCE) \\
& \quad + (BDE) + (CDE) - (AB) - (AC) - (AD) - (AE) - (BC) \\
& \quad - (BD) - (BE) - (CD) - (CE) - (DE) + (A) + (B) + (C) \\
& \quad + (D) + (E) - (U) \tag{4}
\end{aligned}
$$

$$
\begin{aligned}
(ABDE) + (ACDE) - (BCDE) \gtrless\ & (ADE) \tag{5} \\
\lessgtr\ & (ABD) + (ABE) + (ACD) + (ACE) + (ADE) - (BCD) - (BCE) \\
& \quad - (AB) - (AC) - (AD) - (AE) + (BC) + (A) \tag{6} \\
\lessgtr\ & (ABE) + (ACE) + (ADE) - (BCE) - (AE) \tag{7} \\
\lessgtr\ & (ABD) + (ACD) + (ADE) - (BCD) - (AD) \tag{8} \\
\lessgtr\ & (BDE) \tag{9}
\end{aligned}
$$

$$
\begin{aligned}
ABDE - (ACDE) + (BCDE) \gtrless\ & (ABD) + (ABE) - (ACD) - (ACE) + (BCD) + (BCE) + (BDE) \\
& \quad - (AB) + (AC) - (BC) - (BD) - (BE) + (B) \tag{10} \\
\neq\ & (ABE) - (ACE) + (BCE) + (BDE) - (BE) \tag{11} \\
\neq\ & (ABD) - (ACD) + (BCD) + (BDE) - (BD) \tag{12} \\
& (CDE) \tag{13}
\end{aligned}
$$

$$
\begin{aligned}
(ABDE) + (ACDE) + (BCDE) \gtrless\ & (ABD) - (ABE) + (ACD) + (ACE) + (BCD) + (BCE) + (CDE) \\
& \quad + (AB) - (AC) - (BC) - (CD) - (CE) + (C) \tag{14} \\
-\ & (ABE) + (ACE) + (BCE) + (CDE) - (CE) \tag{15} \\
\lessgtr\ & (ABD) + (ACD) + (BCD) + (CDE) - (CD). \tag{16}
\end{aligned}
$$

II.

[1] (ABCD) + (ABCE) + (ABDE) + (ACDE) + (BCDE) ≮ (ABC) + (ABD) + (ABE) + (ACD) + (ACE) + (ADE) + (BCD) + (BCE) + (BDE) + (CDE) − (AB) − (AC) − (AD) − (AE) − (BC) − (BD) − (BE) − (CD) − (CE) − (DE) + (A) + (B) + (C) + (D) + (E) − (U) .

[2] (ABCD) + (ABCE) + (ABDE) + (ACDE) − (BCDE) ≯ (ABC) + (ABD) + (ABE) + (ACD) + (ACE) + (ADE) − (AB) − (AC) − (AD) − (AE) + (A) .

[3] (ABCD) + (ABCE) + (ABDE) − (ACDE) + (BCDE) ≯ (ABC) + (ABD) + (ABE) + (BCD) + (BCE) + (BDE) − (AB) − (BC) − (BD) − (BE) + (B) .

[4] (ABCD) + (ABCE) − (ABDE) + (ACDE) + (BCDE) ≯ (ABC) + (ACD) + (ACE) + (BCD) + (BCE) + (CDE) − (AC) − (BC) − (CD) − (CE) + (C) .

[5] (ABCD) − (ABCE) + (ABDE) + (ACDE) + (BCDE) ≯ (ABD) + (ACD) + (ADE) + (BCD) + (BDE) + (CDE) − (AD) − (BD) − (CD) − (DE) + (D) .

[6] −(ABCD) + (ABCE) + (ABDE) + (ACDE) + (BCDE) ≯ (ABE) + (ACE) + (ADE) + (BCE) + (BDE) + (CDE) − (AE) − (BE) − (CE) − (DE) + (E) .

[7] −(ABCD) − (ABCE) + (ABDE) + (ACDE) + (BCDE) ≮ (ADE) + (BDE) + (CDE) − (ABC) − (DE) .

[8] −(ABCD) + (ABCE) − (ABDE) + (ACDE) + (BCDE) ≮ (ACE) + (BCE) + (CDE) − (ABD) − (CE) .

[9] −(ABCD) + (ABCE) + (ABDE) − (ACDE) + (BCDE) ≮ (ABE) + (BCE) + (BDE) − (ACD) − (BE) .

[10] −(ABCD) + (ABCE) + (ABDE) + (ACDE) − (BCDE) ≮ (ABE) + (ACE) + (ADE) − (BCD) − (AE) .

[11] (ABCD) − (ABCE) − (ABDE) + (ACDE) + (BCDE) ≮ (ACD) + (BCD) + (CDE) − (ABE) − (CD) .

[12] (ABCD) − (ABCE) + (ABDE) − (ACDE) + (BCDE) ≮ (ABD) + (BCD) + (BDE) − (ACE) − (BD) .

[13] (ABCD) − (ABCE) + (ABDE) + (ACDE) − (BCDE) ≮ (ABD) + (ACD) + (ADE) − (BCE) − (AD) .

[14] (ABCD) + (ABCE) − (ABDE) − (ACDE) + (BCDE) ≮ (ABC) + (BCD) + (BCE) − (ADE) − (BC) .

[15] (ABCD) + (ABCE) − (ABDE) + (ACDE) − (BCDE) ≮ (ABC) + (ACD) + (ACE) − (BDE) − (AC) .

[16] (ABCD) + (ABCE) + (ABDE) − (ACDE) − (BCDE) ≮ (ABC) + (ABD) + (ABE) − (CDE) − (AB) .

§ 10. The conditions of superior congruence given in VII. only become impossible if either of the minor limits for any one set of fourth-order frequencies be greater than either of the major limits (*e.g.*, [3] or [4] greater than [1] or [2]). But on expressing the condition that each minor limit must be less than the major, we see we are simply led back to limits of the form V., § 7. That is to say—"A congruence of the fifth degree and third order is self-consistent, if each of the five congruences of the fourth degree, into which it can be resolved, is self-consistent." But we have seen that comparison of the minor and major limits of conditions V. again leads back simply to conditions III. Therefore we must have also—"A congruence of the fifth degree and second order is self-consistent, if each of the ten congruences of the third degree, into which it can be resolved, is self-consistent." The conditions of consistence for the congruence of any degree are then, so far as we have gone at all events, thrown back on the conditions for the simple congruence, the degree of which only exceeds its order by unity.

<center>REMARKS ON THE PRECEDING SECTIONS.</center>

<center>*General Solution.*</center>

§ 11. The elementary method employed in the preceding sections is the one best adapted for exhibiting clearly the physical meaning of the conditions of consistence. It is perfectly adapted for finding the conditions for a congruence of any degree, though the number of comparisons of limits to be made appears at first sight to make the work extremely lengthy. A few considerations, however, rapidly reduce the number of necessary comparisons. Thus all comparisons of expansions due to two groups that are contrary to each other in one term only give *inferior* conditions.

Again, all comparisons of expansions due to groups that are contrary to each other in three terms, give conditions simply derivable from those of the congruence of the third degree. Take for instance the first condition of § 5, III.

$$(AB) + (AC) + (BC) \not< (A) + (B) + (C) - (U).$$

The universe in which this inequality is to hold good is not specified at all. Let it be a universe in which all things are D. Then the condition becomes

$$(ABD) + (ACD) + (BCD) \not< (AD) + (BD) + (CD) - (D).$$

But this is simply the second condition of superior consistence for a congruence of the fourth degree (V., § 7). Again let the universe be not D, but δ. Then the condition becomes

$$(ABδ) + (ACδ) + (BCδ) \not< (Aδ) + (Bδ) + (Cδ) - (δ),$$

getting rid of the negative terms by expansion, that is

$$(ABD) + (ACD) + (BCD) \not> (AB) + (AC) + AD) + (BC) + (BD) + (CD)$$
$$- (A) - (B) - (C) - (D) + (U).$$

But this is again simply the first condition of V., § 7. The whole of the conditions V. for the congruence of fourth degree may in fact be derived by writing down the conditions III. for each aggregate ABC, ABD, ACD, BCD, inserting the universes D or δ in the first case, C or γ in the second, and so on. Evidently this will give the right number of conditions, there being four aggregates and four conditions for each, while each condition must hold good in two universes, total $4 \times 4 \times 2 = 32$.

A precisely similar theorem holds good for the 160 conditions of § 9, VII., for the congruence of the fifth degree. The congruence of second order and fifth degree may be resolved into *ten* congruences of the third degree. The *four* conditions of III. must hold good for each of these congruences in *four* universes, e.g., for the aggregate ABC, the universes DE, Dε, δE, δε. The whole number of conditions so derived is $10 \times 4 \times 4$, or 160. The sixteen conditions of VIII., § 9, cannot be so derived, they involve five, not three, fifth-order frequencies each, and are quite new conditions derived from the comparison of expansions of groups contrary to each other in five attributes.

The results suggest, however, that to obtain the *new* conditions for a congruence of degree $2m + 1$, order $2m$, we have only to consider the 2^{2n} possible comparisons of contraries. A congruence of even degree, say $2m$, is subject to no conditions beyond those immediately derivable from the congruence of degree $2m - 1$.

But this result is only suggested, not proved, by the few cases taken ; nor are the general theorems corresponding to those given at the end of § 8 and § 10 demonstrable from the mere particular cases.

§ 12. By slightly changing the point of view, and remembering that any frequency may be expanded by considering A, B, C, U, &c, as "elective operators," subject to the ordinary laws of multiplication, and to the special laws

$$U.A = A$$
$$A^\epsilon = A,$$

the conditions of congruence may be obtained in a simple general form.

Referring back to the earlier sections (§ 5, § 7, or § 9), it will be seen that in considering the congruence of degree m order $\overline{m-1}$, all the groups of the mth order, containing an even number of negative terms (or attributes), were first taken and expanded, and the expansion put not less than zero, thus giving a system of minor limits for (ABCD . . M). The remaining groups, containing an odd number of negatives, were similarly expanded, giving a system of major limits. The expansions in the two cases were of the forms

$$+ (ABCD \ldots M) - (\text{terms of lower order than } m)_1$$
$$- (ABCD \ldots M) + (\text{terms of lower order than } m)_2.$$

We took this in the previous sections as giving

$$(ABCD \; . \quad . \; M) \not< (\text{terms of lower order than } m)_1$$
$$(ABCD \; . \quad . \; M) \not> (\text{terms of lower order than } m)_2,$$

and deduced

$$(\text{terms of lower order than } m)_2 \not< (\text{terms of lower order than } m)_1.$$

But precisely the same result is arrived at by adding the two expansions and putting the sum not less than zero,* and this is a much more convenient conception from which to obtain the general conditions.

Let the two groups contain positive attributes $K_1 K_2 \; . \quad . \; K_p$ and negative attributes $\lambda_1 \lambda_2 \; . \; . \; \lambda_q$ in common; positive attributes $M_1 M_2 \; . \; . \quad M_r$ in the one, with their negatives $\mu_1 \mu_2 \quad . \; \mu_r$ in the other; and negative attributes $\nu_1 \nu_2 \; . \; . \; \nu_s$ in the one, with their positives $N_1 N_2 \; . \; . \; N_s$ in the other. Using the symbol $\varPi$ to denote "the continued operator-product of all quantities like . . . " the expansions of the two frequencies may be written

$$\prod_{p=1}^{p=p} [K_p] \; \prod_{q=1}^{q=q} [U - L_q] \; \prod_{s=1}^{r=r} [M_r] \; \prod_{s=1}^{s=s} [U - N_s]$$
$$\prod_{p=1}^{p=p} [K_p] \; \prod_{q=1}^{q=q} [U - L_q] \; \prod_{r=1}^{s=r} [U - M_r] \; \prod_{s=1}^{s=s} [N_s].$$

In these expressions it must be remembered that as the one group is to contain an even, the other an odd, number of negatives, if $q + r$ be odd $q + s$ must be even, and *vice versâ*. Hence $r + s$ must in any case be odd, *i.e.*, the two groups must be contrary as regards an odd number of attributes, for if

$$q + s = 2x$$
$$q + r = 2y + 1$$
$$r + s = 2(x + y - q) + 1,$$

which is necessarily odd. The general condition of consistency for a congruence of degree

$$m = p + q + r + s,$$

* Every frequency must be greater than zero, and *à fortiori* the sum of any two. But it is only by taking the sum of two frequencies, the one containing an even, and the other an odd, number of negatives that an expression is obtained from which the mth-order term is eliminated.

and order $(m - 1)$ may then be written

$$\overset{p}{\underset{1}{\varPi}}[K_r] \ \overset{q}{\underset{1}{\varPi}}[U - L_q] \left\{ \overset{r}{\underset{1}{\varPi}}[M_r] \ \overset{r}{\underset{1}{\varPi}}[U - N_r] + \overset{r}{\underset{1}{\varPi}}[U - M_r] \ \overset{r}{\underset{1}{\varPi}}[N_r] \right\} \nless 0 \quad . \quad (1).$$

All the conditions of consistence, whether inferior or superior, given in the preceding sections may be readily verified from this general expression.*

§ 13. If the two groups compared be contrary in c terms $(c = r + s)$, the expansions will give rise to c terms of the $(m - 1)$th order, viz., s from the first term, and r from the second term within the curly bracket. Thus the conditions VII., § 9, with *three* fourth-order frequencies on the last, were all obtained by comparison of expansions of fifth-order frequencies contrary in *three* attributes; the conditions VIII. on the other hand by comparison of expansions due to fifth-order frequencies contrary in all five attributes.

The term outside the bracket in (1) may be regarded as a mere specification of the universe within which the simple condition

$$\overset{r}{\underset{1}{\varPi}}[M_r] \ \overset{r}{\underset{1}{\varPi}}[U - N_r] + \overset{r}{\underset{1}{\varPi}}[U - M_r] \ \overset{r}{\underset{1}{\varPi}}[N_r] \nless 0 \quad . \quad . \quad . \quad . \quad (2),$$

is to hold good. For consider the conditions of the general form (1) in which the contrary terms (those within the bracket) are the same, but in which the universe-terms outside the bracket are contrary as regards one attribute, say K_p. Then the universes are specified by

$$K_p \ \overset{p-1}{\underset{1}{\cdot \ \varPi}}[K_r] \ . \ \overset{q}{\underset{1}{\varPi}}[U - L_q].$$

$$[U - K_p] \ \overset{p-1}{\underset{1}{\varPi}}[K_r] \ . \ \overset{q}{\underset{1}{\varPi}}[U - L_q],$$

and if the corresponding conditions (1) be added, the term in K_p goes out, leaving a condition of one degree lower. By addition of successive pairs of conditions in this way it is evident that the universe-terms may be entirely eliminated, and only condition (2) left. By the converse process of *specification* of the universe the conditions involving c terms of the $(m - 1)$th order may *always* be obtained from the conditions for the congruence of the cth degree and $(c - 1)$th order, a property on which we have remarked while considering the congruences of low orders investigated in § 4—§ 10 ; the conditions (2) merely require to be specified for all possible universes.

If instead of proceeding to the entire elimination of the universe-terms we stop short at conditions of degree n $(n \nless c)$, the whole series of conditions so obtained may be grouped into sets arranged according to the attributes that have *not* been

* They were actually first obtained by the method there described.

eliminated, one such set corresponding to each possible congruence of the nth degree and $(n-1)$th order. Hence the general theorem—"A congruence of the $(n-1)$th order and mth degree is self-consistent if all the possible congruences of the nth degree, into which it may be resolved, are self-consistent." This is the generalisation of the theorems given in § 8 and § 10.

§ 14. The number of conditions to which any congruence of the mth degree is subject is readily obtained. Consider first the congruence of order $(m-1)$. The number of combinations of c (positive) attributes that can be selected from m is

$$\text{- 1)}$$

From each of these combinations can be generated 2^{c-1} contrary pairs, by negativing one or more of the attributes: e.g., from (ABC) can be formed the *four* contrary

$$\left.\begin{array}{llll}\text{ABC} & \alpha\text{BC} & \text{A}\beta\text{C} & \text{AB}\gamma \\ \alpha\beta\gamma & \text{A}\beta\gamma & \alpha\text{B}\gamma & \alpha\beta\text{C}\end{array}\right\}.$$

From the remaining $(p+q)$ or $(m-c)$ attributes can be formed 2^{m-c} different universes. Hence the number of conditions involving c terms of the $(m-1)$th order is

The *whole* number of such conditions (including those of inferior congruence) will be given by inserting all possible values of c in the above expression (viz., all odd numbers not greater than m), and summing. That is

$$2^{m-1}\left(m + \frac{m(m-1)(m-2)}{1\cdot2\cdot3} + \frac{m(m-1)(m-2)(m-3)(m-4)(m-5)}{1\cdot2\cdot3\cdot4\cdot5} + \ldots\right),$$

or, substituting $1 + \overline{m-1}$ for m and rearranging

$$2^{m-1}\left(1 + (m-1) + \frac{(m-1)(m-2)}{1\cdot2} + \frac{(m-1)(}{}\right.$$

The number of conditions is thus quadrupled for every unit rise in the degree of the congruence, and grows with extreme rapidity. The actual numbers, up to the congruence of ninth degree and eighth order are given in Table I. below. For more than five or six attributes the actual arithmetic discussion of any particular case seems to pass the bounds of practical possibilities

TABLE I.—Number of Conditions of Consistence for a Congruence of the mth degree and $(m-1)$th order; specifying separately the number involving 3, 5, 7, or 9 $(m-1)$th order terms.

Terms involved ($= c$).	Degree of congruence $= m$.						
	3	4	5	6.	7.	8.	9
Inferior congruence, 1	12	32	80	192	448	1024	2304
3	4	32	160	640	2240	7168	21504
5	—	--	16	192	1344	7168	32256
7	—	—	—	–	64	1024	9216
9					—	—	256
Total superior congruence only .	4	32	176	832	3648	15360	63232
Grand total	16	64	256	1024	4096	16384	65536

The whole number of conditions for a congruence of the nth order and mth degree is

$$2^{2n} \cdot \frac{m(m-1)(m-2) \cdots (m-n)}{(n+1)!},$$

viz., the number of conditions for a congruence of order n, degree $(n+1)$, multiplied by the number of $(n+1)$th degree congruences into which the mth-degree congruence can be resolved. The actual figures are—

TABLE II.—Whole Number of Conditions of Consistence for a Congruence of degree m order n.

Order of congruence $= n$	Degree of congruence $= m$						
	3.	4.	5.	6.	7.	8.	9.
2	16	64	160	320	560	896	1344
3		64	320	960	2240	4480	8064
4			256	1536	5376	14336	32256
5				1024	7168	28672	86016
6					4096	32768	147456
7						16384	147456
8							65536

GEOMETRICAL REPRESENTATION OF THE CONDITIONS OF CONSISTENCE.

Congruence of the Third Degree.

§ 15. Since the conditions of consistence for a congruence of the third degree only involve three second-order frequencies each, it is possible to construct geometrical models to represent them, the first-order terms being treated as constants. These models exhibit in such a beautiful manner the nature of the conditions, and the limiting character of the cases dealt with in ordinary logic, that it is worth while to treat a few special cases at length as illustrations. It will be convenient to use for the present the abbreviated notation

$$x = (AB)/(U) \; . \; y = (AC)/(U) \; . \; z = (BC)/(U).$$
$$p_1 = (A)/(U) \; . \; p_2 = (B)/(U) \; . \; p_3 = (C)/(U).$$

Then, treating x, y, z as rectangular co-ordinates, all sets of consistent values of x, y, z must determine points within the space bounded by planes (§ 4 and § 5).

$$\left. \begin{array}{lll} x = 0 \text{ or } x = p_1 + p_2 - 1, & x = p_1 \text{ or } x = p_3 \\ y = 0 \text{ ,, } y = p_1 + p_3 - 1, & y = p_1 \text{ ,, } y = p_3 \\ z = 0 \text{ ,, } z = p_2 + p_3 - 1, & z = p_2 \text{ ,, } z = p_3 \end{array} \right\} \qquad . \quad . \quad \text{(A).}$$

$$\left. \begin{array}{ll} x + y + z = p_1 + p_2 + p_3 - 1 & (\alpha) \\ x + y - z = p_1 & (\beta) \\ x - y + z = p_2 & (\gamma) \\ - x + y + z = p_3 & (\delta) \end{array} \right\} \quad . \quad . \quad . \quad . \quad \text{(B).}$$

It is convenient to regard the equilateral tetrahedron bounded by the four planes $(\alpha) - (\delta)$, representing the superior conditions of consistence, as the fundamental "congruence surface," its edges being truncated more or less by the planes (A). Only six of the planes (A) at most can of course come into account at one time, the remaining six lying outside the surface.

The lines $(\alpha\beta)$, $(\alpha\gamma)$, $(\alpha\delta)$, &c., in which the planes α, β, γ, δ meet, are all parallel to one or other of the co-ordinate planes : thus we have for

$$\left. \begin{array}{lll} (\alpha\beta) & z = \tfrac{1}{2} \left(p_2 + p_3 - 1 \right) \\ (\alpha\gamma) & y = \tfrac{1}{2} \left(p_1 + p_3 - 1 \right) \\ (\alpha\delta) & x = \tfrac{1}{2} \left(p_1 + p_2 - 1 \right) \\ (\beta\gamma) & x = \tfrac{1}{2} \left(p_1 + p_3 \right) \\ (\beta\delta) & y = \tfrac{1}{2} \left(p_1 + p_3 \right) \\ (\gamma\delta) & z = \tfrac{1}{2} \left(p_2 + p_3 \right) \end{array} \right\} \quad . \quad . \quad . \quad . \quad . \quad \text{(C).}$$

The plan and elevation of the *complete* tetrahedron are thus both squares of side $= 0.5$. But, comparing equations (C) with (A), the edge

$(\alpha\beta)$ is truncated unless $p_2 + p_3 = 1$.
$(\alpha\gamma)$,,   ,, $p_1 + p_3 = 1$.
$(\alpha\delta)$,,   ,, $p_1 + p_2 = 1$.
$(\beta\gamma)$,,   ,, $p_1 = p_2$.
$(\beta\delta)$,,   ,, $p_1 = p_3$.
$(\delta\gamma)$,,   ,, $p_2 = p_3$,

§ 16. Case (1).

$$p_1 = p_2 = p_3 = 0.5.$$

In this case the congruence-surface reduces to the fundamental tetrahedron, fig. 1*a*,* the planes (A) only passing through its edges and not truncating them. The form is more clearly shown by fig. 1*b*, which is drawn from a photograph of an actual

Fig. 1*a*.   Fig. 1*b*.

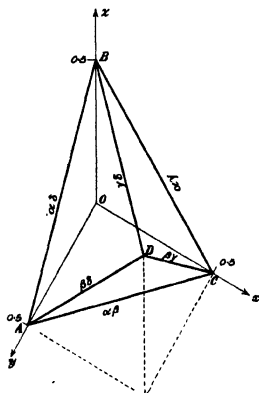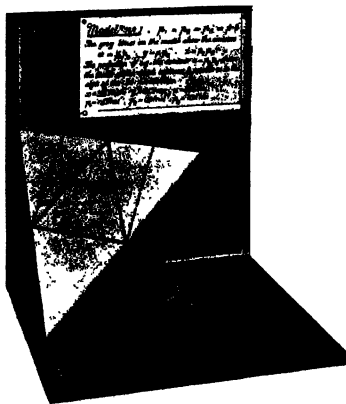* This and the similar following figures are drawn in orthographic projection. The picture-plane is parallel to the axis of z, and its trace on the plane of xy makes an angle of 30° with the x axis. The generators lie in planes perpendicular to the picture-plane and the plane of xy, but make an angle of 45 with the latter plane. The observer must therefore imagine himself to be looking *down* on the model

model. The contours shown in this figure and in the subsequent figs. 2b, 4b, and 4c, do not at present concern us.

The bounding planes are

$$x + y + z = 0\cdot5 \quad (\alpha)$$
$$x + y - z = 0\cdot5 \quad (\beta)$$
$$x - y + z = 0\cdot5 \quad (\gamma)$$
$$- x + y + z = 0\cdot5 \quad (\delta).$$

If the ordinate z corresponding to given values of x and y be drawn, it will in general cut the surface in two points. These determine the upper and lower limits to values of z consistent with the given values of x and y. If however x and y determine a point on the plan of one of the edges of the tetrahedron, z only cuts the surface in one point, and its value can be inferred. Thus we have the following cases of logical inference :—

Given.	Inferred.
$x = 0$	$y = 0\cdot5 - z$
$x = 0\cdot5$	$y = z$
$y = 0$	$z = 0\cdot5 - x$
$y = 0\cdot5$	$z = x$
$z = 0$	$x = 0\cdot5 - y$
$z = 0\cdot5$	$x = y.$

Each of these cases corresponds, it will be noticed, to a single-infinity of special inferences from two data—one to every point on each edge. It is the only instance in which six such infinite series of possible inferences occur, that is, six series of *exact* inferences—inferences of a " universal affirmative " or " universal negative," to use the logical terms.

§ 17. The limits to x, y, or z given for this case by equations (α)-(δ) at the beginning of last section, are precisely those deducible from quite different considerations for the quadrant frequencies (above and below average) in the case of normal correlation. In that case, if r_{12}, r_{13}, r_{23} be the three correlation coefficients, we have for the limits to r_{23}*

$$r_{12}r_{13} \pm \sqrt{1 + r_{12}{}^2 r_{13}{}^2 - r_{12}{}^2 - r_{13}{}^2}.$$

But by the theorem due to Mr. W. F. SHEPPARD†

$$r_{12} = - \cos 2x\pi$$
$$r_{13} = - \cos 2y\pi$$
$$r_{23} = - \cos 2z\pi.$$

* ' Roy. Soc. Proc.,' 1897, vol. 60, p. 486. In the first line of the table on p. 486, for " 0 " on the right *read* ± 1. A similar correction is to be made in the ' Journal of the Roy. Stat. Soc.,' vol. 60, p. 834.

† ' Phil. Trans.,' A, 1898, vol. 162, p. 101.

Therefore the limits to $-\cos 2z\pi$ are

$$\cos 2x\pi \cdot \cos 2y\pi \pm \sqrt{1 + \cos^2 2x\pi \cos^2 2y\pi - \cos^2 2x\pi - \cos^2 2y\pi}$$
$$= \cos 2x\pi \cdot \cos 2y\pi \pm \sin 2x\pi \sin 2y\pi$$
$$= \cos(x \pm y)\,2\pi.$$

Also
$$-\cos 2z\pi = \cos(z \pm \tfrac{1}{2})\,2\pi.$$

Therefore the limiting values to z are given by

$$\pm \tfrac{1}{2} \pm (x \pm y).$$

Here we need not take all three signs positive, for by the inferior conditions z cannot be greater than 0.5; nor all three negative, for x cannot be less than zero. Hence the limits given are

$$\left.\begin{array}{r} 0.5 + x - y \\ 0.5 - x + y \end{array}\right\}$$
$$\left.\begin{array}{r} 0.5 - x - y \\ -0.5 + x + y \end{array}\right\}$$

These are precisely the limits given by the conditions of consistence stated at the commencement of § 16. Thus the limits to one correlation coefficient in terms of the two others are most simply regarded as functions of the limits to the quadrant frequencies. The table below gives the limits to z for different values of x and y; the table is of course symmetrical with regard to x and y.

TABLE showing the limits to z in terms of x and y, $x = (AB)/(U)$, $y = (AC)/(U)$, $z = (BC)/(U)$, for the case of Equality of Contraries.

Value of y.	Value of x.					
	0.	0·1.	0 2.	0·3	0 4.	0·5.
0	0·5	0 4	0·3	0 2	0·1	0
0 1	0·4	0·5 0·3	0 4 0 2	0·3 0·1	0 2 0	0 1
0·2	0·3	0·4 0·2	0·5 0·1	0 4 0	0·3 0·1	0 2
0·3	0·2	0 3 0·1	0·4 0	0 5 0·1	0·4 0·2	0·3
0·4	0·1	0 2 0	0 3 0·1	0·4 0·2	0·5 0·3	0·4
0·5	0	0·1	0·2	0·3	0·4	0·5

§ 18. Case (2).

$$p_1 = p_2 = p_3 = 0\cdot4.$$

The equations to the bounding planes are

$$x = 0 \quad y = 0 \quad z = 0.$$

$x + y + z = 0\cdot2$	(α)
$x + y - z = 0\cdot4$	(β)
$x - y + z = 0\cdot4$	(γ)
$- x + y + z = 0\cdot4$	$(\delta).$

The surface is shown in fig. 2a below, and in fig. 2b from a photograph of a model.
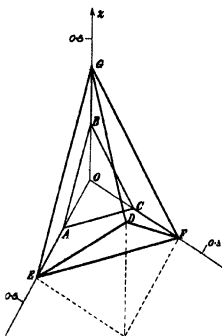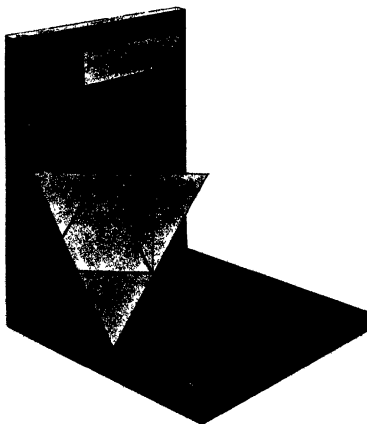
Fig. 2a.   Fig. 2b.

The three edges $\alpha\beta$, $\alpha\gamma$, $\alpha\delta$ of the tetrahedron, that in fig. 1 lay in the co-ordinate planes, are now truncated by them. Thus only three of the six infinite series of exact inferences occurring in the last case are left, viz.,

Given.	Inferred.
$x = 0\cdot4$	$y = z$
$y = 0\cdot4$	$z = x$
$z = 0\cdot4$	$x = y.$

It should be noticed that from $x = 0$ $y = 0$ we can *in the present case*, infer $z > 0.2$, *i.e.*, from "no A's are B" and "no A's are C" infer "some (at least half) of the B's are C." But the reason why the ordinary rule of logic, "from two negative premises no conclusion can be drawn," comes into play is, however, fairly obvious. If

$$p_1 + p_2 + p_3 < 1$$

the plane α disappears behind the origin of co-ordinates, and the rear of the surface is bounded solely by the co-ordinate planes. The value of z may then be anywhere between zero and p_1 or p_3 at the point $x = 0$ $y = 0$, *i.e.*, there is, *à priori*, no inference.

I do not propose to enter into the discussion of numerically indefinite inferences— *i.e.*, the "particular affirmative" and "particular negative" conclusions of the ordinary syllogistic treatment. I would, however, suggest that such numerically indefinite inferences may be regarded as mere degradations, owing to the truncation of the tetrahedron, of the series of exact or definite inferences possible in the last case.

§ 19. Case (3).

$$p_1 = 0.45 \qquad p_2 = p_3 = 0.4.$$

The equations to the bounding planes are

$$x = 0 \qquad y = 0 \qquad z = 0$$
$$x = 0.4 \qquad y = 0.4 \qquad z \; 0.4$$
$$x + y \; + z = 0.25 \qquad (\alpha)$$
$$x + y \; - z = 0.45 \qquad (\beta)$$
$$x - y \; + z = 0.4 \qquad (\gamma)$$
$$- x + y \; + z = 0.4 \qquad (\delta).$$

The form of the surface is shown in fig. 3. Only one edge of the primitive tetrahedron is now untruncated, viz., GK or $\gamma\delta$. Hence only one of the original six series of definite inferences is left, viz.—

Given.	Inferred.
$z = 0.4$	$x = y.$

In addition to this infinite series there are, however, two special cases of inference corresponding respectively to the points H and M of the figure.
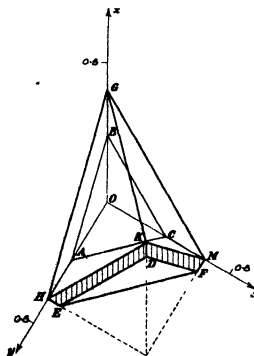
Fig. 3.

	Given.		Inferred.
$x = 0$	. $y = 0\cdot4$		$z = 0$
$x = 0\cdot4$	. $y = 0$		$z = 0.$

These are " universal-negative" conclusions of the ordinary syllogistic type

All C's are A,	All B's are A,
No A's are B,	No A's are C,
∴ No C's are B.	∴ No B's are C.

§ 20. Case (4).

A type of the most general case possible.

$$p_1 = 0\cdot35 : p_2 = 0\cdot4 : p_3 = 0\cdot45.$$

The equations to the bounding planes are

$$x = 0 \qquad y = 0 \qquad z = 0$$
$$x = 0\cdot35 \quad y = 0\cdot35 \quad z = 0\cdot4$$
$$x + y + z = 0\cdot20 \qquad (\alpha)$$
$$x + y - z = 0\cdot35 \qquad (\beta)$$
$$x - y + z = 0\cdot40 \qquad (\gamma)$$
$$- x + y + z = 0\cdot45 \qquad (\delta).$$

The form of the surface is shown in fig. 4a opposite, and in figs. 4b and 4c from photographs of a model. All the edges of the tetrahedron are now truncated by planes representing the conditions of inferior congruence. No infinite series of definite inferences are left, but only four special cases corresponding to the points KEGF of the figure :—

	Given.		Inferred.
(1) $x = 0\cdot35$	. $z = 0\cdot4$		$y = 0\cdot35$
(2) $y = 0\cdot35$	$z = 0$		$x = 0$
(3) $y = 0$	$z = 0\cdot4$		$x = 0$
(4) $x = 0\cdot35$	$z = 0$		$y = 0.$

The corresponding syllogisms are—

(1.)	All A's are B,	(3.)	All B's are C,
	all B's are C,		no C's are A,
	∴ all A's are C.		∴ no B's are A.
(2.)	All A's are C,	(4.)	All A's are B,
	no C's are B,		no B's are C,
	∴ no A's are B.		∴ no A's are C

§ 21. In extreme cases the congruence-surface presents the appearance of a right six-face with its corners truncated rather than a tetrahedron with its edges cut down.

Fig. 4a.

Fig. 4b.

Fig. 4c.

Q 2
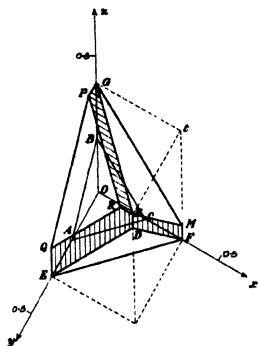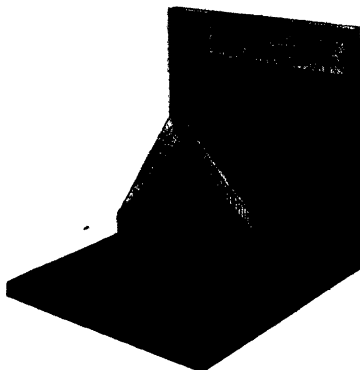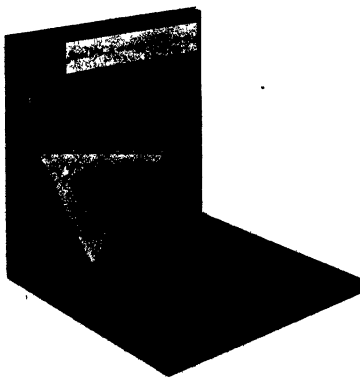
Illustrations of such surfaces will be found in figs. 16-23 (§ 36, p. 130-131). The inferior instead of the superior conditions then predominate in importance.

Two of the planes of superior congruence—but not more than two—may simultaneously pass outside the planes of inferior congruence and so disappear. Thus to determine the planes of inferior congruence suppose

$$p_1 < p_2 < p_3 \quad \text{and} \quad p_1 < 0.5,\ p_2 < 0.5,\ p_3 < 0.5.$$

The plane α vanishes if

$$p_1 + p_2 + p_3 < 1.$$

The plane β cuts the plane $z = 0$ in the line

$$x + y = p_1 ;$$

the limiting values of x and y being $x = p_1$, $y = p_1$ it cannot disappear.

The plane γ cuts the plane $z = p_2$ (the limiting plane) in the line

$$x - y = 0,$$

and therefore it also cannot disappear.

Finally, the plane δ cuts the plane $z = p_2$ in the line

$$y - x = p_3 - p_2.$$

But the greatest possible value of $y - x$ is p_1. Therefore the plane δ disappears if

$$p_3 - p_2 > p_1.$$

An illustration of a surface of this character will be found in fig 19, § 36, p. 130.

In any case there are four "syllogism points" (like KEGF of fig. 4) to the surface.

§ 22. In all the preceding examples of figs. (1)-(4) the values assigned to p_1, p_2, and p_3 have been less than 0.5, so that the edges $\alpha\beta$, $\alpha\gamma$, $\alpha\delta$ of the primitive tetrahedron were truncated, if truncated at all, by the co-ordinate planes. If

$$p_1 + p_3 > 1$$
$$p_2 + p_3 > 1$$
$$p_3 + p_1 > 1$$

this ceases to be the case (*vide* equations A § 15), the truncating planes then move inwards, and the congruence-surface stands clear of the co-ordinate planes.

A little consideration will show, however, that no new features are introduced into the surface itself. Thus suppose that p_1, p_2, p_3 are all less than 0.5, but that we then substitute $(1-p_3)$ for p_3 so as to make one ratio greater than 0.5. How is the original surface altered ? Substituting $(1-p_3)$ for p_3 amounts to substituting γ for C. Then if x_1, y_1, z_1 be the original co-ordinates, x_2, y_2, z_2 the co-ordinates after the substitution, we must have

$$x_3 = x_1$$
$$y_4 = p_1 - y_1$$
$$z_3 = p_2 - z_1,$$

since

$$(A\gamma) = (A) - (AC). \qquad\qquad (B\gamma) = (B) - (BC).$$

The first surface is therefore changed into the second by a simple transformation of co-ordinates, *i.e.*, if we are dealing with an actual model of the surface, by turning it over and shifting it.

Fig. 5 is drawn to illustrate the nature of an actual transformation. It is drawn for the values

$$p_1 = 0\cdot35 \quad p_2 = 0\cdot4 \quad p_3 = 0\cdot55,$$

substituting $(1-0\cdot45)$ for the $0\cdot45$ assigned to p_3 in fig. 4*a*, p. 115. The two figures are similarly lettered. The model of fig. 4 has been turned over, round an axis parallel to the axis of x, through a half revolution.
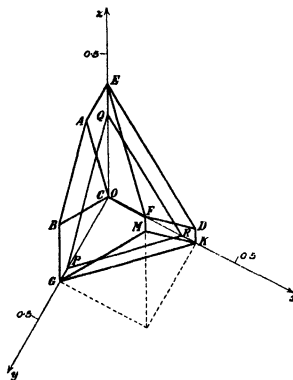
Fig. 5.

§ 23. If $(1-p_3), (1-p_2), (1-p_1)$ be *successively* substituted for p_3, p_2, p_1 the transformations are as follows :—

(1.) Substituting $(1-p_3)$ for p_3

$$x_2 = x_1$$
$$y_3 = p_1 - y_1$$
$$z_3 = p_2 - z_1$$

(2.) Substituting $(1-p_2)$ for p_2

$$x_3 = p_1 - x_2 \quad = p_1 - x_1$$
$$y_3 = y_2 \quad\quad\;\, = p_1 - y_1$$
$$z_3 = 1 - p_3 - z_2 = 1 - p_2 - p_3 + z_1.$$

(3.) Substituting $(1-p_1)$ for p_1

$$x_4 = 1 - p_2 - x_3 = 1 - p_1 - p_2 + x_1$$
$$y_4 = 1 - p_3 - y_3 = 1 - p_1 - p_3 + y_1$$
$$z_4 = z_3 \quad\quad\;\; = 1 - p_2 - p_3 + z_1.$$

The second and third cases are obtained, like the first, by simply expanding. Thus $z_3 = (\beta\gamma)/(u)$, and

$$(\beta\gamma) = (\gamma) - (B\gamma) = (U) - (C) - (B\gamma)$$
$$= (U) - (B) - (C) + (BC)$$

or, dividing by (U)

$$z_3 = 1 - p_3 - z_2 = 1 - p_2 - p_3 + z_1$$

as above.

§ 24. The correctness of the transformations given may, of course, be verified directly. Thus suppose

$$p_1 < p_2 < p_3 < 0 \cdot 5$$

then the equations to the bounding planes of the congruence-surface are

$$\left.\begin{array}{lll} x = 0 & y = 0 & z = 0 \\ x = p_1 & y = p_1 & z = p_2 \\ x + y + z = p_1 + p_2 + p_3 - 1 \\ x + y - z = p_1 \\ x - y + z = p_2 \\ -x + y + z = p_3 \end{array}\right\} \qquad . \qquad . \qquad \text{I.}$$

If $(1-p_2)$ be substituted for p_2, and $(1-p_3)$ for p_3, then

$$p_1 < 1 - p_3 < 1 - p_2,$$

and the equations must be

$$\left.\begin{array}{lll} x = 0 & y = 0 & z = 1 - p_2 - p_3 \\ x = p_1 & y = p_1 & z = 1 - p_3 \\ x + y + z = p_1 - p_2 - p_3 - 1 \\ x + y - z = p_1 \\ x - y + z = 1 - p_3 \\ -x + y + z = 1 - p_3 \end{array}\right\} \qquad . \qquad . \qquad . \qquad \text{II.}$$

The set of equations II. may be obtained from the set I. by the transformations (2) above, thus affording the verification

The fact that the model for *any* values of p_1, p_2, and p_3 can always be transformed into a model for the case $p_1 < 0.5$, $p_2 < 0.5$, $p_3 < 0.5$ justifies our terming case (4), § 20, "a type of the most general case." The geometrical transformations here suggested would seem to correspond to "reductions" of the syllogisms. Thus in fig. 5 the point F stands where the point K stood in fig. 4a. But the point F of fig. 4a corresponds to a syllogism in *Celarent, Cesare, Camenes* or *Camestres*; F in fig. 5 to a syllogism in *Barbara*. The transformation of co-ordinates corresponds to a reduction of either of the first four forms to the last.

§ 25. In case any of those who read this memoir should care to construct models of the congruence-surfaces illustrated in figs. 1–4, I give dimensioned sketches of their developments below, figs. 6–9. These developments are on half the scale of the projections shown in the preceding figures. An angle with one arc across it is an angle of 60°, with two arcs 45° or 135°; an angle blocked in is a right angle.
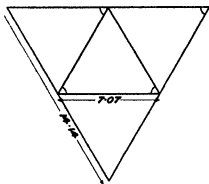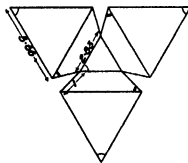
Fig 6.   Fig. 7.
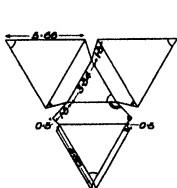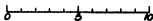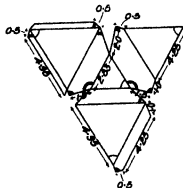
Fig. 8.   Fig. 9.

Congruence of the Fourth Degree.

§ 26. The conditions of consistence for the congruence of the fourth degree in general require space of four dimensions for their direct representation (*cf.* V., § 7).

Since, however, each of the bounding hyper-planes is parallel to one or other of the axes, four surfaces in three dimensions may be substituted for the hyper-surface in four dimensions. If we denote

$$(ABC)/(u), \qquad (ABD)/(u), \qquad (ACD)/(u), \qquad (BCD)/(u),$$

by x, y, z, w, then the surfaces representing the consistence of [1] xzy, [2] xyw, [3] xzw, [4] yzw, respectively, for given values of the frequencies of lower orders, will in general differ from each other. If we desire to find the limits to values of w for given values of x, y, z, we must first see that the values of x, y, z are consistent from surface [1], then find the three pairs of major and minor limits to w given by surfaces [2], [3], and [4]. The lowest of the major limits and highest of the minor limits so given are the true limits to w.

We may take as examples of fourth-degree congruence-surfaces three of the very simplest cases, in which (1) equality of contraries subsists for the first and second-order frequencies, and (2) actual equality subsists between all the second order frequencies, i.e., $(AB) = (AC) = (AD) = (BC) = \&c.$ These are highly specialised examples, but are of some interest for their bearing on the theory of normal or quasi-normal correlation.

It must be remembered that in a *normal* distribution of frequency, where the divisions between A and a, B and β, &c., are taken at the means, the frequencies of all orders are definitely determined by the second-order frequencies, and complete equality of contraries subsists for frequencies of all orders. This is not the case where the only datum is that equality of contraries subsists for frequencies of the first and, therefore, also of the second order. The equality of contraries need not, as shown in my previous memoir,[*] spread to the frequencies of the third order, but if it be assumed to do so the latter become determinate as in the "normal" case.

§ 27. For the simple type of cases considered, the equations to the bounding-planes of the congruence-surfaces reduce to the forms given below. The four component surfaces for xyz, xyw, &c., are, of course, all identical, so it is only necessary to consider one in each case. The first conditions given are those of inferior congruence (*cf.* § 5), and for brevity we have written

$$(AB)/(u) = (AC)/(u) = \&c. = q$$

$$
\begin{array}{lll}
x = 0 & y = 0 & z = 0 \\
\quad = 2q - 0\cdot 5 & \quad = 2q - 0\cdot 5 & \quad = 2q - 0\cdot 5 \\[4pt]
x = q & y = q & \\
\quad = 3q - 0\cdot 5 & \quad = 3q - 0\cdot 5 &
\end{array}
$$

[*] "On the Association of Attributes in Statistics," 'Phil. Trans.,' A, vol. 194, pp. 263 *et seq.*

$$x + y + z = 3q - 0.5 \quad (\alpha)$$
$$x + y - z = q \quad (\beta)$$
$$x + y + z = q \quad (\gamma)$$
$$- x + y + z = q \quad (\delta)$$
$$x + y + z = 6q - 1.0 \quad (\epsilon)$$
$$x + y - z = 2q - 0.5 \quad (\zeta)$$
$$x + y + z = 2q - 0.5 \quad (\eta)$$
$$- x + y + z = 2q - 0.5 \quad (\theta).$$

§ 28. Case (1)—

$$(AB)(u) = (BC)(u) = \&c. = q = 0.25$$

All the attributes ABCD are independent *pair and pair*.
The conditions of inferior congruence are

$$x = 0 \qquad y = 0 \qquad z = 0$$
$$x = 0.25 \quad y = 0.25 \quad z = 0.25,$$
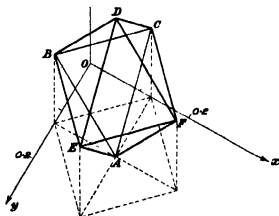
Fig. 10

but do not come into account, the planes they represent only passing through the angles ABCDEF (fig. 10)* of the equilateral octahedron bounded by the planes—

* Figs. 10, 11, 12 are drawn to twice the scale of figures 1-5

$$x + y + z = 0.25 \quad (a)$$
$$x + y - z = 0.25 \quad (\beta)$$
$$x - y + z = 0.25 \quad (\gamma)$$
$$-x + y + z = 0.25 \quad (\delta)$$
$$x + y + z = 0.5 \quad (\epsilon)$$
$$x + y - z = 0 \quad (\zeta)$$
$$x + y + z = 0 \quad (\eta)$$
$$-x + y + z = 0 \quad (\theta).$$

There are twelve somewhat interesting infinite series of definite inferences corresponding to the twelve sides of the octahedron. They may be grouped in three divisions following the contours EBCF, EDCA, BDFA, viz.:—

$$x - y = 0.125$$
$$\text{or } y - x = 0.125$$
$$\text{or } x + y = 0.125$$
$$\text{or } x + y = 0.375$$

for the contour EBCF, and the two similar systems that may be written down by cyclical substitution. These are all, it will be seen, inferences of *independence* ($z = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$), and therefore somewhat strikingly different to the usual definite inferences which are all inferences of complete association (all A's are B — no A's are B). From the same values of x and y we could, of course, infer $w = \cdot125$, so the theorem may be expressed in words thus—

"In any case where cross-equality* and independence both subsist for the second-order frequencies, independence must subsist for two at least of the four positive third-order frequencies if either (1) the sum of the remaining two is equal to $\frac{1}{8}$th or $\frac{3}{8}$ths of the total frequency; or (2) the difference between the remaining two is equal to $\frac{1}{8}$th of the total frequency."

I should, perhaps, note that in the present case the fact that the criterion of independence holds for the positive fourth-order class necessitates its holding for all the remaining classes of the aggregate. This is not so in general. It may further be remarked that the independence of the attributes, pair and pair in the second-order classes, does not connote independence for the groups of the third order. I hope to return to the logic of independence on a future occasion.

§ 29. Case (2)—

$$(AB)/(U) = (BC)/(U) = \&c. = q = 0.20.$$

The pairs of attributes are now all *negatively* associated.

---

* I propose to use this term in lieu of the more lengthy " equality of contraries

The planes representing conditions of inferior congruence are

$$x = 0 \qquad y = 0 \qquad z = 0$$
$$x = 0\cdot1 \qquad y = 0\cdot1 \qquad z = 0\cdot1.$$

The conditions of superior congruence are represented by planes

$$x + y + z = 0\cdot1 \qquad\qquad (\alpha)$$
$$x + y - z = x - y + z = -x + y + z = \quad 0\cdot2 \quad (\beta, \gamma, \delta)$$
$$x + y + z = 0\cdot2 \qquad\qquad (\epsilon)$$
$$x + y - z = x - y + z = -x + y + z = -0\cdot1 \quad (\zeta, \eta, \theta),$$

but on drawing the figure (fig. 11) it will be seen that the planes $(\beta, \gamma, \delta, \zeta, \eta, \theta)$ do not come into account, the surface being an octahedron bounded by (α) (ϵ) and the six planes representing conditions of inferior congruence.
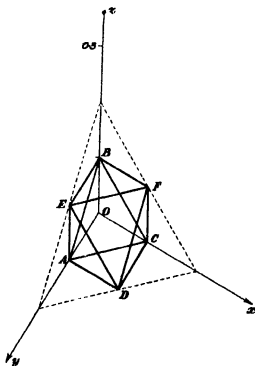
There are no infinite series of definite inferences but only six special cases, corresponding to the points ABCDEF. They are all of the form

Given.   Inferred.

$$\left.\begin{array}{ll} x = 0 & y = 0 \\ \text{or} \quad x = 0\cdot1 & y = 0\cdot1 \end{array}\right\} \ z = 0\cdot1.$$

It must be noted that in the type of cases at present under discussion q cannot be less than $\frac{1}{6}$th, or 0·166 . .   , for, by the *third*-degree conditions of consistence,

$$(AB) + (AC) + (BC) \not< (A) + (B) + (C) - (U)$$

that is it

$$(AB) = (AC) = (BC) = q \cdot (U)$$
$$(A) = (B) = (C) = \tfrac{1}{2} \cdot (U)$$
$$3q \not< \tfrac{1}{2}, \qquad q \not< \tfrac{1}{6}.$$

For this value of q the planes (α) and (ϵ) fall together into the origin; the whole surface, so to speak, closes up.

§ 30. Case (3)—

$$(AB)/(U) = (BC)/(U) = \&c. = 0·3$$

In this case the pairs of attributes are all *positively* associated. The first case is therefore intermediate between the second case and the present one.

The planes representing the conditions of inferior congruence are

$$x = 0·1 \quad y = 0·1 \quad z = 0·1$$
$$x = 0·3 \quad y = 0·3 \quad z = 0·3,$$

but, as in Case (1), these do not actually come into account, only passing through the *edges* AC, BP, GD, and FM of the octahedron bounded by planes

$$x + y + z = 0·4 \qquad\qquad\qquad (\alpha)$$
$$x + y - z = x - y + z = -x + y + z = 0·3 \quad (\beta, \gamma, \delta)$$
$$x + y = z = 0·8 \qquad\qquad\qquad (\epsilon)$$
$$x + y - z = x - y + z = -x + y + z = 0·1 \quad (\zeta, \eta, \theta).$$

The surface is represented in fig. 12 below. To facilitate identification, but avoid overcrowding the small-scale figure with lettering, the planes may be stated to be

α	.	.	.	ABC	ϵ	. . . EDF.
β	.	.	.	GDFMH	ρ	. . . PQRCB.
γ	.	.		QRMFE	η	. . . PBAHG.
δ	.	.	.	PQEDG	θ	. . . CRMHA.

There are eighteen infinite series of definite inferences corresponding to the eighteen edges, all reducible to one or other of the types

Given,	Inferred.
$\pm (x - y) = 0·1$	$z = 0·2$
$x = 0·3$	$z = y$
$x + y = 0·25$	$z = 0·15$
$x + y = 0·55$	$z = 0\,25.$
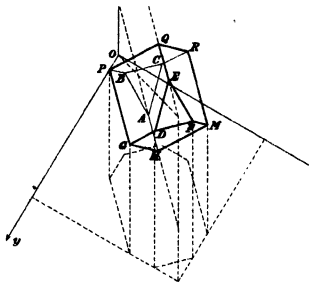
Fig. 12.

§ 31. For the benefit of those who would like to construct actual models of the surfaces shown in figs. 10, 11, 12, I again give below sketches of their developments, figs. 13—15. These sketches are drawn to half the scale of the preceding projections; the notation for the angles is the same as in figs. 6-9.
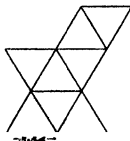
Fig. 13.
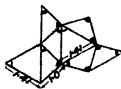
Fig. 14.
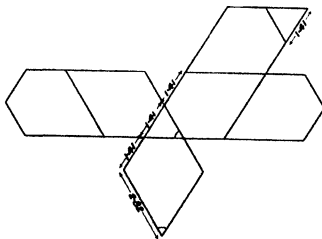
Fig. 15.

THE LIMITS TO ASSOCIATIONS GIVEN BY CONDITIONS OF CONSISTENCE.

§ 32. The general conditions of consistence give limits to the frequencies of any one aggregate in terms of the frequencies of two or more given aggregates of the same congruence.   Hence they give limits also to the possible associations between attributes in the unknown aggregate.

Thus, to take an imaginary example, suppose we have

(AB)	11	(AC)	8
(Aβ)	22	(Aγ)	25
(αB)	39	(αC)	52
(αβ)	28	(αγ)	15
	———		———
	100		100.

Required the limiting values of the BC aggregate.   From the conditions of consistence, § 5, we have

$$(BC) \not< 33 + 50 + 60 - 100 - 11 - 8$$
$$\not< 24$$
$$\not< 11 + 8 - 33 \not< - 14$$
$$\not> 60 + 11 - 8 \not> 63$$
$$\not> 50 - 11 + 8 \not> 47.$$

Therefore the limiting values to (BC) are 24 and 47.   But we know (B) = 50, (C) = 60, (U) = 100, therefore the limiting values to the frequencies of the aggregate are

(BC)	24	47
(βγ)	26	3
(βC)	36	13
(βγ)	14	37
	———	———
	100	100.

Hence we may calculate, if desired, the limiting values to the association |BC|. For the association coefficient suggested in my previous memoir the values are -0.47 and -0.96. The values of |AB| and |AC| are -0.47 and -0.83 respectively.

For the great majority of cases occurring in practice the limits thus inferred from known associations are, as in the imaginary example, pretty wide.   Very high values must be assigned to the two given associations before it is possible to infer even the sign of the third.   Arguments of the vague type, "so many A's are B, and so many A's are also C, that clearly we must expect to find B and C frequently occurring

together," are not uncommon, but the speaker seldom has any conception of the limits to (BC) actually implied by given values of (AB) and (AC).

§ 83. To enforce the danger of rashly inferring, we take some figures from the illustrations of my previous memoir borrowed from the material of the Childhood Society.

The following are the proportions,* per 10,000 cases observed, of those with given defects and given combinations of defects, for boys and girls of all ages.

A = development defects, B = nerve signs, C = low nutrition, D = mental dulness.

	Boys.	Girls.
(U)	10,000	10,000
(A)	878	682
(B)	1,085	850
(C)	285	325
(D)	789	689
(AB)	296	248
(AC)	142	180
(AD)	296	307
(BC)	134	141
(BD)	455	363
(CD)	123	132
(ABC)	57	66
(ABD)	153	128
(ACD)	179	80
(BCD)	64	63
(ABCD)	30	33

These figures give the following values for the associations†—

	Boys	Girls
|AB| . .	0·750	0·784
|AC| . .	0·848	0·916
|AD| . .	0·846	0·900
|BC| . . .	0·783	0·814
|BD| . . .	0·897	0·905
|CD| . .	0·823	0·835

A hasty arguer might think he was safe in inferring from the values, e.g., of |BC| and |AC| that at least some A's must be B, if not that A and B were

* "On the Association of Attributes in Statistics," &c., 'Phil. Trans.,' A, vol. 194. Table on p. 318, but the figures reduced to proportion per 10,000 cases observed.

† Pages 306–307 of the same memoir.

positively associated. Either inference would be quite incorrect. Let us proceed to discuss the conditions of consistence.

§ 34. We have shown (§ 8) that a congruence of the fourth degree and second order is self-consistent when the conditions of consistence hold for each of the four congruences of the third degree into which it may be resolved. These conditions are as follows in the present instance ; writing for brevity

$$x_1 = (AB)/(u) \qquad x_2 = (AC)/(u) \qquad x_3 = (AD)/(u)$$
$$x_4 = (BC)/(u) \qquad x_5 = (BD)/(u) \qquad x_6 = (CD)/(u).$$

Congruence.   I. Boys.

ABC $x_1 = 0$ $x_2 = 0$ $x_4 = 0$
        $x_1 = 0.0878$ $x_2 = 0.0285$ $x_4 = 0.0285$
        $(x_1 + x_2 + x_4 = -0.7752)$
        $x_1 + x_2 - x_4 = \quad 0.0878$
        $x_1 - x_2 + x_4 = \quad 0.1085$
        $-x_1 + x_2 + x_4 = \quad 0.0285$

ABD $x_1 = 0$ $x_3 = 0$ $x_5 = 0$
        $x_1 = 0.0878$ $x_3 = 0.0789$ $x_5 = 0.0789$
        $(x_1 + x_3 + x_5 = -0.7248)$
        $x_1 + x_3 - x_5 = \quad 0.0878$
        $x_1 - x_3 + x_5 = \quad 0.1085$
        $-x_1 + x_3 + x_5 = \quad 0.0789$

ACD $x_2 = 0$ $x_3 = 0$ $x_6 = 0$
        $x_2 = 0.0285$ $x_3 = 0.0789$ $x_6 = 0.0285$
        $(x_2 + x_3 + x_6 = -0.8051)$
        $x_2 + x_3 - x_6 = \quad 0.0875$
        $x_2 - x_3 + x_6 = \quad 0.0285$
        $-x_2 + x_3 + x_6 = \quad 0.0789$

BCD $x_4 = 0$ $x_5 = 0$ $x_6 = 0$
        $x_4 = 0.0285$ $x_5 = 0.0789$ $x_6 = 0.0285$
        $(x_4 + x_5 + x_6 = -0.7841)$
        $(x_4 + x_5 - x_6 = \quad 0.1085)$
        $x_4 - x_5 + x_6 = \quad 0.0285$
        $-x_4 + x_5 + x_6 = \quad 0.0789.$

Congruence.   II. Girls.

ABC $x_1 = 0$ $x_2 = 0$ $x_4 = 0$
        $x_1 = 0.0682$ $x_2 = 0.0325$ $x_4 = 0.0325$
        $(x_1 + x_2 + x_4 = -0.8142)$
        $x_1 + x_2 - x_4 = \quad 0.0682$
        $x_1 - x_2 + x_4 = \quad 0.0850$
        $-x_1 + x_2 + x_4 = \quad 0.0325$

Congruence.   II. Girls.

ABD $x_1 = 0$ $x_3 = 0$ $x_5 = 0$

$x_1 = 0.0682$ $x_3 = 0.0682$ $x_5 = 0.0689$

$(x_1 + x_3 + x_5 = -0.7779)$

$x_1 + x_3 + x_5 = 0.0682$

$x_1 - x_3 + x_5 = 0.0850$

$-x_1 + x_3 + x_5 = 0.0689$

ACD $x_2 = 0$ $x_3 = 0$ $x_6 = 0$

$x_2 = 0.0325$ $x_3 = 0.0682$ $x_6 = 0.0325$

$(x_2 + x_3 + x_6 = -0.8304)$

$x_2 + x_3 - x_6 = 0.0682$

$x_2 - x_3 + x_6 = 0.0325$

$-x_2 + x_3 + x_6 = 0.0689$

BCD $x_4 = 0$ $x_5 = 0$ $x_6 = 0$

$x_4 = 0.0325$ $x_5 = 0.0689$ $x_6 = 0.0325$

$(x_4 + x_5 + x_6 = -0.8136)$

$x_4 + x_5 - x_6 = 0.0850$

$x_4 - x_5 + x_6 = 0.0325$

$-x_4 + x_5 + x_6 = 0.0689.$

§ 35. If the limits to each class frequency given by these relations be worked out it will be found that the lower limit to every class, in terms of the others, is zero without exception. That is to say, any pair whatever of the given defects might exhibit complete "disassociation" (association coefficient $= -1$), without this being in any way inconsistent with the high associations exhibited by other pairs. In two cases for the boys and three for the girls *upper* limits could, however, be inferred, as given below.

Group.	Limit given by congruence.	Limits.		Corresponding associations.	
		Boys.	Girls.	Boys.	Girls.
(AB) . .	ABC	870	643	0·9996	0·9972
(AD) . .	ABD	630	574	0·9861	0·9953
. .	ACD	—	634	—	0·9991
(BD) . .	ABD	—	630	—	0·9955

Thus our imaginary "hasty arguer," if he attempted to infer from the given values of (BC) and (AC) that "some A's must be B" or, worse still, that "A and B must be

positively associated," would be inferring almost the direct contrary of the truth. The real inference is "some A's are *not* B," or "A and B are not completely associated." Similarly from the given values of (AB) and (BD), or of (AC) and (CD) in the case of the girls, he could only infer "some D's are not A"; and from (AB) and (BD) in the case of the girls again, "some D's are not B."

§ 36. In order to illustrate the case completely I give sketches of the congruence-surfaces in figures 16—19 for the boys and 20—23 for the girls,* and have marked in each figure the co-ordinates corresponding to the *actual* values of (AB)/(u), &c.
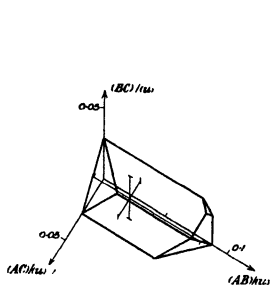
Fig. 16.
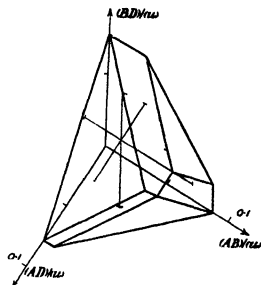
Fig. 17.
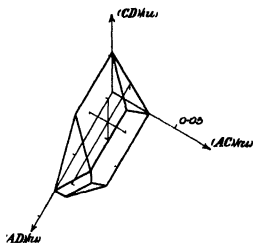
Fig. 18.
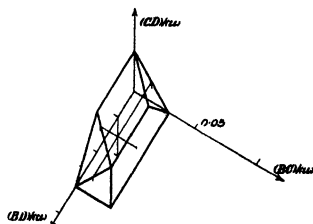
Fig. 19.

---

* Note that the figures are on a much larger scale than those previously given. In figs. 21 and 22 the width of the narrow strip cut off on the plane (AD)/(u) = ·0682 has been somewhat exaggerated.

Only in the case of figs. 17 and 21 does the general appearance of the surface resemble that shown in fig. 4; in all the other cases the greater part of the surface is given by the conditions of inferior congruence. Mere inspection of the figures shows
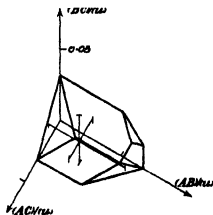
Fig. 20.
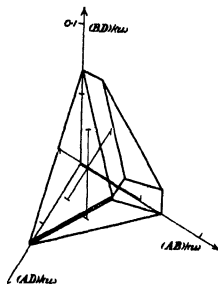
Fig. 21.
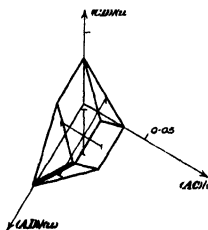
Fig. 22.
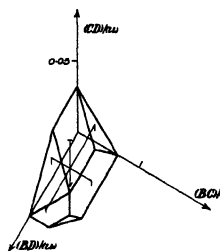
Fig 23

that while, e.g., a very slight alteration of the actual values of (BC) and (AC) would lead to a lower limit for (AB) (cf. figs. 16 and 20) the associations between A and B and A and C would have to be very largely increased indeed before we could infer (BC) > 0. A study of such figures as these would, I think, lead to a good deal more caution in statistical inference.

§ 87. It may be remarked that while the conditions of inferior congruence for second-order groups always give the limits ± 1 to the corresponding associations,

the inferior conditions for the groups of third or higher orders only assign these extreme values to some, but not all, of the partial coefficients. Thus, for example, take the figures of § 32. Applying the inferior conditions of consistence (§ 5, II.) we find as limits to (ACD) for the boys 0 and 123. The remaining frequencies of the aggregate must then be—

$$
\begin{array}{lccc}
\text{(ACD)} & . & . & . & . & 0 & 123 \\
\text{(ACδ)} & . & . & . & . & 142 & 19 \\
\text{(AγD)} & . & . & . & . & 296 & 173 \\
\text{(aCD)} & . & . & . & . & 123 & 0 \\
\text{($a\gamma$D)} & . & . & . & 370 & 493 \\
\text{(aCδ)} & . & . & . & 20 & 143 \\
\text{(A$\gamma\delta$)} & . & . & . & 440 & 563 \\
\text{($a\gamma\delta$)} & . & . & . & 8609 & 8486.
\end{array}
$$

The aggregate corresponding to the minimum value of (ACD) will evidently give minimum values to the partial associations in *positive* universes like $|\text{AC}|\text{D}|$, but maximum values to those in negative universes $|\text{AC}|\delta|$, $\text{AD}|\gamma|$, &c. Then inspection will show that only for $|\text{AC}|\text{D}|$ and $|\text{AD}|\text{C}|$ are the limits ± 1; for the remaining associations the limits are—

$$
\begin{array}{lcc}
|\text{CD}|\text{A}| & -1 & +\cdot 91 \\
|\text{AC}|\delta| & +\cdot 33 & +\cdot 99 \\
|\text{AD}|\gamma| & +\cdot 68 & +\cdot 88 \\
|\text{CD}|a| & -1 & +\cdot 99.
\end{array}
$$

In the discussion of the Childhood Society's material in my previous memoir I remarked on the fact that all the partial coefficients of association between defects in positive universes were small, while those in negative universes were large and positive. I at first thought that this might be a *logical* consequence of the given values of the second-order groups, but this is not the case. The values of the associations like $|\text{AC}|\text{D}|$, $|\text{AD}|\text{C}|$, &c., are almost indeterminate; certain of the coefficients with negative universes ($|\text{AC}|\delta|$, $\text{AD}|\gamma|$) are necessarily positive, but others ($|\text{CD}|a|$) may fall to the extreme limit -1.

§ 38. It may be useful to remark that if two associations $|\text{AB}|$ and $|\text{AC}|$, in a third-order congruence, are both equal to zero, the limits to the third association $|\text{BC}|$ are necessarily ± 1, *whatever the values of* p_1, p_2, p_3. If we write $p_1 p_2$ for (AB)/(U) or x, $p_1 p_3$ for (AC)/(U) or y, the limits to (BC)/(U) or z are

$$
\begin{array}{lll}
z \not< (1-p_1)(p_2+p_3-1) & (1) \\
 \not< p_1(p_2+p_3-1) & (2) \\
 \not> p_2+p_1(p_3-p_2) & (3) \\
 \not> p_3+p_1(p_2-p_3) & (4).
\end{array}
$$

Now if $p_2 + p_3 > 1$ the lower limit to z is, by the inferior conditions of consistence, $p_2 + p_3 - 1$. But (1) and (2) both give lower limits still, and therefore do not come into account. Again, if $p_2 + p_3 < 1$, the lower limit to z, by the inferior conditions, is zero. But (1) and (2) give negative limits, and therefore again do not come into account.

As regards the major limits, z must not, by the inferior conditions, be greater than the least of p_2 and p_3. But if p_2 be the less (3) and (4) both give limits greater than p_2; if p_3 be the less (3) and (4) both give limits greater than p_3. In neither case then do they come into account. We have therefore proved the theorem.

IV. *The Action of Magnetised Electrodes upon Electrical Discharge Phenomena in Rarefied Gases.*

By C. E. S. Phillips.

In two previous communications which I have had the honour to lay before the Royal Society, reference is made to the action of a magnetic field, either constant or changing, upon the distribution of ions within a highly evacuated space.* The chief result dealt with in the first case was the formation of a luminous ring which appeared to be in rapid rotation about the lines of magnetic induction. The other note described a case in which a positively electrified body placed in a rarefied gas became diselectrified when a magnetic field was created in its neighbourhood. I now beg to submit an account of some further experiments which have been made in this matter, with a view to obtaining evidence as to the cause of both these phenomena.

The Luminous Ring.

A detailed account of the apparatus most suitable for the production of the luminous ring in rarefied gases has already been given in the first of the papers just referred to ; nor has it so far been found possible to materially improve upon the method there described.

The bulb (fig. 1) having been exhausted to a pressure of about ·005 millim. of mercury (care being taken to drive off the greater part of the gases held by the electrodes), has a strong discharge passed through it for a few seconds from the secondary circuit of an induction coil.

The discharge is then stopped and the magnetisation of the soft iron electrodes, E_1E_2, by means of the external electro-magnets, M_1M_2, gives rise to a luminous ring which suddenly appears within the bulb, circumscribing the pointed ends of the magnetic electrodes, and revolving about the magnetic axis at a considerable velocity.

* 'Roy. Soc. Proc.,' vol. 64, p. 172 ; 'Roy. Soc. Proc.,' vol. 65, p. 320.

The conditions most favourable to the formation of a luminous ring have been found to be—

(*a.*) A sufficiently powerful electrical stimulation of the residual gas within the bulb to ensure ionisation.

(*b.*) A magnetic field of at least 2000 lines per square centimetre between the oppositely magnetised pointed pole pieces (which may be electrically insulated).

(*c.*) Exhaustion should be continued until the discharge while passing in the bulb is oscillatory.
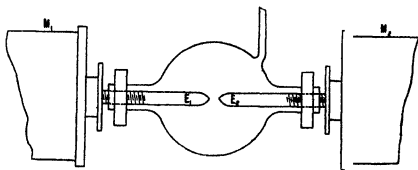
Fig. 1.

The brightness of the light emitted by the ring may be momentarily increased by—

(*a.*) Connecting the outside of the bulb (preferably at its equator) to either of the magnetic electrodes, or by merely touching it with a finger.

(*b.*) Electrifying the exterior surface of the bulb positively.

(*c.*) Electrifying the magnetic electrodes negatively.

(*d.*) Having oxygen gas in the bulb instead of air.

(*e.*) Suddenly destroying the magnetic field while the ring is faintly visible.

A ring which had once formed could be instantaneously extinguished by—

(*a.*) Electrifying the exterior of the bulb negatively.

(*b.*) Electrifying the electrodes positively.

(*c.*) Magnetising the electrodes so that like poles faced each other.

(*d.*) Destroying the magnetic field. In no instance did the ring remain visible after the current was cut off from the electro-magnets.

Satisfactory results have also been obtained when only one magnetic electrode was used, the arrangement being indicated in fig. 2.

In such cases it was generally found convenient to facilitate the oscillatory nature of the discharge within the bulb by gumming a piece of tinfoil. T, on to the outside,

and connecting that with the positive knob of the induction coil or influence machine.

The plane in which the luminous ring appeared was controlled by the position of the tinfoil during stimulation. Fig. 2 shows the appearance of the ring for two respective positions of the tinfoil patch. On no occasion did a luminous ring form when T was placed behind the point of the electrode. Sometimes, when the electrode was magnetised, a glowing gas encircled it at an inch or so from the pointed end, and this faintly luminous cloud moved rapidly forward, gaining in brightness until it circumscribed the point itself. Then it slowly faded away. It must be understood, however, that this effect was seldom seen, the ring generally appearing to suddenly encircle the extremity of the electrode, where it remained visible for some seconds unless disturbed by unequal attractions exerted upon it through alterations in the electrostatic charges residing, either upon the magnetic electrode (which was insulated) or upon the walls of the bulb. This effect was more noticeable, however, with the apparatus shown in fig. 1, for, in that case, by touching different parts of the bulb, or by slightly electrifying the electrodes by means of an influence machine, the
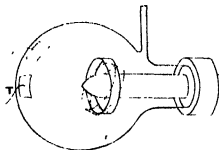
Fig. 2.

luminous ring could be moved backwards or forwards—threaded first on to one electrode and then the other—in a manner consistent with the view that it was mainly composed of a negatively electrified cloud of gaseous particles.

Intermittent Luminosity.

A stroboscopic examination of the ring proved that its luminosity was not continuous. The number of flashes of light per second also varied from time to time. The luminosity of the ring therefore appeared to depend upon an intermittent discharge occurring within the bulb.

Residual Gases.

The nature of the gas in which the ring appeared did not markedly influence its brightness or colour. It must however be pointed out that with the apparatus

shown in fig. 1, the large mass of metal exposed within the bulb, made it difficult to
ensure that the gases experimented with were in a pure state. The iron electrodes
continued always to give off a little contaminating gas at very low pressures, even after
a week's pumping. When the characteristic spectrum of the gas used was visible
while the discharge passed freely (the bulb having been evacuated and filled with the
gas alternately six times), observations on the appearance of the ring at still lower
pressures were begun. In the case of oxygen obtained from steel cylinders, and also
from potassic chlorate, the ring appeared to be somewhat whiter than in air. With
hydrogen and carbon dioxide, however, no change of any kind was noticeable.
Mercury vapour was present in every case. As the object of the investigation was
to determine the process by which the luminous ring formed, it was not thought
advisable to devote too much time to more detailed experiments upon the gases.

The Spectrum.

It was considered sufficient to ascertain whether the composition of the light
emitted by the ring was similar to that of the faint blue haze visible in the bulb
while the induction-coil discharge was passing. This was found to be the case over a
range from about $\lambda 4365$ to $\lambda 5451$. Great difficulty, however, was found in distin-
guishing the lines owing to their feeble intensity, nor was it possible to obtain
satisfactory photographs of the ring itself.

The Ring in Rotation.

The want of uniformity in the density of the glowing nebulous stream constituting
the ring enabled its rotation to be plainly made out. The following experiment,
however, placed the fact beyond a doubt.

A bulb, similar to that shown in fig. 3, was exhausted and a discharge passed
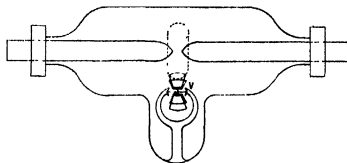
Fig. 3.

through it. It will be seen that a luminous ring forming under the influence of the
magnets would, in this case, partially envelop the wings of the light mica vane, V,
so that, if the ring revolve, the vane might also turn, and by the direction of its

rotation show the sense in which the luminous ring itself was rotating. Under favourable circumstances it was found that whenever the luminous ring was adjusted (by touching the glass at various points) to slightly envelop the wings, as indicated by the dotted line in the figure, a very violent rotation of the vane resulted. In this way it was seen that the ring rotated in the same direction as that in which a current would momentarily be induced in a coil of wire if it were suddenly moved towards a magnetic north pole, namely clockwise, looking through the coil at the pole. It was often noticed however, that a very faintly luminous haze hung about the ring, and when that, instead of the brightest portion, enveloped the mica wings, a rotation of the vane, also very rapid, again took place, but in the opposite direction to that observed in the first case.

Distribution of Ions.

Assuming that the path of an electrified particle rapidly moving towards a magnetic field is modified thereby in accordance with laws known to hold good in the case of the mutual action of such a field and current-carrying wires, the direction of rotation of the luminous ring indicated that it consisted of negative ions streaming inwards towards the strongest part of the magnetic field.

The next question to consider therefore was the change (if any) produced in the distribution of the electrostatic charges upon the bulb, through the action of the magnetic field, and to observe the sign of the electrification upon the ring itself. A piece of tinfoil fastened to the outer surface of the glass bulb invariably became negatively electrified when the magnetic field was suddenly created. The experiment was performed in the following order: Subsequently to the passage of a strong discharge through the bulb, the electrification on the tinfoil patch (placed at the equator of the bulb) was tested in the usual way by means of an electroscope. It was generally positive. This electrification having been neutralised by connection to earth, the current through the electro-magnets was "turned on" while the leaves of the electroscope were in contact with the patch; the tinfoil was then found to have suddenly acquired a strong negative electrification.

This result appeared to be consistent with the view that, after the stimulation of the residual gas by the induction coil discharge, there resided upon the inner surface of the bulb a layer of positively electrified gas particles which, through the action of the magnetic field, had been removed. On the other hand, such an inrush of positive ions from the inner surface of the bulb towards the denser portion of the magnetic field, did not appear to support the idea that the luminous ring was negatively electrified, nor could it account for the direction of rotation of the ring as determined by the mica vane. That the actual luminosity of the ring was mainly due, however, to the incoming radial streams of positive ions was probable from the fact that when a positively electrified body was brought into contact with the glass of the bulb, a faintly luminous ring in the interior was greatly brightened. A pause sometimes

occurred between the magnetisation of the electrodes, and the appearance of a luminous ring. But in such a case positive electrification of the outer surface of the bulb immediately caused the ring to appear within. Under these circumstances the evidence that the luminosity was due to the radial streams of positive ions was also supported by the appearance of the ring as it became gradually built up of luminous arcs in the manner shown in fig. 4. The positively electrified body touched the bulb
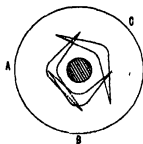
Fig. 4.

at the points A, B, and C. These luminous curves were in a few rare cases clearly visible; they revolved round the magnetic axis very slowly at first, then more rapidly for a few revolutions, blending together into a continuous ring as they moved, which widened out and soon faded completely away. On a few occasions when the electrodes were suddenly magnetised the whole of the residual gas within the bulb appeared luminous—a bluish-white glow—while at the central portion as before, the ring, slightly more luminous than the rest, could be seen rapidly revolving. An electroscope attached to the insulated magnetic electrodes showed that when a positively electrified body was brought into contact with the outer surface of the glass bulb a certain amount of positive electrification was produced upon the iron electrodes themselves. In the case of one particular bulb, on a very dry day and under correct conditions as to pressure and stimulation, the sudden magnetisation of the electrodes caused a spark to leap across a 6-mm. air gap between two wires, one connected to a tinfoil patch at the equator of the bulb, and the other in contact with either of the iron electrodes. This remarkable effect took place after all external residual electrification upon the apparatus had been removed by carefully connecting to earth each part of the bulb as well as the electrodes. By means of a tinfoil patch, fastened to the surface of the bulb at various places, some useful results were also obtained with a bulb containing only one magnetic electrode. Fig. 5 shows the distribution of charges upon the bulb at the moment the discharge through it was stopped. At a pressure scarcely low enough to give a luminous ring the action of the magnet was to increase the positive electrification around the equator, and also slightly that upon the tinfoil ring T at the end. But in this case no luminous ring appeared, nor was there evidence of the withdrawal of positive ions from the interior glass surface. At a lower pressure, however, the ring became visible, and then the sign of the

electrification upon the glass at the equator reversed to negative while the tinfoil became more strongly positive, as before. The range of pressures over which the best results were obtained with bulbs nearly spherical and 6 centims. in diameter extended from 0·0120 to 0·0004 millim. of mercury, as measured by means of a

Fig. 5.
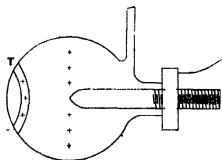
MacLeod gauge. The sign of the electrification upon the luminous ring itself was then found to be invariably negative by means of the following experiment :—

A ring of platinum wire stood within the bulb (fig. 1). It was supported by a thin metallic stalk, which terminated in a small exterior loop, and was placed so as to occupy the position in which under suitable conditions a luminous ring might have been expected to appear. The diameter of the platinum ring was approximately that of the luminous ring when at its best. A discharge was then started and stopped and the electrodes magnetised, with the result that two luminous rings appeared within the bulb, one on each side of the platinum circle and very close to it. On slightly electrifying the platinum circle negatively both these rings flew apart, each threading itself on one of the electrodes. On the other hand, when the circle was positively electrified both rings moved towards each other, and filled the space bounded by the wire with a brightly luminous whitish-blue glow. The luminous ring has, in this way, been shown to invariably be negatively electrified. Recourse was then had to a set of experiments dealing in more detail with the existence of the radial incoming streams of positively electrified gas particles previously referred to.

Diselectrification Effects.

In the first place a new apparatus was constructed, in which the soft iron rods E₁ E₂, were sheathed in a glass tube (fig. 6), which was continuous throughout and melted at the ends on to the short necks projecting from the bulb. The glass tube carried, wrapped round its central portion, a piece of aluminium foil, 2·5 cm. wide, and connected by means of a fine wire to an external terminal loop T₃. Concentrically with this thin aluminium cylinder stood a larger, though much narrower band A of stouter aluminium foil, supported upon a metallic stalk which terminated in an external platinum loop T₁.

The iron electrodes, E_1 E_2, in this case no longer used as such, but merely as prolongations of the electro-magnet cores, were screwed towards each other through the brass framework W W, adjusted to have their pointed ends 2 mm. apart, and arranged centrally within the glass tube and aluminium cylinder just described.

The results obtained at the correct gas pressure with this apparatus were as follows :—A was positively electrified while the gold leaves of an electroscope were connected with it. T_2 was joined to the case of the electroscope. When E_1 and E_2 were oppositely magnetised the leaves collapsed, showing A to have lost its charge. On the other hand, when A was negatively electrified no appreciable change occurred
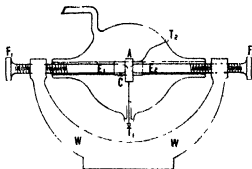
Fig. 6.

in the amount of the charge upon it, through the action of the magnetic field. Neither was there dielectrification of A, even if it were charged positively when the magnets had similar poles at the pointed ends.

[I find that Professor E. WARBURG has already drawn attention to this effect, and in 1896 published some results, with which my own observations are in complete agreement. (See ' Sitzungsber d. k. pr. Akad. d. Wissensch. zu Berlin,' p. 223, 1896, and ' Annalen der Physik,' N. F., 62, p. 385).]—*July* 1, 1901.

The aluminium cylinder C was next positively electrified, and the connection with the electroscope transferred from A to C. In this experiment A was connected to the case of the electroscope, and no change occurred in the electrification of the cylinder either on creating or destroying the magnetic field. The leaves of a second electroscope were then connected to A, as in the first case, while the other electroscope remained attached to C. A was then positively electrified, while C was temporarily to earth. On the magnets being excited the leaves attached to A collapsed, while those in connection with C slightly diverged—positively electrified ; thus confirming the idea that an incoming radial stream of positive ions does, under these conditions, exist. Luminosity generally accompanied these effects. With C negatively electrified while A was temporarily connected to earth no change occurred in the electrification of C when the magnets were excited. Now, if the motion of charged particles either towards or away from the denser portion of the magnetic field and at right angles to the magnetic axis is directly facilitated by either a constant or changing magnetic

field, it is reasonable to suppose that the incoming radial streams of positive ions would have their equivalent in outgoing (that is, from the centre of the bulb towards the equator) streams of negative ions. But such was not found to be the case, as this last experiment showed. Acting upon a suggestion kindly made by Lord KELVIN. this point was still further investigated by means of the following experiment :—Two plane pieces of metal were fixed into the bulb as symmetrically as possible, one on each side of the 2-mm. air gap between the pointed ends of the magnetic electrodes. There was no glass covering over these electrodes, the bulb used being similar to that shown in fig. 1. These metal planes were 1·5 centims. long, ·5 centim. wide, and 1 millim. thick, and were placed parallel to one another with a space of 2 millims. between them. The question to be tested was whether the previous diselectrification effects were in any way due to want of symmetry. But with this apparatus all the results previously obtained were easily repeated. The two metallic planes were then replaced by a finely pointed wire, which projected into the bulb, and this when charged positively became diselectrified when the magnet was excited.

On returning to the arrangement shown in fig. 6, it was found, as a general rule, that the higher the potential to which the body A was charged, the greater for a constant pressure became the strength of the magnetic field necessary to produce diselectrification—more especially was this noticeable at very low pressures. The relationship is shown by the continuous curve in fig. 7, where the abscissæ represent the degree to which A was electrified, and the ordinates are proportional to the strength of the magnetic field sufficient to produce diselectrification. On the other hand, when A in each experiment was charged to the same potential the magnetic strength sufficient to produce diselectrification varied with the pressure of the residual gas in the manner indicated by the dotted line in the same figure.

The results graphically shown by these curves were obtained in the following manner :—A was connected with the quadrants of a Kelvin electrostatic voltmeter, while a fine wire from T_2 was securely screwed to the case of the instrument. The current supplied to the electro-magnets passed through a variable resistance, and also an ammeter which could be read to two places of decimals. It was generally found to be an advantage to introduce a small condenser between T_1 and T_2 in order to steady the needle of the voltmeter. The current through the electro-magnets was always increased very slowly, and the exact moment at which diselectrification occurred could easily be determined by the sound of the sharp snap which was heard when A suddenly lost its charge.

The regularity of the readings of the ammeter for a constant gas pressure within the bulb may be seen from the following examples, where P represents the potential in volts to which A was charged, and C the strength of the current in amperes through the electro-magnets, just sufficient to produce diselectrification (pressure was ·006 millim.).

Each reading was repeated four times.

P = 1200.
C = 3·00, 3·20, 3·20, 3·15.

. . . .

P = 2200.
C = 4·40, 4·50, 4·50, 4·80.

. . . .

P = 2600.
C = 5·10, 5·12, 5·10, 5·10.
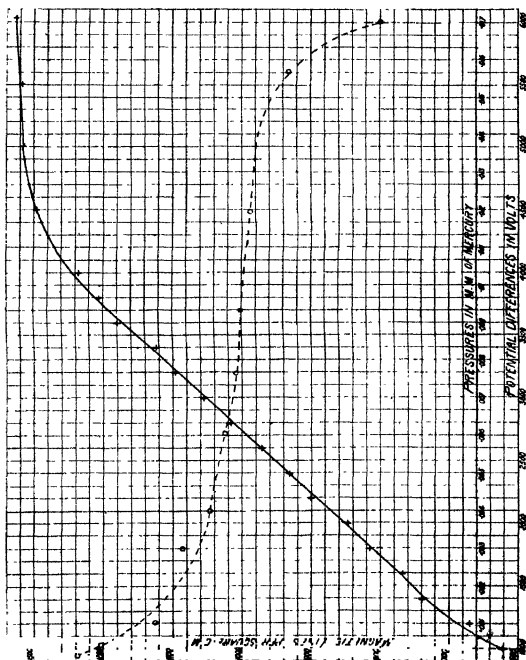
Fig. 7.

It is obvious, however, that the above observations can only refer to a particular bulb and a certain set of conditions. The principal difficulty lay in determining the conditions. Readings taken at the beginning of the week could not be accurately repeated at the end of it, although the curves plotted from such records were in the main parallel. Those shown in fig. 7 were obtained with oxygen as the residual gas within the bulb, and are typical of the results obtained with a particular bulb, irrespective of whether the residual gas was air, hydrogen, carbonic dioxide, or oxygen ; the only difference in all the various cases being a shifting, parallel to itself of the complete curve.

Reference has already been made to the interval which occasionally elapsed between the excitation of the magnet and the appearance of a luminous ring within the bulb. A similar pause was noticed in connection with experiments upon the diselectrification effects. It should also be pointed out that the sign of the electrification of the bulb, both in the case of the luminous-ring experiments and those relating to diselectrification, was invariably negative through the action of the magnetic field. Moreover it has already been seen that luminous flashes usually accompanied diselectrification. The question as to the origin of the luminous ring appeared therefore to be intimately involved in any explanation that might be found to account for the diselectrification phenomenon, and it became important to see finally whether conditions favourable to the one but unfavourable to the other could be found to exist simultaneously.

A general survey of the results up to this stage pointed to a concentration of negative ions at the centre of the bulb, and a series of independent experiments were then carried out in order to ascertain what was the distribution of charged gas within the bulb at the moment the induction coil discharge ceased to pass through it. For instance, the action of the magnetic field upon the residual gas particles was examined not only subsequently but also during the passage of the discharge. A screw thread was cut upon each electrode, so that by its means the uniform green fluorescence of the glass might appear as a luminous spiral and show, by widening out or twisting, the paths of the electrified particles shot off from the metal. The action of the magnetic field upon jets of gas* which, at the lowest pressures and even after continued exhaustion, were found to be given off from metal exposed within the bulb, was also examined. The results of these experiments are embodied in fig. 8, which shows the probable distribution of the electrified gas to be such that the central portion of the bulb was fairly uniformly filled with negative ions, while a layer of positively electrified gas resided upon the interior surface of the glass : the former being indicated by the small dots and the latter by the larger circles. The diagram represents the state of affairs at the moment an oscillatory discharge had passed through the bulb, and I desire to call attention to the accumulation of negative ions at either end.

With the object of ascertaining whether these clouds of negative ions were affected

* 'Electrician,' vol. 41, p. 425 ; 'B. A. Report,' 1900, p. 639.

by the sudden production of a magnetic field at the centre of the bulb, the following experiment was then arranged :—The apparatus shown in fig. 9 was intended to answer two purposes. In the first place, it would test whether the negative ions tended to concentrate and be drawn from the ends of the bulb in towards the central portion, and also, if that action were prevented, whether the ball A would lose a positive charge when the magnets were excited.
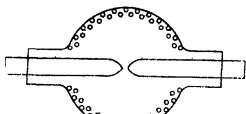
Fig. 8.
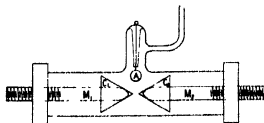
Fig. 9.

Two thin copper cones C_1C_2 were soldered, one upon the pointed end of each of the magnetic electrodes M_1M_2. They were fitted into the glass tube as shown, and the pressure lowered until a discharge passed between the cones with difficulty. When M_1 was connected with the negative terminal of the induction coil and M_2 to the positive, and the discharge had been started and stopped, the magnetisation of the electrodes gave rise to a luminous ring behind and close to the cone C_2. The reverse conditions produced a ring behind the cone C_1. Subsequently to an oscillatory discharge through the tube a luminous ring appeared simultaneously behind each cone immediately the magnets were excited. In all these cases the luminosity visible between the apices of the cones was slight, but it was not to be expected that all the negative ions could be prevented from concentrating at the centre of the tube by this device. A was then positively electrified, and the electrodes magnetised. It became only partially diselectrified—a very different result from that obtained in previous experiments where no cones were attached to the electrodes. This result was repeated a great many times, and, although in each case the diselectrification was enfeebled by the introduction of the cones, a further experiment was made with a view to completely preventing diselectrification of the ball A while the conditions remained suitable for the magnets to act upon the electrified gas particles as before. The cones were removed from the electrodes. An aluminium cylinder C was then slipped in before the tube was sealed up, and arranged so as to be capable of sliding along the fine wire W_1W_2 as shown in fig. 10. The ball A remained as before. As a trial experiment the pressure within the tube was lowered and A positively electrified. When the magnets were excited A was completely diselectrified. The cylinder C was then tapped into the position indicated by the dotted line and the wire W_1W_2 connected to earth. A was again positively electrified, but when the electrodes M_1M_2 were magnetised there was no diselectrification of the ball whatever. A proof that

the new position of the cylinder had not disturbed the action of the magnetic field upon the electrified gas within the tube consisted in charging the cylinder itself positively and then suddenly magnetising the electrodes. When that was done the cylinder completely lost its charge in a very satisfactory manner.
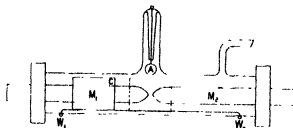
Fig. 10.

Varying Magnetic Fields.

It has long been thought that an electrified body would tend to move, through the action upon it of a rapidly varying magnetic field. The path of the body should lie along a direction at right angles to that in which the magnetic lines are moving, and also at right angles to the direction in which they spread. The conditions accompanying the appearance of the luminous rings, and also the diselectrification already described, appeared suitable for the detection of the action which a variation in strength of the magnetic field might be expected to have upon the practically weightless and very minute electrified gaseous particles freely floating about within the bulb.

It has already been pointed out that a luminous ring having become dim and almost invisible could be momentarily brightened by suddenly destroying the magnetic field. But with the apparatus shown in fig. 6 the following striking results were obtained :—The aluminium band A was positively electrified to a potential of 6000 volts. When the current supplied to the electro-magnets was very slowly increased, as much as 11 amperes was necessary to produce diselectrification of A, while on the other hand, if the current were very rapidly increased in value A lost its charge when only 6 amperes flowed through the coils.

Further than this, it was always noticed that diselectrification occurred when the magnetic field was suddenly destroyed rather than when it was created. A particular rarefaction was found for each bulb, such that the band A could be electrified, and, irrespective of the fact that the electrodes were magnetised, retain its charge. *In all such cases the effect of rapidly destroying the magnetic field was to produce diselectrification, whereas when the current supplied to the magnet coils was broken under water, and in consequence the magnetic field caused to gradually diminish in strength, no trace of diselectrification was observed.* It was therefore clear that variations in the

strength of the magnetic field played an important part both in the formation of the luminous ring and also in the diselectrification process.

Conclusion.

The preceding experiments show that the principal effect of the magnets is to produce a concentration of negative ions at the strongest portion of the magnetic field, and centrally within the bulb. There is also experimental evidence to prove that this concentration of negatively electrified gas is responsible for the diselectrification of a positively electrified body placed in its neighbourhood. The experiment with the metal cones (fig. 9) and also that with the electrostatic screen (fig. 10) demonstrated that, either when the concentration of negative ions occurred between the points of the electrodes or when an earth-connected metallic screen came between the negative cloud of gaseous particles and the electrified body, no diselectrification took place. I consider that this concentration of negative ions is due to two main causes. In the first place, it is partly produced by the action of the magnetic field upon ions already in motion within the bulb. The pause of about two seconds sometimes found to occur between the excitation of the magnets and the appearance of a luminous ring or diselectrification of a positively charged body supports this view. And secondly, owing to the reaction resulting from the sudden excitation of the magnets,* the comparatively dense clouds of ions situated at the ends of the bulb (fig. 8) would, in rapidly turning about the magnetic axis, tend to move towards the pointed ends of the electrodes and so concentrate as observed. Owing to the viscosity† of the gas, however, it is not to be expected that such a rotation of the ions would be other than momentary. But that the rate of change of the magnetic lines affects the distribution of the electrified particles within the bulb is clear from the results of the experiments already given.

[Professor S. P. Thompson‡ has already pointed out that the preliminary experiments seemed to indicate the existence of some such action taking place within the bulb.]—*July* 1, 1901.

Referring again to fig. 8, we see the distribution of ions at the moment prior to exciting the magnets. The negative ions in concentrating as explained give rise to an increase in the potential difference between the electrified gas upon the inner surface of the glass and that at the central portion of the bulb. At a gas pressure just too high to give the luminous ring, the effect of creating a magnetic field was found to be an increase in the positive electrification upon the outer surface of the bulb (fig. 5), and this is now seen to be consistent. That state of things is represented in fig. 11, where the positive ions are supposed to accumulate opposite to the

---

* 'Electrician,' vol. 25, p. 35.
† 'Phil. Trans.,' vol. 172, p. 387.
‡ 'Electrician,' vol. 43, 1899, p. 412.

negative concentration until the potential is such that they suddenly flow inwards, while the superficial layers of the negatively charged gas-mass at the centre of the bulb move rapidly outward. In this manner is explained the occasional appearance of a ring of green fluorescence upon the equator of the bulb at the moment the electrodes are magnetised, and also the loss of positive electrification, not only from the walls of the bulb but from objects placed in the interior as well. The similarity
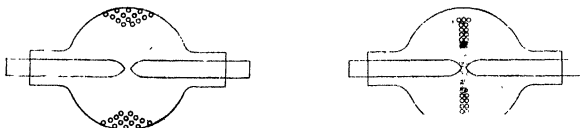
Fig. 11.

between both the diselectrification effects and the luminous ring phenomena has been also experimentally demonstrated, and it now remains to offer an explanation of the luminous ring itself. The final action of the magnetic field, the incoming streams of positive ions, and the production of the luminous ring, are represented by fig. 12. The ring is most luminous at its outer edge or surface owing to that being the boundary at which most of the incoming positive and outgoing negative ions collide.* But while the concentration of negative ions can be explained for a growing magnetic field, it is not clear why the sudden cessation of the field should also produce a concentration. The influence of the rate of change of the magnetic lines was more marked in this than in the former case. Whether Mr. WALKER'S ingenious theory† of the luminous ring satisfactorily explains that result is not quite certain. I could get no direct experimental proof either one way or the other. The direction in which the luminous ring rotates has been already referred to in detail. It must now be pointed out that the sense is opposite to that in which a negative ion might be expected to move owing to the sudden growth of the magnetic field. On the other hand, it is evident that the outer surface of the ring is more positive than the surrounding negatively electrified gas particles, which would in consequence flow inwards and be deviated in a manner consistent with the observed direction of rotation of the whole. It is seen also that the positive ions felt the action of the magnetic field to a lesser degree, probably owing to their greater mass and comparative want of mobility. We have, therefore, to imagine the initial cloud of concentrated negative ions not necessarily in rotation, but that, owing to the incoming streams of positively electrified particles, the gas-mass at the centre of the bulb

* 'Proc. Roy. Institution,' vol 9, p. 142.
† 'Electrician,' August 25, 1899, p. 634.

becomes luminous, and is then, as it were, blown upon by streams of negative ions which are attracted to the outer surface of the luminous ring, and thus cause the whole to revolve at a high velocity.

The expansion of the luminous ring is accounted for by the outward flying streams of negative ions which go forth to meet the incoming positive ones. Moreover, this view receives strong support from the experiment in which a mica vane, when partly immersed in the haze sometimes visible about the ring itself, rapidly rotated in the opposite direction to that in which it turned when the brighter portion enveloped the wings. It is also seen why the size and shape of the bulb itself exerts an influence upon the effects and why the distribution of charged gas in the interior plays so important a part. The other minor phenomena, such as the lateral movement of the ring as a whole, from one part of the bulb to another, as well as the formation of luminous rings in planes other than at right angles to the magnetic axis, may all be explained in accordance with the theory here put forth.

V. *Total Eclipse of the Sun, January 22, 1898.—Observations at Viziadrug.*

By Sir NORMAN LOCKYER, *K.C.B., F.R.S., Captain* CHISHOLM-BATTEN, *R.N., and Professor* A. PEDLER, *F.R.S.*

[PLATES 4–8.]

CONTENTS.

PART I.—GENERAL ARRANGEMENTS. By Sir NORMAN LOCKYER, K.C.B., F.R.S

PART III.—SPECTROSCOPIC OBSERVATIONS MADE WITH A 6-INCH SHORT-FOCUS LENS AND A GRATING. By Professor A. PEDLER, F.R.S.

PART IV.—THE PRISMATIC CAMERAS. By Sir NORMAN LOCKYER, K.C.B., F.R.S.

ILLUSTRATIONS.

Part I.—General Arrangements.

By Sir Norman Lockyer, K.C.B., F.R.S.

Objects of the Expedition.

For the expedition to Kio Island in 1896 I had arranged an extensive programme of observations which unfortunately could not be completely carried out on account of bad weather,[*] and I did not find it necessary to materially modify this programme for the expedition of 1898. In view of the large number of observers which was expected to be available in 1898, as in 1896, the observations proposed included records of nearly all the phenomena of a total eclipse, in addition to spectroscopic researches.

Three associated series of spectroscopic observations were provided for :

(1.) Prismatic cameras, to give records of the phenomena with a larger dispersion than any previously employed. The special features of this form of spectroscope, and the important points to be considered in the interpretation of the photographic records which they give, were described in the Report on the Eclipse of 1893.[†]

(2.) An integrating spectroscope of large dimensions to attempt to get stronger photographic indications of the coronal radiations, by utilising the whole area of the corona, which the prismatic camera cannot do.

·(3.) A grating spectroscope, in conjunction with a telescope of large aperture and short focus, for eye observations of the inner corona. The spectrum of the chromosphere, as photographed in 1893 and 1896, showed that the Fraunhofer lines were not completely represented, and it seemed likely that the missing lines would be found above the chromospheric layer photographed, and associated with the coronal layers, of which we had photographed only a few of the brightest radiations. I certainly saw some of them in Egypt in 1882,[‡] and it was accordingly decided to attempt to fill the blank in the photographic record by visual observations.

The present Report deals with all the observations except those made with the prismatic cameras, in which case the work of reducing the photographs is very heavy and will not be completed for some time.

[*] ' Phil. Trans.,' A, vol. 190 (1897), p. 2.
[†] ' Phil. Trans.,' A, vol. 187 (1896), pp. 551–618.
[‡] ' Roy. Soc. Proc.,' vol. 34, p. 297.

In the first part of the Report I give an account of the general arrangements. The second part is Captain CHISHOLM-BATTEN'S report on the observations made by the officers and men of H.M.S. "Melpomene."

Professor A. PEDLER, F.R.S., gives an account, in the third part, of his visual observations with the grating spectroscope.

Details of the meteorological observations made at Viziadrug, under the direction of Mr. JOHN ELIOT, M.A., F.R.S., C.I.E., Meteorological Reporter to the Government of India, have already been published.*

The Observing Station and Preparations.

After various inquiries which I had made respecting the suitability of Viziadrug for observations of the total eclipse, I informed the Eclipse Committee that I was prepared to take charge of an expedition to that locality, and it was agreed that the observations at this station should be placed in my charge.

The latitude and longitude of the part of the fort at Viziadrug finally occupied were 16° 33' 26" N. and 73° 18' 58" E. respectively, and the duration of totality was estimated at 127 seconds.

In connection with the work at this station the Admiralty was asked for a ship of war to convey the observers from Colombo to Viziadrug, and to permit the use of the ship, if possible, as a base, to enable me to repeat the observations attempted in Norway in 1896 with the assistance of H.M.S. "Volage," which ship supplied twenty-four assistants during the eclipse and fifty volunteers for general observations.† As a result of the Royal Society's application, H.M.S. "Melpomene," in command of Captain CHISHOLM-BATTEN, R.N., was told off to join the expedition.

The expedition, which left England on December 10, consisted of Mr. A. FOWLER, Dr. W. J. S. LOCKYER, and myself, together with the Marquis of GRAHAM, who joined as a volunteer. Some little time after reaching Viziadrug Professor PEDLER, F.R.S., from Calcutta, joined the party, and shortly before the eclipse Mr. JOHN ELIOT, Meteorological Reporter to the Government of India, joined from Simla. On arrival at Colombo we found H.M.S. "Melpomene" waiting there, and at once proceeded to the selected spot of observation—Viziadrug.

During the three days' voyage to our station a call for volunteers was made by Captain BATTEN, and 120 came forward. Lectures and demonstrations were therefore at once commenced by Lieutenants BLACKETT, COLBECK, and DUGMORE, Second Engineer MOUNTIFIELD, and myself to the several parties of men who had undertaken to perform special pieces of work. Twenty-two separate groups of observers were formed. On our arrival at Viziadrug we were received very kindly by Mr. BOMANJI, the Collector of Ratnagiri, and an Overseer of the Public Works

---

* ' Indian Meteorological Memoirs,' vol. 11, Part I. Calcutta, 1899.

† ' Phil. Trans.,' A, vol. 190 (1897), pp. 1–21.

Department, who was on the spot in charge of some most excellent masons and carpenters, picked men from Ratnagiri as we later ascertained, and plenty of material for the construction of the necessary concrete bases and huts. It was important to erect the huts as soon as possible, not only to shelter the instruments, but also the observers, from the sun. Several screens were made which could be moved and placed in any required position; these were found to be invaluable while the instruments were being erected. A considerable number of coolies was also present to do such work as carrying packing-cases, sawing wood, clearing the camp, &c.

In the fort was also a police guard sent from Ratnagiri. The camp was watched, both by day and night, so effectively by them that no damage to any instrument was reported.

On the arrival of the "Melpomene" at Viziadrug, Mr. BOMANJI came on board to report the arrangements which had been made for the expedition by the Government of India. As these were not quite complete, it was necessary for the first few days to return to the ship every evening, but afterwards Mr. FOWLER, Dr. LOCKYER, and myself took up our quarters at the Dak bungalow inside the fort, close to the instruments. Meals were provided at the Collector's camp, which was also inside the fort.

A party was landed at the fort on the afternoon of our arrival to inspect the site suggested by Mr. BOMANJI, and it was at once evident that it would satisfy all requirements, provided the fluctuations of temperature of the great masses of masonry composing the fort had no disturbing influence on the steadiness of the air. In order to investigate this point a $3\frac{3}{4}$-inch telescope was erected, and observations of the surrounding landscape, and, at dusk, of various stars, were made, from which it appeared that the atmosphere was sufficiently steady for the observations.

Next morning the instruments were landed and the concrete bases for them were commenced. The erection of the huts was also begun by the native workmen, and continued without intermission.

The instruments were set up as soon as their bases were ready, and by the end of a week all were practically in readiness for the eclipse. Constant clear skies enabled all the adjustments to be made without difficulty.

A plan of the fort showing the arrangement of the instruments is given in Plate 4.

During the week preceding the eclipse the adjustments were frequently tested, and a complete system of drills was established.

As the number of volunteers was so large, I pointed out to Captain BATTEN, who had volunteered to aid in a special branch of the work, the importance of his taking charge of the whole camp and giving all the necessary orders for conducting the operations during the general rehearsals, and the eclipse itself. He eventually agreed to this, and the procedure and time signals were arranged between us.

The development of the photographic plates was commenced immediately after

the eclipse, and it was found that the results were on the whole very satisfactory. No results, however, were obtained with the integrating spectroscope, and the kinematograph films taken by Lord GRAHAM were too badly fogged to serve any useful purpose.

The dismantling of the instruments was commenced very soon after the eclipse, and the packing, together with the development and copying of the negatives, kept the party fully occupied until the morning of January 25, when the expedition left Viziadrug.

Half of the negatives and glass copies of the remainder were conveyed to England in charge of Mr. FOWLER, while the remaining half of negatives and positives were sent home *viâ* Bombay.

Local Conditions of Eclipse.

In response to my request, the Superintendent of the ' Nautical Almanac' office kindly supplied me with the following local particulars of the eclipse for the station which I proposed to occupy :—

Assuming the position of Viziadrug to be 16° 32′ N. and 73° 22′ E., the times of contact will be

		d.	h.	m.	s.	
1898.	January	21	23	12	20	
		22	0	46	9	} Duration 125 secs.
		22	0	48	14	
		22	2	14	33	

These are local mean times, and to reduce to G.M.T., 4h. 53m. 28s. must be subtracted.

The position angle of the point of 2nd contact is 51°, and that of 3rd contact 241°, reckoned from the north point towards the east in each case.

On reaching India, however, it was ascertained that, according to the most recent Admiralty chart, the position of the selected station was 16° 33′ 26″ N., and 73° 18′ 58″ E. This being nearer to the central line than the position adopted by Dr. DOWNING, the duration of totality was estimated at 127 seconds, and the programme of work arranged accordingly. The altitude of the sun at mid-eclipse was calculated to be very nearly 53°, and the azimuth nearly 14°.

The apparent semi-diameters of the sun and moon at the time of eclipse were respectively 16′ 16″·4 and 16′ 39″·5, while the relative motion was approximately 0″·36 per second.

The angle between the vertex and the north point at the time of mid-eclipse was calculated to be 14° 9′, the north point of the limb lying towards the west. The position angle of the sun's axis being 8° west of north, the axis would therefore be inclined 22° 9′ from the vertical towards the west. The heliographic latitude of the

sun's centre was 5° 20′ S., so that the direct image was oriented as shown in fig. 1, which also shows the points of 2nd and 3rd contact.
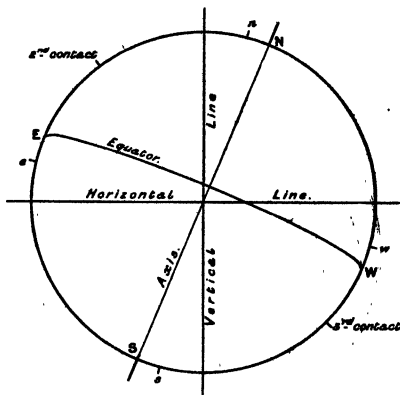
Fig. 1. Orientation of Sun's Disc at Mid-eclipse.

In connection with the instruments using the light from cœlostat mirrors, it was further necessary to calculate the sun's amplitude at the time of rising. This was found to be 20° 31′ south of east.

Time Arrangements.

The general time signals were given by a bugler under Captain BATTEN's orders. The chronometer was in charge of Lieutenant DE WET, R.N.

One of the ship's chronometers was fortunately available for use on shore on the day of eclipse, but previously watches were frequently corrected by means of signals sent by Lieutenant QUAYLE from the ship.

The special signals during totality were given every 10 seconds, beginning at 127 —the assumed period of totality—by means of the eclipse clock (which was started at the signal " Go " by cutting a thread, thereby releasing the pendulum), by two timekeepers, one during the first half the other during the second half of totality.

In the system adopted not only was the time left called out every tenth second, but other signals were interpolated to guide the work in the photographic huts. In

order that there might be no mistake about the calls, a spiral was drawn on the clock-face and the seconds left plainly marked at the points which the second hand would occupy during its two revolutions (fig. 2). The time calls were repeated by two assistants, who stood about 50 yards from the eclipse clock, for the benefit of the observers in the outlying parts of the camp.
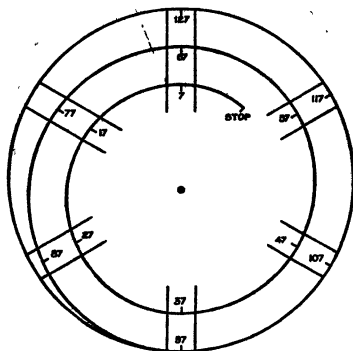
Fig. 2. Dial of Eclipse Clock.

For the work of the prismatic cameras it was important to get a signal as nearly as possible 5 seconds before the beginning of totality, and, in order to eliminate the possible error of the chronometer, it was arranged to determine this by direct observations. Two methods were adopted. In one of them a boat was moored at a distance of 2 miles from the camp, in the direction of approach of the shadow, which would pass this point 5 seconds before totality. This failed because of the indefinite boundary of the shadow.

The other method was to determine when the visible remaining crescent subtended an angle of 45°; calculation showed that this would occur at the desired interval from totality. This method was completely successful.

A signal at 25 seconds before commencement of totality was also given from observations, the cusps then subtending an angle of 90°. These observations of the cusps were made by Lieutenant DE WET.

The chronometer was 5h. 8m. 35·5s. slow on mean time at Viziadrug, and 0h. 15m. 20s. slow on Greenwich mean time, as determined by Lieutenant QUAYLE.

Acknowledgment of Assistance.

The extraordinary interest and the skill displayed by the officers and men of H.M.S. "Volage" under Captain KING HALL in 1896, and of H.M.S. "Melpomene" under Captain CHISHOLM-BATTEN in the present year, prove beyond all question that, in eclipses in which a man-of-war can be employed, the most effective and the most economical means of securing observations is to depend upon the naval personnel, one or two skilled observers being sent out to help in the final adjustments of instruments according to the number it is intended to employ.

At Viziadrug, Mr. FOWLER and Dr. LOCKYER were enabled to report all the fixed instruments and huts, eight in number, erected and all but the final adjustments made after six days' work, a long break being necessary in the middle of the day in consequence of the heat. Such an achievement as this is beyond all eclipse precedent, and was only rendered possible by the help of a large staff of highly trained men. Of the 150 engaged in the operations only three originally formed the expedition.

It is, therefore, quite inappropriate that I, on the part of the expedition, should here tender thanks to Captain BATTEN, the officers and men of H.M.S. "Melpomene" for their assistance, for as matters turned out we assisted them; but we are anxious to place on record the kindness we received from them both afloat and ashore, and since the great success of the recent observations is due almost entirely to Captain CHISHOLM-BATTEN and the ship's company of the "Melpomene," I trust that the President and Council of the Royal Society may be pleased to communicate this fact to the Lords Commissioners of the Admiralty.

Among those to whom thanks are specially due are the following, representing the Indian Government :—

E. GILES, Esq., Director of Public Instruction, in charge of arrangements made by Bombay Government.
K. R. BOMANJI, Esq., Collector of Ratnagiri.
J. L. JENKINS, Esq., Collector of Salt.
E. H. AITKEN, Esq., Assistant Collector of Salt.
F. R. BADER, Esq., Assistant Engineer, P.W.D.
GANGADHAR ANANT BHAT, Executive Engineer, P.W.D.
GOVIND GOSHI, Overseer, P.W.D.
SADASHI GOVIND JOSHI, Clerk to the Overseer, P.W.D.

Thanks are also due to the Officers of the Police, Telegraph, and Customs Departments, and others representing the Bombay Government, for their unceasing efforts to help us in every way.

Everybody was struck by the admirable and smart manner in which the subordinates of the Public Works Department accomplished their respective tasks.

I took upon myself, when leaving Viziadrug, to write an unofficial letter to Mr. BOMANJI, thanking him, in the name of the expedition, for his great personal kindnesses to us as well as for the valuable assistance we had received from him and the other local representatives of the Government.

L. LEE, Esq., Collector of Customs for Ceylon, and other Customs officials at Colombo, rendered valuable assistance to the expedition by granting special facilities and providing means for transhipping the instruments.

The Orient Steam Navigation Company very kindly conveyed the instruments free of charge to and from Colombo.

To W. H. SINCLAIR, Esq., a former Collector of the district (now retired), I was indebted for the supply of much valuable local information before leaving England.

My own personal thanks are due to Mr. FOWLER and Dr. LOCKYER, who assisted me in the preliminary work of organisation, and who, while at Viziadrug, worked hard both day and night to further the objects of the expedition; and also to Mr. BOURNE, Midshipman, attached to me as Aide-de-Camp, who was indefatigable in helping me to carry out the various details of the local organisation.

PART II.—OBSERVATIONS MADE BY THE OFFICERS AND MEN OF H.M.S.
"MELPOMENE."

By Captain CHISHOLM-BATTEN, R.N.

Admiralty Instructions.

H.M.S. "Melpomene" was ordered to receive on board Sir NORMAN LOCKYER and party at Colombo, convey them to Viziadrug, and bring them back to Colombo after the eclipse. During her stay at Viziadrug such assistance as lay in her power was to be rendered to the party. It was to be quite understood that it was very desirable that the officers and men should interest themselves in the work, and be encouraged to come forward as assistant observers, if required, by Sir NORMAN LOCKYER.

By later orders, H.M.S. "Melpomene" was ordered to convey Sir NORMAN LOCKYER from Viziadrug to Goa, instead of returning direct to Colombo.

Diary of the Expedition.

On the arrival of the "Lusitania," the Torpedo Instructor was sent on board to make himself, as much as might be, familiar with the eighty cases of eclipse gear, in order that he might attend to them throughout.

On the 4th January the party and the gear were transferred from the mail steamer and H.M.S. "Melpomene" left Colombo.

On the 5th January Sir NORMAN LOCKYER gave a lecture, with illustrations by lantern slides, on Eclipses in general, and afterwards volunteer observers were called for. All the officers, and over 100 men, gave in their names as volunteers for special work, or for general assistance. The same evening the sketching party were instructed and exercised, and daily afterwards.

Friday, January 7.—The party arrived at Viziadrug. The Collector came on board, and on landing, the party selected a site, a piece of flat ground, the highest inside the old Castle, on which there was plenty of room for the instruments (see Plan, Plate 4), and near which was the Traveller's bungalow, built over the old powder magazine of the pirates, which was used as a dark room, mess room for the men, lecture room when lantern slides were required, and general resting-place, on account of its being cool and shaded.

Saturday, January 8.—All instruments were landed, and a fair start made in setting them up, the leading hand of each party from the ship assisting at his own instrument or job.

Instruction was carried out on board on the spectrum, in corona drawing, and on observing the colours of the landscape.

Sunday, January 9.—During Sunday the native workmen, masons, carpenters, and labourers provided by the Public Works Department of India, prepared cement foundations, shelter huts, and the site generally.

Monday, January 10.—Good work was done in setting up the instruments, which were in place by the evening. Instruction was carried out on board on hand spectro-scopes.

A signal-house had been built, and a signal pole set up for communication with the ship. Larger parties every day were now required on shore for setting up gear, and for practice with it. As Viziadrug has the reputation of being a favourite place for snakes, men always landed in gaiters. The dark room for the British party was set up in the magazine, and one for Mr. TURNER, the photographer, sent from Calcutta by the Government of India, was set up in a shady place close to the bungalow on his arrival on Tuesday.

Tuesday, January 11.—Lectures on spectra colours, landscape, and corona drawings were given.

Sir NORMAN LOCKYER, Mr. FOWLER, and Dr. LOCKYER took up their quarters at Norman Castle.

Wednesday, January 12.—Sketching corona, with discs, party were under instruc-tion, and the usual duties of various parties independently.

Thursday, January 13.—Professor PEDLER arrived, and found his telescope in place, and a party ready to assist him.

In the evening, about five o'clock, the first rehearsal took place of the parties ready.

Instruction in colour of landscape, and corona sketching with discs, was carried out on this day.

Friday, January 14.—Instruction was carried out on colours of landscape and the observation of stars during totality.

A second rehearsal took place, and the most important photography was found to be well ahead of time.

Saturday, January 15.—A rehearsal took place, and the usual instruction went on.

Monday, January 17.—The first rehearsal at eclipse time took place, in order that practice might be obtained in directing the instruments to the proper point in the heavens; a second rehearsal took place at 5 P.M. These rehearsals in the evening were at first repeated two, or even three, times the same evening, but later on one drill was found enough.

Instruction by Mr. TURNER on landscape cameras, and lectures on stars, with a view to their observation during totality, took place.

Tuesday, January 18.—Rehearsals took place at eclipse time, at 5 P.M., and after sunset, the latter for the purpose of testing the lighting arrangements, which were found to be sufficient.

Lecture on spectra was given.

Wednesday, January 19.—Mr. AITKEN, who was to take charge of the naturalists' party, arrived but left again before the eclipse, and only returned on the last morning. Professor PEDLER started a thermometer class in readiness for Mr. ELIOT. Lecture was given on spectra, and rehearsal took place at 5 P.M.

Thursday, January 20.—Mr. ELIOT arrived and was supplied with a party for making meteorological observations under his guidance.

Lecture on observations of temperature.

Rehearsal at 5 P.M. The party was now complete.

Friday, January 21.—Rehearsal of duties at eclipse time. Half-holiday.

List of Instruments and Observers.

The observers were as follows :—

1. *Time Signals.*
   Captain A. W. CHISHOLM-BATTEN, R.N
   F. DOWNTON, Leading Seaman.
   W. WOODS, Yeoman of Signals.

   W. GROVES, Shipwright.
   F. T. MAREY, Private, R.M.L.I.
   G. S. FULLILOVE, Private, R.M.L.I.
   G. CLEARY, Private, R.M.L.I.

2. *6 inch Prismatic Camera.*
   Mr. FOWLER.
   Lieutenant O. DE WET, R.N
   C. IRONSIDES, G.M.
   J. TURNER, T.I.

   F. BRADING, A.B.
   J. INNES, A.B.
   G. SALT, Boy, 1st Class.

5. *9-inch Prismatic Camera.*
   Dr. LOCKYER.
   Lieutenant PERCIVAL JONES, R.N.R.
   A. RAMAGE, A.B.
   W. BRAY, Ch. Arm.

   A. WILKINS, Shipwright.
   E. ASHFORD, A.B.
   F. FENTON, A.B.
   A. CARR, Boy, 1st Class.

4. *Integrating Spectroscope.*
    Lieutenant G. C. QUAYLE, R.N.   G. TRAVILL, P.O., 1st Class.
    J. BIRD, Ch. E.R.A.

5. *Coronagraph.*
    Staff-Engineer A. KERR, R.N.   C. MOSELEY, Leading Stoker, 1st Class.
    W. HOLMES, E.R.A.   G. COLLIER, Stoker.

6. *Discs.*

A. RUBB, Ship's Corporal, 1st Class.	R. SUTHERLAND, Leading Signalman.
G. PINK, Qualified Signalman.	W. WEBB, A.B.
J. HENRY, Boy, 1st Class.	W. CORNEY, Stoker.
B. BROOK, Stoker.	G. PRICE, A.B.
A. McDONALD, P.O., 1st Class.	J. JONES, A.B.
A. TULL, Ship's Steward's Boy.	F. DIBBINS, Ordinary Seaman.
L. PETTINGALE, Leading Signalman.	L. GATES, A.B.
W. BROOKER, A.B.	R. DAVIS, A.B.
S. DREW, Ordinary Seaman.	P. McKENNA, A.B.

7. *Sketches of Corona without Discs.*

A. RICHARDSON, P.O., 1st Class.		T. WELLS, A.B.	N.W.
W. PANKHURST, A.B.	General.	H. BRINSTEAD, A.B.	
H. LACK, Boy, 1st Class.		E. DANN.	S.E.
W. ANDERSON, A.B.	N.E.	W. EVANS.	
E. WILSON, Ordinary Seaman.		W. CLAYTON.	S.W.

8. *3¾-inch Equatorial.*
    Sir NORMAN LOCKYER, K.C.B.   M. MOORE, Stoker.
    Mr. H. WILLMORE, Assistant Engineer, R.N.

9. *Observations on Stars.*
    Lieutenant HENRY BLACKETT, R.N.   T. SUTTON, Stoker.
    J. McDONALD, A.B.   J. FITZROY, Boy, 1st Class.
    F. STEVENS, A.B.   G. RUSSELL, Private, R.M.L.I.
    R. BUCKLAND, Plumber's Mate.

10. *Hand Spectroscopes.*
    Lieutenant C. E. B. COLBECK, R.N.   P. MANNING, Ordinary Seaman.
    C. KITCHINGHAM, Private, R.M.L.I.   H. MITCHELL, Stoker.
    C. WOODLEY, P.O., 1st Class.   J. DOBSON, Sergeant, R.M.L.I.

11. *Prisms for Observations of Ring Spectra.*
    Mr. J. MOUNTIFIELD, Senior Engineer, R.N.   R. COATES, Stoker.
    W. MORRIS, E.R.A.   G. TARRANT, Stoker.
    A. HOWE, E.R.A.   H. WARREN, Stoker.
    C. STACEY, Leading Stoker, 2nd Class.   J. INCH, Stoker.
    H. KNIGHT, Leading Stoker, 2nd Class.   G. GRAY, Chief Stoker.
                                    J. CROSS, Stoker.

12. *Polariscope.*
    Staff-Surgeon C. L. NOLAN, R.N.

13. *Landscape Colours.*
    Lieutenant E. N. R. DUGMORE, R.N.   P. DARVIL, Boy, 1st Class.
    G. FARRELL, Boy, 1st Class.   H. RHODES, Ordinary Seaman.
    W. JACOBS, A.B.   H. ATTREE, Signalman.

14. *Landscape Cameras.*
    Mr. TURNER, Survey Department, Calcutta.   J. COLLINS, Chief Stoker.
    E. GYNGELL, A.B.   J. KEARNEY, Leading Stoker, 1st Class.
    H. CHILDS, Chief Stoker.   E. CROSS, Leading Stoker, 2nd Class.

15. *Shadow Phenomena.*
    W. KEENAN, Chief Carpenter's Mate.   G. RILEY, Stoker.
    A. REYNOLDS, Stoker.   B. CRUNDEN, Stoker.
    W. WEEKS, Shipwright.   C. CARPENTER, Stoker.

16. *Observations of Shadow-bands.*
    Staff-Surgeon C. L. NOLAN, R.N.   A. PURKINGTON, 2nd S. B. Steward.
    C. HESTER, Private, R.M.L.I.

17. *Kinematograph for Eclipse.*
    The Marquis of GRAHAM.   C. THOMAS, Seedie.
    A. SHILCOCK, E.R.A.   P. KING, Ordinary Seaman.
    E. GREEN, Boy, 1st Class.   W. CRONEN, Stoker.

18. *Kinematograph for Shadow.*
    Mr. H. P. BARNETT, Paymaster, R.N.   A. GIDNEY, E.R.A.

19. *Contact Observations.*
    Lieutenant O. DE WET, R.N.   C. IRONSIDES, G.M.

20. *Observations on Natives, Animals, &c.*
    W. J. C. SLOCOMBE, Ordinary Seaman.   F. BEAL, Ordinary Seaman.
    G. WHITTINGSTALL, Ordinary Seaman.

21. *6-inch Equatorial with Grating Spectroscope.*
    Sir NORMAN LOCKYER, K.C.B.   P. ROSS, Ch. E.R.A.
    Professor A. PEDLER, F.R.S.   G. VANSTONE, Ch. E.R.A.
    Mr. R. C. STEELE, Gunner, R.N.   H. BROWN, Ship's Steward's Boy.

22. *Meteorological Observations.*
    Mr. JOHN ELIOT, C.I.E., F.R.S.   J. BARTLETT, Stoker.
    J. RUSSELL, Chief Stoker.   T. McCARTHY, Stoker.
    C. BUTT, Leading Stoker, 1st Class.   E. PERRY, Stoker.
    H. ROCKETT, Stoker.   G. WOOLSTON, Stoker.
    A. WALLACE, Stoker.   G. GARRARD, Stoker
    G. PRATT, Stoker.   C. MINTRAM, Stoker.
    H. WALLBURN, Stoker.   P. KEEFE, P.O., 1st Class.

Aides-de-Camp to Sir NORMAN LOCKYER, K C.B., F.R.S.
    Mr. W. H. P. BOURNE, Midshipman, R.N.   J. HUNT, P.O., 2nd Class.

Integrating Spectroscope.

This instrument was intrusted to Lieutenant G. C. QUAYLE, R.N., the Navigating Officer, who reports as follows :—

"This instrument was erected in the same hut as the coronagraph, and was also fed by the same mirror. It was placed in a horizontal position, bearing S. 70 E., and

secured on to a flat board resting on three packing cases filled with stones, the
packing cases standing on solid cement bases.

"Three exposures were made—

$$\begin{array}{lll}
\text{1st, 15 secs.} & \ldots & 127\text{--}112 \\
\text{2nd, 90 ,,} & \ldots & 107\text{--}17 \\
\text{3rd, 8 ,,} & \ldots & 12\text{--}4
\end{array}$$

"This instrument was worked by myself and two assistants, one as exposer and
one as timekeeper."

To this it may be added that the instrument had two prisms each, with an angle
of 60°, and faces $2\frac{7}{8} \times 5$ inches, and they were set to minimum deviation for H_γ. The
focal length of the collimating lens was 72 inches, and its effective aperture $2\frac{7}{8}$ inches,
so that a part of the sky, a little more than two degrees in diameter, would be
embraced.

Light was reflected into the slit by the mirror of a 12-inch cœlostat kindly lent to
the expedition by Dr. A. A. COMMON.

The collimator being horizontal, it was set to the bearing S. 70° E. corresponding
to the bearing of the sun at rising. Some little trouble was experienced at first with
the driving clock of the cœlostat, but owing chiefly to the skill of Engineer MOUNTI-
FIELD and W. HOLMES, engine-room artificer, it was made to work very satisfactorily
a few days before the eclipse.

The adjustments of the instrument were thoroughly tested by Mr. FOWLER on the
morning of the eclipse during the partial phases, and successful photographs were
taken ; but although the three exposures were duly made during totality, no results
were secured.

There is reason to fear, however, that the slit was closed before totality, so that we
are left in doubt as to the efficiency of a photographic integrating spectroscope.

The Coronagraph.

Staff-engineer KERR reports as follows :—

"During totality five photographs were taken by this instrument. Throughout
this time the plates were inserted and exposed continuously, the insertions being as
rapidly effected as possible.

"Commencing at totality, the first exposure lasted 5 seconds, the second was a
'snap,' the third lasted 75 seconds, the fourth occupied 22 seconds, the fifth and last
was a 'snap' made at 5 seconds before the end of totality.

"Attention to instrumental details precluded observation of the totality phase of
the eclipse before and after taking the third photograph, but, during the liberal
interval afforded by the comparatively long exposure for the latter, a good view of
the eclipse was obtained."

The plates employed were as follows :—

        (1.) MAWSON's " Castle " . . . . 5 seconds.
        (2.) EDWARDS' " Snap Shot " (isochromatic) . Instantaneous.
        (3.)   ,,   ,,   ,,   . . 75 seconds.
        (4.) Lumière.   Series A 22   ,,
        (5.)   ,,   ,, ... Instantaneous.

The plates were developed with pyro-soda very shortly after the eclipse by Mr.
TURNER and Mr. FOWLER, and very successful results were obtained.   Numbers 1, 2,
3, and 4 are reproduced in Plate 5.

The orientation of the photographs may be readily derived from fig. 3, showing the
orientation of the sun's image in the cœlostat mirror from calculations made before
the eclipse by Mr. FOWLER.

Fig. 3.   Orientation of Sun's Image in Cœlostat, reflected along a horizontal S.E. line.

(n. s. e. w. = north, south, east, and west points.)   N. S. E. W. = points where Sun's axis and equator

Comparing the two shortest exposures with the longer ones, it appears that the
brightest parts of the inner corona shown by the former have no obvious connection
with the long streamers shown in the latter, and, as was pointed out in Sir NORMAN

LOCKYER's preliminary report,[*] the green coronal image depicted by the prismatic cameras corresponds very closely with the image of the inner corona.

Discs.

The arrangement introduced by Professor NEWCOMB in 1878, in which a circular disc is used to protect the eye from the glare of the inner corona, was adopted to facilitate the search for faint extensions of the corona.

The six discs were manned by eighteen volunteers. Of each party of three, one was pointer, one was observer, and the third the writer. A convenient seat being arranged for the observer from which he could comfortably get the disc and sun in line through his eyepiece, he made way for the pointer, who kept the eyepiece in line with disc and sun until the last moment.

On the "Alert" being sounded at ten minutes before totality, the observer was blindfolded, and the bandage was only taken off 10 seconds after totality commenced. He then dictated what he saw, taking the most delicate and faintest effects first, and afterwards going round the moon's disc clockwise to ensure nothing was omitted.

After the eclipse was over, the observers, with the aid of the notes, made a sketch from memory.

The observers had been selected from volunteers, and had been trained from the 5th of January, a period of seventeen days. The method of selection and training was to show on the screen a slide of a corona from a previous eclipse, and allow duration of totality for copying it, and a few minutes after the slide had been removed for touching up from memory.

Lieutenant QUAYLE, who adjusted the discs, reports as follows:—

"The discs varied in size from 6 to 2 inches, and were made of thin wood, painted a dead black. Six discs were used, placed at from 40 to 15 feet from the eye, and covering from 7 to 3 minutes of arc.

"As the altitude of the sun at the time of eclipse reached 53°, some difficulty was at first experienced in finding a suitable site so as to give the necessary elevation, but eventually the inner wall of the fort was selected, and out of a varied collection of spars, uprights were chosen ; these were either driven into crevices in the wall, or fixed with staples and lashed.

"The correct position of each disc was obtained by means of a sextant, prismatic compass, and a measuring tape.

"The eyepieces were suspended on cross-bars, lashed between two uprights about 3 feet apart, thus giving a considerable horizontal motion, the eyepieces themselves supplying the correction for altitude. Lieutenant COLBECK assisted in erecting the discs."

In order to simplify the orientation, the vertex or "top" of the sun was to be

read as north, the "bottom" as south, while the right and left were to be read as east and west respectively, as in ordinary maps. The lengths of the streamers were stated in terms of the diameter of the disc employed in each case.

The Superintendent of the Trigonometrical Branch Survey of India very kindly forwarded six photo-zincographed cards conveniently designed for sketches, and these were used by the disc observers. Each observer was provided with a written form containing the following questions :—

(1.) Longest streamer—
    Length in diameters of disc =
    Bearing =
(2.) Is there a streamer in S.S.W. ?
(3.)   „   „   „ N.N.E. ?
(4.) What do you see in N.E. quadrant ?
(5.)   „   „   S.E.   „
(6.)   „   „   S.W.   „
(7.)   „   „   N.W.   „
(8.) What are your impressions as to colours ?
(9.) General remarks.

The records of the observers are sufficiently accordant, but it is not considered necessary to give the details of the observations and drawings, as nothing was noticed which was not photographed.

It is somewhat remarkable that even the longest streamer was only traced to a little more than 2 diameters by any of the observers who were provided with discs, while Sir NORMAN LOCKYER, Mr. FOWLER, and Dr. LOCKYER agreed in estimating it at not less than 4 diameters.

The most important result, however, is the absence of any record of faint equatorial extensions.

Sketches of Corona, without Discs.

Lieutenant DUGMORE reports as follows :—

"The corona party, consisting of ten men, was selected during the ' Melpomene's ' passage to Viziadrug from Colombo, and from January 14th to the day of the eclipse they were instructed daily, making sketches in the limited time during which totality would last, viz., 127 seconds, from paintings of past coronas, until they attained a high degree of proficiency. These drills were carried out by exposing the paintings on a screen, and while the timekeeper called out every ten seconds, the men made their sketches.

"For the eclipse the party was stationed close to the meteorological shed, and each man was provided with a large sheet of pasteboard with a disc to represent the moon,

and concentric circles of one moon's diameter each marked on it, and also the lines to divide the quadrants. One man, W. PARKHURST, A.B., used the ship's long-distance telescope, and made a sketch of the whole corona. Another man also made a general drawing of the whole corona, using the naked eye only; whilst, of the remaining eight, two sketched each quadrant."

Most of the sketches are excellent, but as they give no certain indications of details which are not shown in the photographs, it is not considered necessary to reproduce them.

The 3¾-inch Equatorial Telescope.

This telescope was used by Sir NORMAN LOCKYER to observe the exact time of second and third contacts to give the signals "Go" and "Over" to the timekeepers. A diagonal eyepiece was employed, in which a silvered reflecting surface could be substituted for one of plane glass by the simple operation of sliding a frame containing these reflectors. For the first 50 seconds of totality he employed this instrument to minutely observe the structure of the rifts and streamers. In his absence the instrument was used by Assistant Engineer H. H. WILLMORE for the examination of the structure of the corona. Mr. WILLMORE took down the notes of Sir NORMAN LOCKYER's observations which are included in his report which follows :—

"From 127 seconds to 70 seconds Sir NORMAN observed through the telescope, and I took the following notes at his dictation :—

"'Distinct structure at poles; many brilliant prominences; much curvature; long horn as in 1896 eclipse; tracery at both poles quite exquisite; four distinct streamers, one of three diameters.'

"At 70 seconds I took Sir NORMAN's place. The corona appeared very different from anything I had seen in books, appearing more homogeneous and less well-defined in its various parts than any of the illustrations I had practised with. At the solar poles plume-like structures of great beauty were observable, intercepted by darker curved rifts. The inner corona was discernible from both the chromosphere and the outer corona, which latter appeared less bright. I particularly noticed a very bright prominence jutting out at the lower right-hand portion of the sun's limb. The outer corona in its main features appeared to consist of four long conical streamers. I could discern no difference in the structures of these streamers. At 17 seconds Sir NORMAN again observed up till the expiration of totality.

"For some 5 seconds after third contact the corona appeared to lose hardly any of its beauty or brilliance. Then it slowly faded, commencing from its more remote portions towards the sun's limb, this phenomena taking another 3 or 4 seconds."

Observations of Stars during Eclipse.

Lieutenant BLACKETT reports as follows on the drills and actual observations :—

"The star observation party consisted of six observers, in my charge, making seven in all. It was found more convenient to tell them off in different parties, as follows :—

"*Lieutenant in charge.*—To make a hasty sketch of the whole, noting times of appearance and disappearance of planets only.

"*Two observers.*—To take eastern semicircle.

"*Two observers.*—To take western semicircle.

"*Two observers.*—To observe stars near the sun.

"The party was supplied with star atlases and photographic copies of a chart showing the stars (from mag. 1 to 4) and planets which would be in the sun's vicinity during the eclipse.

"After some drills with stars, such as making correct drawings of the different constellations (Orion was found a good subject, the three stars in the belt being used as a base to work on), the observers were found to be quite efficient.

"For the actual eclipse, charts on a larger scale were used and nailed to boards, stars being marked down as they appeared or disappeared. These charts were photozincographed at the office of the Trigonometrical Branch, Survey of India, Dehra Dun, and were found handy and most useful.

"It was found that the best position for observing the stars, the sun being at an altitude of 53°, was to lie down full length, the sailor's hat or a round helmet being held up and used to screen the eyes from the bright rays.

"Only a few stars were visible, all of which appeared on the western side of the sun.

"*Venus* was noted particularly both before and after totality, and was the first of the planets to appear and the last to disappear, being seen altogether for 20 minutes.

"*Mars* was visible for a short time during totality, and then disappeared.

"*Mercury* was not seen.

"*α* and *β Capricorni*, though close to the sun, were not visible.

"*θ Aquilæ*, much higher up, came out before totality for a short period, and then as suddenly disappeared, and was not seen again.

"*λ Aquilæ* appeared and disappeared in the same manner and time as *θ*, but appeared again, was visible for a short time, and then disappeared.

"*Another star* was seen by all the party, some of whom seemed to find a difficulty in locating it on the charts. It is placed on the chart as being midway between Venus and Mars by two of the observers, and marked as *π* Sagittarii by the other five ; it was seen only for a short time, and that before totality. This star was about second magnitude.

"It is to be noted that the stars that appeared before totality disappeared, as far

as could be judged, at totality. No stars except Venus were visible for any length of time after totality. All the observations were made with the naked eye."
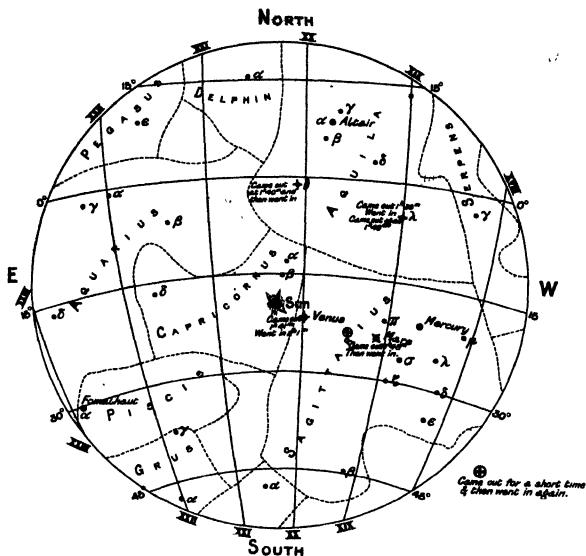
Fig. 4. Chart showing Stars in Sun's vicinity and the Stars actually observed during the Eclipse.

The observations of stars may be conveniently expressed as follows :—

Name.	Appearance.	Disappearance.	Reappearance.
λ Aquilæ	8 m. before totality	Shorly afterwards	At end of totality
θ Aquilæ	7 m. „ „	„ „	
Venus	6 m. „ „	12 m. after end of totality	
Mars	1 m. after beginning of totality	Shortly afterwards	
?	Seen only for a short time before		

It thus appears that more stars were seen just before the commencement of totality than during the actual period of totality; that is, they were logged as disappearing just before the total phase commenced. A similar observation was made by Admiral Don Ulloa in the eclipse of 1778.* The body seen between Venus and Mars was certainly not a star, although its behaviour was similar to that of θ Aquilæ. Sir Norman Lockyer has suggested that it may have been a collision of meteor swarms in the sun's neighbourhood.

The chart summarising the observations of stars is reproduced on a small scale in fig. 4 (p. 171).

Hand Spectroscopes.

Three small integrating spectroscopes were provided for eye observations. Two of these were direct-vision pocket spectroscopes, while the third was an ordinary prism of 30° attached to the end of a tube about 10 inches long, which was provided with a slit at the other end.

Lieutenant Colbeck was in charge of these observations, and the preparatory work consisted of observations of the spectra of various salts when burned in a spirit lamp flame, in addition to joining in the general drills.

Sir Norman Lockyer has examined the records, and suggests that Lieutenant Colbeck's observation at 25 seconds before totality possibly indicates a faint continuous spectrum, with the hydrogen C and F bright. Further, that the lines seen during totality were probably :

<div style="padding-left:3em">

Yellowish-green, seen by three observers—Corona line.

Blue   ,,   ,,   ,,   ,, —Hydrogen line (F).

Red, seen by one observer—$\begin{cases} \text{He } 7065\cdot5. \\ \text{He } 6678\cdot2. \\ \text{Hydrogen line (C).} \end{cases}$

Yellow   ,.   ,, — Helium line (D_3).

Green   ,,   ,, —$\begin{cases} \text{Mg group } (b). \\ \text{Iron, 5018.} \end{cases}$

</div>

After totality the Fraunhofer lines reappeared in the following order :—

<div style="padding-left:3em">

Line in the orange (D).

Lines in the green (? b, e).

Lines in the violet and blue (? F, G).

</div>

Prisms for observation of Ring Spectra.

These observations were chiefly intended to supplement the photographic work of the prismatic cameras, by providing a record of the phenomena in the green and

* 'Phil. Trans.,' 1799, p. 105.

red part of the spectrum. Two of the chief questions on which such observations might be expected to throw light were (1) whether the green ring image of the corona corresponded with the outline of the corona ; (2) whether hydrogen was present or absent from the corona, as indicated by the presence or absence of a coronal ring at Hα.

In addition to the visual observations, an experiment was made as to the efficiency for photographic work of a small camera fitted with a transmission grating.

Senior Engineer MOUNTIFIELD, R.N., who was in charge of this work, reports as follows :—

"The instruments used in these observations were five in number, and consisted of (1) A direct-vision hand spectroscope-prism section only of about 1¼ inch diameter, giving a spectrum of about 10° visual angle.

" (2) A 'meteor spectroscope' of about ¾ inch diameter, giving a spectrum of about half the length of the first.

" (3) A prism with a refracting angle of 60°.

" (4) and (5) Prisms with refracting angle of 45°.

"Instruction was given during the 14 days preceding that of the eclipse, the general nature of continuous and discontinuous spectra explained and illustrated by means of small, straight, and circular slits in tinfoil placed before the flame of a spirit lamp, and observed through the instruments to be used.

"The probable variations in the actual spectrum to be observed were also explained.

"The observers were instructed to note specially shapes which appeared to be repeated in different parts of the spectrum, and the number of parts in which they were repeated.

"The actual observations were commenced at 12.30, local mean time. Each observer was provided with an assistant, who noted down on a form, ruled in columns —one for each 10 seconds of totality—what was seen.

"At the commencement of observations, the greatest breadth of the sun's crescent was about one-sixth of its diameter, and only a bright continuous spectrum could be seen.

" My own observations were as follows :—At 12.33 the crescent had become so narrow that two curved dark lines appeared faintly in the red.

"Other lines soon appeared, and gradually increased in distinctness, until about 12.43, when those in blue and green began to disappear, followed by yellow, green, orange, and red in the order named. No dark lines were distinguishable by me half a minute before totality.

"At the commencement of totality a number of bright arcs flashed out suddenly —the number could not be estimated reliably, but two arcs in red, one in yellow, one in greenish-yellow, one in green, and one in blue were noted.

" In from 1 to 1½ seconds these had faded, leaving a few irregular bright arcs. These

consisted of a patchy arc, extending over about 150° of the sun's circumference, on the N.E. Nearly continuous, but varying in height from about 1' to ½', broken near the equator, continued on the S.E., where there was a peculiar whorl-shaped prominence about 2 feet in height. This was shown complete in two red, one orange, and three yellowish-green images (close together). In two greenish-blue images and one blue one, patches only of these arcs were observed, very disconnected in the north-east, and only slightly more continuous in the south-east, where the arcs were broadest, thus appearing to consist of the bases of the greater projections. The yellowish-green arcs were very bright, the red somewhat less so, the remainder being relatively faint. The red arcs appeared to be the most complete, the yellow-green slightly less so, the orange still less continuous, the blue and others very incomplete.

"At about 65 seconds after the commencement of totality, rings were observed, discontinuous, but extending round the whole of the sun's circumference, of the same general character as the arcs before mentioned, but slightly larger in diameter. Two rings in the red appeared broader and more regular than the rest; three in the yellowish-green, very close together, were brighter, but not so extended as the first; and one in the orange was much less complete and fainter, but similar in shape to the red and yellow-green. These rings were represented in the greenish-blue and blue by only a few isolated patches, principally in the north-east.

"Seventy seconds after the commencement of totality, these rings had disappeared on the east side, leaving arcs surrounding the west side, similar in appearance, omitting the whorl-shaped prominence, to those first observed. These arcs gradually became broader and more connected, the blue and green-blue becoming nearly continuous.

"Two minutes after commencement these rings were all much more connected and broader, while preserving their relative breadth and brightness.

"About 125 seconds after commencement a number of bright arcs suddenly appeared, covering the structure of the arcs previously described. The number seen might be estimated at about twenty; but this is not considered reliable, as the arcs rapidly faded; in about one second they were obscured by bright continuous spectrum.

"In half a minute two dark lines appeared in the red, one in the orange, a broad one in the yellowish-green, one in the green-blue, and one in the blue, gradually becoming broader and losing definition until five minutes after totality, when they were not distinguishable.

"During the whole period of totality, a continuous spectrum was observed, faint at the ends, but increasing in brightness about the yellow and green, where it was nearly half as bright as the bright arcs observed in that part of the spectrum. No details of structure were observed in this spectrum, but its width was about twice the apparent diameter of the sun.

" No bright arcs were observed before or after totality.

" The records of the other four observers are similar, though somewhat less complete. The appearance of dark Fraunhofer arcs as the crescent became thinner, their replacement by light arcs at the commencement of totality, the change of the arcs from the east to the west side, and the reappearance of dark arcs was noted by all.

" The half-plate camera, with diffraction grating, was manipulated by G. GRAY, chief stoker. It was supported on a stand near the finder of the 6-inch prismatic camera, a portion of the light from the siderostat of that instrument being intercepted and reflected into the lens of the camera by a right-angle prism. A photographic diffraction grating was fitted to the front of the Cooke lens of the camera, and the whole was so arranged as to show three orders of the spectrum above and one below the image of the sun. Ordinary plates were used, as isochromatic ones of the proper size were not available. The exposures were made by means of a sliding shutter fitted in front of the grating. Four plates were exposed, with varying exposures, namely, 10 secs., 65 secs., 15 secs., about $\frac{1}{4}$ sec. For all the exposures $F/A = 6.5$."

Remarks by Sir NORMAN LOCKYER.

In discussing these observations, it is important to distinguish between rings produced by the light of the corona and those produced by the succession of prominences which surround the dark moon. An examination of the photographs taken with the prismatic cameras shows that the conditions with regard to the visibility of the chromosphere and prominences were as follows :—

During first 30 seconds of totality. .	Chromospheric arcs in N.E , prominences in S.E., N.W., and W.
During next 60 seconds	No chromospheric arcs visible, but the succession of prominences appear as dotted rings.
During last 30 seconds of totality . .	Chromospheric arcs in S.W., prominences in S.E., N.W., N., and E.

The green ring of the corona was visible throughout totality, and must have been visible to all the observers. Its variation from the fragmentary rings, due to chromosphere and prominences, however, appears to have almost escaped notice. None of the observations suggest the presence of a red coronal ring, so that there was no indication of hydrogen in the corona.

The flashing out of bright lines at the commencement and end of totality was observed, and also the spectrum of the chromosphere and of the largest prominence. The lines seen in the latter were probably as follows :—

In red $\left\{\begin{array}{l}\text{He 7065 or 6678.}\\ \text{H}_{\alpha}\text{ (C).}\end{array}\right.$

Orange . He (D$_3$).

Yellowish- $\left\{\begin{array}{l}\text{Enhanced Fe 5316·79 (1474 K).}\\ \text{``Cool'' Fe (E).}\\ \text{Mg (2).}\end{array}\right.$
green

Greenish- $\left\{\begin{array}{l}\text{Enhanced Fe 5018.}\\ \text{,,}\qquad\text{,,}\quad 4924.\end{array}\right.$
blue

Blue . . H$_\beta$ (F).

The appearances described by Mr. MOUNTIFIELD in the middle of totality were clearly due mainly to the fragmentary rings of prominences which were then visible, and do not represent the spectrum of the corona. The observations do not give any information as to the relation of the green coronal ring to the form of the corona, but information on this point is given by the photographs taken with the prismatic cameras; these show that the green ring resembles in form the inner corona, and has little or no relation to the longer streamers. The photographs taken with the small camera show the principal chromospheric and prominence lines very distinctly in two orders of spectra, representing on a small scale what was photographed with the larger instruments. No. 2 is especially interesting, as showing in one of the first order spectra a well-defined coronal ring, more refrangible than K, which is probably identical with the line recorded by Captain HILLS at λ 3801. Traces of other coronal rings are also seen. The first order spectrum of No. 3 shows an arc of the coronal ring λ 4231 in a position corresponding to the north-western limb of the sun.

The Polariscope.

The instrument employed for observations of polarisation phenomena was that used by Sir NORMAN LOCKYER in India in 1871. It consists of a telescope having an object glass 1·4 inches in diameter and 14 inches focus, with a double eyepiece; one contains a Savart and the other a biquartz. A simple rotation of the eye end sufficed to readily bring either of the eyepieces into position. The eyepiece of the biquartz, which was alone employed, gives a magnification of about nine times, and a field of about two degrees diameter. Dr. NOLAN states his results as follows: —

"With the junction of the crystals vertical the corona was seen of a uniform bright silvery-steel colour; the large streamer in the S.W. was followed to its end; it became less bright and distinct from the centre. At E.S.E. a prominence was noted, bright glowing red. The interspaces were seen of a dark leaden blue, and were uniform.

"On rotating the crystals through 180°, no change of colour was noted; the contrast between the colours of the streamers and rifts became more marked, and an

increase of intensity in the colour of a rift between N.W. and N.N.W. was noticed. Colours, chiefly violet, were noted in the S.W. just at the end of totality."

Landscape Colours.

Lieutenant Dugmore has supplied the following statement as to the work in this section :—

This party consisted of six—one officer and five men—and was exercised every day from January 12th to January 21st inclusive.

The period during which the landscape most resembled the landscape of an eclipse was between 5 and 10 minutes after sunset, and this was the time selected each day for practice.

For the eclipse four of the party were stationed on a small hill inside the Fort of Viziadrug, and one of them was told off for each quadrant of the horizon, while the other two made notes of landscape in all directions from the " Melpomene."

At the beginning of the eclipse little change took place in the appearance of the landscape. It gradually grew darker till about 5 minutes before totality occurred, when it appeared as in a very bright moonlight night, and continued so up to the instant of the commencement of totality. Then the land became a dense black colour, except the distant hills, which assumed a delicate neutral tint. The heavens darkened to a very deep blue ; the sky by the horizon became pink and yellow, with bright blue above the streaks, except in N.W., where a neutral tint appeared under a yellow streak. The sea changed from a bright greenish-blue to a rich dark blue, except the calm spots, which were of a very pale blue tint. The wind changed from east to west at about 8 minutes before totality, it being almost calm during the phenomenon. There were no clouds.

Landscape Cameras.

A number of ordinary cameras being available, it was determined to utilise them in attempts to secure records of a variety of phenomena.

Mr. A. W. Turner, the photographer sent down by the Survey Office at Calcutta, undertook to give the men the necessary instruction, and himself took charge of one of the instruments.

A 10″ × 8″ camera was placed in position to photograph the corona by using a spare part of the coelostat mirror, a small diagonal mirror being introduced so that the camera did not interfere with the coronograph and integrator. Two exposures were made by Mr. Turner during totality, one with an exposure of 100 seconds, and the other with an exposure of 20 seconds. The long exposure gave an excellent result, the great streamer in the south-west being depicted through a distance of three lunar diameters.

A 5" × 4" camera, with Zeiss lens, operated by J. Collins, Chief Stoker, was also utilised for photographs of the corona. This was fixed in a convenient position, and had no arrangement for following the sun during exposure. Three exposures of 15 seconds each were made during totality, and one photograph shows very clearly the form of the corona and the relative lengths of the four streamers. On this negative the diameter of the sun is 0·05 inch, and the longest streamer, in the south-west quadrant, extends 0·25 inch from the moon's limb. Photographically, therefore, this streamer was five lunar diameters in length. Another exposure of 10 seconds, 5 seconds after totality, shows the crescent strongly solarised, and a faint representation of the corona on the opposite side.

A 5" × 4" folding Kodak, operated by J. Kearney, Leading Stoker 1st class, was set in a position to include the corona and a portion of the wall of the fort in the field of view. Three exposures were made for periods of 20 seconds each. The first began just after totality had commenced, and gives a good representation of the corona—the two longest streamers extending only one lunar diameter. The second photograph shows clearly the lower corona, notwithstanding the fact that part of the sun had begun to appear. During the third exposure the camera was slightly shaken, so that the image is duplicated.

Two cameras fitted with instantaneous shutters were set up with the view of photographing the shadow passing over the landscape in case its boundary should be sufficiently well defined. One was a 7½ × 5, and the other a half-plate, and they were operated respectively by E. Gyngell, A.B., and H. Childs, Chief Stoker. It turned out that the shadow on its approach appeared very gradually, so that the plates exposed simply resulted in under-exposed photographs of the general landscape with no features of interest.

The images of the sun seen inside the various observatories which were covered with matting were so striking throughout the preparations that it was decided to attempt to photograph them during the partial phases of the eclipse.

A quarter-plate camera for this purpose was accordingly placed in charge of E. Cross, Leading Stoker 2nd class. A bamboo screen was erected, with its plane roughly perpendicular to a line directed to the sun, and behind this a white cloth was spread to receive the images. Three successful exposures were made with this instrument.

Shadow Phenomena.

The party which was organised for the observation of the shadow phenomena other than the "shadow bands" was in charge of Lieutenant Blackett, who reports as follows :—

"The observers state that the approaching shadow could not be seen, though carefully watched for. During totality the boat with mast and large flag could

scarcely be seen ; it was anchored 2 miles off, and the height of the eye above sea-level was 70 feet.

"The darkness at totality was of about bright twilight. A cool westerly breeze came in suddenly, which made the air chilly and the temperature fall considerably."

Observations of Shadow Bands.

Staff-Surgeon NOLAN, R.N., observed these phenomena with the help of two assistants. He reports as follows :—

"To observe these a large white tablecloth was spread on a flat piece of ground, and the corner of a wall which gave two planes at an angle of about 115° to each other was whitewashed. The angle of these vertical planes was towards the sun. Two square heavy rods were also provided, 3 feet long, the centre foot being painted black, the ends white. These were to indicate the direction in which the shadow-bands travelled ; they were to be placed on the white sheet parallel to the direction of the bands, and their exact position afterwards determined. One, marked ' Before,' was to be used before totality, the other, marked ' After,' was to be used after. The black foot in the centre was to assist the eye in estimating the breadth of the shadows and interspaces.

The bands commenced to appear at 12ʰ 41ᵐ 35ˢ.

> „   ceased   „   at the second contact.
> „   reappeared at the third contact.
> „   ceased altogether at 12ʰ 51ᵐ 42ˢ.

None were observed during totality, though looked for. As they continued to the very moment of the second contact, and reappeared as rapidly at the third, it was thought that possibly the darkness rendered them invisible.

" *The Direction of Travelling.*—The rods showed that the general direction of the bands before totality was to the west, and the first rod was placed pointing N. 88° W., but the direction of the course of the bands gradually veered round 40°–45° to the south, and remained in this direction for the last two-thirds, or perhaps a little more, of their duration before totality. It was also noted by Private HESTER, who was specially told off to observe the direction and place the rods, that for a few moments before the marked change in direction took place to the *south*, the bands seemed to waver and changed towards the *north* 10°–15° three or four times. The direction of the bands after totality was to N. 60° E., and was constant. The bands, therefore, for most of the time before totality, travelled in exactly the opposite direction to those after totality. This was observed on the white flame, and noted separately by three observers, S. B. S. PURKINGTON, Private HESTER, and myself. In this connection, note a short report below, given by Mr. MOUNTIFIELD, who incidentally observed the bands.

"*Rate of Movement.*—It is certain that the bands increased rapidly in speed from their first appearance to their disappearance at totality, and decreased as rapidly on their reappearance till they finally ceased. S. B. S. PURKINGTON. who was specially told off for this observation, states that they moved too rapidly for him to count them. The other two observers at the white sheet agreed. An observer who noticed the bands at another part of the camp incidentally gave their speed as 4–6 miles per hour ; the exact time of this observation is not known accurately ; it was within 10 minutes before totality. Mr. MOUNTIFIELD in his note below seemed to have seen two sets of shadows ; these were not observed on the sheet at the time of this observation ; I was employed elsewhere.

"*Width of the Bands.*—The three observers at the white sheet agreed in estimating the bands at $\frac{1}{2}$ inch to $1\frac{1}{2}$ inch broad, with one exception (see later). Also that they varied considerably, and were broadest at the commencement of their appearance and when they ceased to appear.

"*Width of the Spaces.*—The observers agreed that these were very variable, much larger, 4 to 6 inches, at the first appearance of the bands, and decreasing to totality when they almost equalled the bands. At the end of totality they were small, as at the commencement of totality, and gradually increased in size until the bands were actually intermittent. At about 1 minute after totality there was a long inter-mittency during which a large band, about 2 inches broad, passed by itself in a most striking manner ; this was noticed separately by each of the three observers.

"*General.*—The bands were described as ' narrow mottled shadows,' ' shadows of bands of flocculent wool,' ' ripples of smoke,' ' ripples of water,' &c. They appeared at times to break in two."

Note by Mr. MOUNTIFIELD.—" About 5 seconds after third contact light shadows were observed of a rippling character, comparing with waves, the crests being in a N. and S. direction, the transverse velocity about 6 miles an hour to westward. Superposed, appeared to be a much fainter series drifting eastward. Distance from crest to crest 6 inches to 8 inches. Motion was irregular, as though an oscillating motion were acting in conjunction with the onward speed of 6 miles an hour.

"In compliance with a request for further information on one or two points, Dr. NOLAN states that the bands moved in a direction perpendicular to their lengths, and that the rods were placed in the direction of motion of the bands—pointing in the direction towards which the bands travelled one after the other. In order to remove all ambiguity the accompanying sketch (fig. 5) was appended, in which the arrow indicates the direction of the rod, and the parallel lines show the direction of the bands themselves."

The following further description of the bands themselves was also given :—

"The bands were simply narrow bands of shadow, parallel to each other, and following each other like ripples on water, and seemed to slightly change their direction, as described in the report, as a whole. The shadows were observed on the

whitewashed wall, as indeed they were all over the ground, but attention was con-
fined to the sheet."
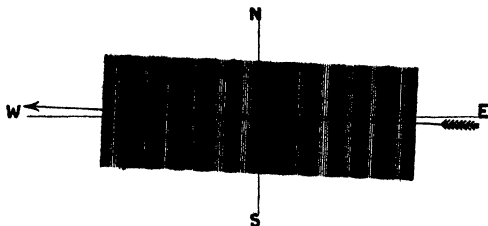
Fig. 5. Observations of Shadow Bands.

It is interesting to note further that Staff-engineer KERR draws attention in his
report to the presence of shadow bands during totality, travelling in a westerly
direction.

Kinematograph for Eclipse.

The Marquis of GRAHAM has drawn up the following statement with regard to
the work of this instrument :—

"By the kind permission of Sir NORMAN LOCKYER and Captain CHISHOLM-BATTEN
I was allowed to accompany the eclipse party as a volunteer. I therefore took out
with me two kinematographs with which I hoped to secure a record of the eclipse
itself as seen by the eye, and one also of the swoop of the shadow across land
and sea.

"The eclipse kinematograph was provided by Mr. HILGER with a fine photographic
objective 3 inches in diameter and 21 inches focal length. The shutter arrangements
were completely altered in order to suit this large aperture, and this also necessitated
certain alterations in the mechanism for the film. The handle gearing was altered to
provide either for 'time' or 'running' exposures, and a Thornton-Pickard shutter
was adapted to the lens. Finally a model cœlostat, with a 4-inch speculum, designed
by Mr. FOWLER, was provided to reflect the sun's rays into the lens during the whole
eclipse.

"Owing to various delays, I did not receive the instrument until December 9th,
so that preliminary experiments were out of the question. During the voyage to
Colombo, however, both instruments were examined, and many defects came to
light. But for the timely assistance and skill of Mr. HARRIS, Chief Engineer of the

'Lusitania,' and Mr. ROBERTSON, Fifth Engineer, the whole undertaking would have been a disaster from the outset.

"A very suitable site was allotted for the eclipse kinematograph, and its erection and adjustment were proceeded with the day after our arrival at Viziadrug.

"Our plan of work was as follows :—Beginning 5 minutes before first contact, and continuing to the second contact, exposures were made at the rate of four a minute, giving 380 pictures in all. From second to third, that is, during totality, twelve exposures of 10 seconds each were to be made. At third contact exposures at the rate of four a minute were to be resumed, and continued until five minutes after the fourth contact. The estimated total number of exposures was therefore 772, which would use 52 feet of film on the scale of fifteen photographs per foot.

"On the day of the eclipse, the kinematograph was firmly screwed to the table after loading, and, as arranged, work commenced 5 minutes before first contact. About the time of contact, the shutter broke down and 10 minutes were occupied in repairing it. A little later the shutter spring broke, and again a few seconds were lost. All then went well until near the end of the eclipse, when the rubber ball of the shutter began to work irregularly

"As I had no means of developing the films in India, they were sent to London for development by the Eastman Co. Unfortunately, probably through leakage, the film was too much fogged to be of any use for the purpose for which it was intended. At the same time, several parts of the film were clear enough to show that the exposures had been successfully made, and that the cœlostat performed its work satisfactorily. With an instrument better adapted to the climate and with more precautions for keeping out stray light, there is every reason to look with confidence for a successful result in a future experiment."

Kinematograph for Shadow.

In his general report, Lord GRAHAM further states that this instrument was provided with a new Dallmeyer lens of wide angle, and was set up on three packing cases filled with stones, resting on one of the outer bastions of the fort.

Seventy-two exposures were made, using $4\frac{3}{4}$ feet of film, but as the advent of the shadow was not strongly marked, the film possesses no feature of interest.

Contact Observations.

In addition to the telescope employed by Sir NORMAN LOCKYER for the observation of contacts, two smaller instruments, primarily acting as finders for the two prismatic cameras, were employed for a similar purpose. The telescope working in connection with the 6-inch prismatic camera was in charge of Lieutenant DE WET, and the time records of the contacts, as indicated by the chronometer, were made

by Gunner's Mate C. IRONSIDES. The other instrument was in charge of Lieutenant PERCIVAL JONES, the times by a deck watch being recorded by A. WILKINS, Lieutenant DE WET reports as follows on the instrument and observations :—

"The instrument used for observing the contacts and cusps was a 2-inch telescope, which acted as finder for and was fed by the same mirror as the 6-inch prismatic camera. A screen, 10 inches in diameter, formed by pasting paper on a copper disc, was fixed inside a box in such a way that the disc could be turned round. On the paper a circle 4 inches in diameter was drawn, and also lines marking 90° and 45° of arc. This screen was placed in such a position, that, when focussed, the finder would throw an image coincident with the circle on the screen. It was calculated that the cusps would form an arc of 90° 25 seconds before totality, and an arc of 45° 5 seconds before totality, when the 6-inch and 9-inch prismatic cameras were to commence taking their series of photographs. The observations of contacts and cusps were very difficult to take accurately, owing to the excessive 'boiling' of the sun's crescent as shown on the screen, especially when nearing totality. This 'boiling' seemed to increase simultaneously with the springing up of the sea breeze.

"The observed times of contact, corrected to local mean time, were :—

1st contact . . .	21 d. 23 h.	12 m.	31 s.
90° (cusps) . . .	0　0	45	31
45°　,, . . .	0　0	45	50
2nd contact . . .	0　0	45	53
3rd　,, . . .	0　0	47	57
4th　,, . .	0　2	14	30

Long. of Viziadrug 4 h. 53 m. 16 s. E."

No separate report has been received from Lieutenant PERCIVAL JONES, but shortly after the eclipse a memorandum of the recorded times was made by Mr. FOWLER.

The two sets of observations, corrected to Greenwich mean time, may be summarised as follows :—

Phase.	DE WET.			JONES.			Mean.		
	h.	m.	s	h	m.	s.	h.	m.	s
1st contact . . .	18	19	15	18	18	50	18	19	25
Cusps, 90° . . .	19	52	15						
,,　45°. . .	19	52	34						
2nd contact . . .	19	52	37	19	52	36	19	52	36·5
3rd　,, . .	19	54	41	19	54	42	19	54	41·5
4th　,, . .	21	21	14	21	21	0	21	21	7

It will be seen that, according to Lieutenant DE WET, totality lasted 2 minutes 4 seconds, and according to Lieutenant PERCIVAL JONES 2 minutes 6 seconds.

The experience gained during this eclipse indicates that the determination of definite intervals before totality by the observation of the cusps is of great value.

Observations on Natives, Animals, &c.

Three observers were told off to note the behaviour of the natives, animals, &c., during the eclipse. The notes were collected by Lieutenant BLACKETT, who has summarised them as follows :—

"The natives, so far as was observed from the fort, were not much disturbed by the eclipse. They had been previously warned not to light fires, the smoke of which might interfere with the observations, and also not to make a noise. On the beach a few—according to a matter-of-fact observer thirteen—ran into the sea, and bathed, as is their custom on these occasions ; they also set up a discordant yell for 3 minutes. It was ascertained afterwards that a large proportion of the natives retired to another beach about 3 miles away out of sight and hearing of the fort, and here, undisturbed by the local police, were able to bathe and howl *ad infinitum.*

"This district being particularly devoid of land birds, owing to the barren nature of the soil, not much was gained from observations. A few small birds twittered a good deal during the partial phase, and especially about 10 minutes before totality, but at 7 minutes before totality a sudden silence fell on them all. Some crows (*Corvus hiemalis*) were observed to roost, and martins were seen to be flying unusually high. Bats were not observed to be disturbed in any way, though carefully looked for in likely places. A few snakes came out as they usually do towards dusk."

A snake was also seen by Mr. BARNETT about 7 minutes before totality.

Lieutenant DUGMORE states that "the gulls, which seemed somnambulant soon after first contact, rose at the commencement of totality and flew off silently towards the light in the south, returning, however, in a few seconds while totality was still on, and again settling on the water, where they remained until about 45 minutes after, when they grew less lively and commenced fishing operations. The live-stock on board did not take much apparent notice."

Observations on animal life, &c., were also undertaken by Mr. E. H. AITKEN, Assistant Collector of Salt, who was good enough to supply the following notes for insertion in this Report :—

"The darkness was neither so intense nor so sudden as I had anticipated, and the effects which it produced on living things were less striking than those described in accounts of former eclipses. At 12.40 a cock began to crow, and it continued doing so at intervals till some minutes after totality was over. Up to 12.42 kites and swallows were flying about as usual, and little birds of several species were chattering. I especially noticed the toot of the little barbet, commonly

known as the 'Coppersmith' (*Xantholæma*), and the shrill cry of the little striped squirrel. About 12.45 several butterflies began to settle in sheltered nooks about a haystack near me, and I saw a pair of kites plainly seeking a place to roost.

" At this time my attention was attracted by a rapidly undulating shadow on the ground. The general effect was like that produced by the shadow of hot, ascending smoke, but more regular. The undulations ran from east to west. I did not at the time note the apparent distance between them, but should say from memory that it was about half a foot. I think this lasted till put out by the darkness, but I am not sure, as I was attending more to other things.

" During the period of totality I heard the voices of birds, but immediately after it there was a striking silence for a short time. This was noticed by a friend as well as myself. Then a large number of crows which had collected to roost in one tree, after their manner, started into the air with much cawing and flew off, and in a few moments everything was going on as if nothing had happened.

" A man whom I set to watch a roosting tree of bee-eaters (*Merops*) reports that the birds gathered to the tree as the darkness came on. Another reports that the fowls in his yard went about all the time as if nothing unusual were happening. A third, who was fortunate enough to find a troop of monkeys (which I had specially directed them to look for), says that they went about feeding until the darkness came on, when they stopped suddenly and sat motionless and silent, each in its place, until the light returned."

PART III.—SPECTROSCOPIC OBSERVATIONS MADE WITH A 6-INCH SHORT FOCUS LENS AND A GRATING.

By Professor A. PEDLER, *F.R.S.*

Introductory.

The Government of Bengal, having been informed that Sir NORMAN LOCKYER, the head of one of the Eclipse Expeditions coming from England, would be glad if it could be arranged that I could be deputed to Viziadrug to help in the observations of the Total Eclipse of the Sun, permission for my deputation was very kindly and willingly given. Hence I left Bengal on January 7th, and arrived in Bombay on the 9th, but I was detained there until the 12th in connection with certain arrangements required by the Expedition.

I left Bombay on the 12th, and arrived at Viziadrug on the 13th of January.

After my arrival, Sir NORMAN LOCKYER placed in my charge the 6-inch equatorially mounted refracting telescope with a grating spectroscope attached, the Rowland grating having about 16,000 or 17,000 lines to the inch. This instrument had been erected in position under the superintendence of Mr. FOWLER previous to my arrival.

Four assistants were detailed to help in working the instrument. Mr. STEEL, R.N., was deputed to work the fine adjustments of the telescope, so as to bring the desired portion of the image of the sun on to the slit of the spectroscope. Mr. ROSS, who had helped in the erection of the telescope, was selected as time keeper; VANSOME was deputed to have charge of the lamp, and give light when required, and BROWN acted as clerk, and recorded all observations that were dictated to him.

Drills were commenced from the day after my taking over charge of the instruments, and they were repeated five or six times each day so as to give smoothness and quickness in the work, and confidence to those employed in it.

At Sir NORMAN LOCKYER'S suggestion, it was arranged that during the eclipse an attempt should be made to repeat certain observations made by him during the total eclipse of 1882 in Egypt, when what are called the enhanced lines of iron, due to iron vapours under the influence of exceedingly high temperature (as given by a powerful spark) were seen short and bright 7 minutes before totality, while 2 minutes before totality some iron lines usually found in the spectrum of sunspots (i.e., due to iron vapour in a cooler condition) were observed long and thin. During totality it was also arranged to examine the spectrum of the corona, and specially to search for any evidence of the existence of carbon, iron, magnesium, hydrogen, &c.

It was arranged that the spectrum of the *first order* should be worked with, and it was of course necessary that only a portion of the spectrum should be observed. The portion selected, which was well visible in the field of the observing telescope of the spectroscope, included easily from 1474 K to beyond " F," both being well within the field of view.

The Reference Spectrum.

Previous to the instrument being brought out from England, some excellent photographic reference spectra had been prepared by Mr. FOWLER under Sir NORMAN LOCKYER'S direction, and these showed the bands of carbon, the quartette of iron lines rather less refrangible than F, the triple magnesium lines *b*, and also F very distinctly. This reference spectrum was fitted into the eye-piece of the observing telescope of the spectroscope, and the lines acted as the means for absolutely identifying these elements should these lines be visible in the corona, and also were used for reference to decide the position of any other lines which might be observed in this part of the spectrum.

Programme of Work.

The spectroscope being rigidly attached to the telescope, it had to remain in one position in relation to the telescope throughout the time of observation during the eclipse. Hence the following was the programme of work adopted for the eclipse, and this programme was practised daily in the numerous drills before the day of the eclipse.

At 25 minutes before totality, the image of the crescent of the disappearing sun was to be thrown by Mr. STEEL, R.N., on the slit of the spectroscope, so that the north edge of the sun was on the slit, and the cusp just out the slit, which was placed almost perpendicularly to the limb of the sun. The cusp was then kept by the driving clock and by the use of the fine adjustments by Mr. STEEL on the desired part of the slit until the call of "totality." During this period the time keeper, Mr. ROSS, was to give the time, calling 10, 7, 5, 3 and 1 minutes before totality, so that BROWN, the recorder, could note the time against every observation which was made by myself at the observing telescope, and dictated to him.

At the call of "127 seconds," or the commencement of totality, Mr. STEEL was to work the fine adjustments of the telescope to bring the slit of the spectroscope gradually round the edge of the dark moon, and to place the slit tangentially to the east of the equator of the sun at a distance of about one-tenth to one-tenth of the diameter of the dark moon from its edge. A card with lines marked on it at fixed distances was placed over the slit of the spectroscope to enable this position to be judged with considerable accuracy. This one-tenth to one-twelfth of the diameter of the moon was taken to the position of the "*inner* corona."

Thirty seconds after, or at 97 seconds call, the slit was to be moved slightly outwards from the edge of the moon, so that it was about one-sixth or one-fifth of the moon's diameter away from its edge, and this was taken as the position of the "*outer* corona."

At 67 seconds call I was to leave the observing telescope, and Sir NORMAN LOCKYER was to observe any spectrum of the outer corona from 62 seconds to 32 seconds call, and at 32 seconds call Mr. STEEL again had to move the slit on to the inner corona within about one-tenth or one-twelfth of the diameter of the moon from its edge, when Sir NORMAN LOCKYER would observe the spectrum of the "inner corona."

· At the call of "totality over," Mr. STEEL was to place the slit perpendicularly on the north point of the cusp of the reappearing crescent of the sun, and was to keep it at the cusp so long as observations were required, the calls of time after totality being similar to those before totality.

Mr. STEEL did his work very successfully, and so did all the other three assistants, and BROWN took down every word I dictated to him.

The Observations made.

The following is a statement of the notes written down by BROWN during the time of observation at the eclipse. These I recopied and made complete on the afternoon of the day of the eclipse, so that no details should be lost by allowing any interval of time to elapse between the observations and their full record.

12 *hrs.* 20 *min.*—Commenced observing edge of the sun close to the point of disappearing crescent. Shortly after "F" came into view, bright and short.

12 hrs. 23 min.—" F " line bright and short. Confirmed by Sir NORMAN LOCKYER. " F " line commenced to extend over bright limb of sun, and continued well visible.

At call—10 *minutes before Totality.*—" F " bright and short, beyond the edge of moon, " F " becoming much longer ; seen clearly also over the bright crescent of the sun. No other bright lines visible.   " F " long and bright.

7 minutes before Totality.—" F " bright and shorter on faint continuous spectrum, with usual Fraunhofer lines. No other bright lines visible.

Iron lines looked for particularly, but not seen.   Carbon-green bands looked for particularly, but not seen.

" F " can be seen bright on the disappearing crescent of the sun.

" F " becoming decidedly brighter.

" F " long and bright.

No other bright lines visible.

Call 5 minutes before Totality.—" F " bright ; also distinctly bright on crescent of the sun.

" F " brighter near the sun, also bright beyond the moon.   At this time the bright " F " line was again confirmed by Sir NORMAN LOCKYER, and was the only bright line seen by him.

4 minutes (about) before Totality.—" F " bright and long.

Magnesium (" *b* ") distinctly bright.

" F " is much brighter than " *b*."

Call 3 minutes before Totality.—" F " bright and long.

Magnesium (*b*) bright and long.

Magnesium and " F " continue bright and long.

From 2 to 1 minutes before totality close to point of disappearing crescent.

> 4871 Iron line (ÅNGSTRÖM) rather long and bright.
> 4890   „   „   bright.
> 4918   „   „   not clearly seen, but apparently a line at that place.
> 4956·5 „   „   bright.

Just before call of—

1 *minute before totality* I saw 4871, 4890, and 4956·5 iron lines distinctly bright at point of disappearing crescent of sun.   No enhanced iron lines at 4924 and 5018 could be seen.

At call 1 *minute before totality* I reported " F " continues very bright.

Call of " *Go*" *for totality*, or 127 seconds call, the slit was removed from its former position, and placed tangentially on the " inner " corona east of equator of the sun by Mr. STEEL.   In passing along the edge of the dark moon eight or nine bright lines at least flashed into the spectrum between " F " and " *b*," most of them being in the half of the field nearer to " F."   They were, however, visible for only a few seconds, and appeared to " flash " in rather than remain steady, and they practically

disappeared when the slit was brought to rest on the inner corona. No measurements could therefore be made with the reference spectrum. When at rest on the inner corona in the spectrum, " F " was seen long and bright ; 1474 K (judged by iron lines 5323 and 5327 in reference spectrum) longer and brighter than " F," and practically filled the whole length of the slit ; four, or possibly five, bright lines very feeble, very close to, if not identical with, the positions of iron quartette, but not sufficiently bright to measure very accurately, and there was not sufficient light to give very good indications of the reference spectrum. I could see no enhanced iron lines. No carbon bands were visible.

At 97 *seconds call*, Mr. STEEL put slit tangentially on " *outer* " corona east of equator, when " F " was seen long and fairly bright ; 1474 K was seen long and bright ; " *b* " was seen very faint, with a faint continuous spectrum. No carbon bands were visible. No iron lines were visible.

At 67 *to* 62 *seconds call*, I gave place to Sir NORMAN LOCKYER, who reported he was unable to detect any lines in the spectrum.

47 *seconds call.*—Again took my place at the spectroscope. Slit still on " outer " corona, east of equator of sun, and I reported no spectrum visible.

Afterwards, faint continuous spectrum, too feeble to make any measurements.

32 *seconds call.*—Slit placed tangentially on " inner " corona (east of equator of sun). " F " line long and bright ; 1474 K line longer and brighter than " F."

Iron lines looked for, but not visible.

Carbon bands looked for, but not visible.

17 *seconds call.*—Slit placed tangentially on " inner " corona* (east of equator of sun). Only " F " and 1474 K visible.

Iron lines looked for, but none visible.

Carbon bands looked for, but none visible.

Call " *Totality over.*"—Slit placed perpendicularly at north edge of the reappearing crescent of the sun.

" F " long and bright.

Magnesium (" *b* ") rather long and bright.

No carbon bands visible.

No iron lines visible.

Call 1 *minute after totality.*—Working near the point of the crescent.

" F " long and bright.

Magnesium (" *b* ") bright.

1474 K rather faint.

Carbon bands looked for, but none visible. 4871, 4890, 4956·5 (ÅNGSTRÖM) lines of iron flashed in. A bright iron line about 4956·5 of iron (ÅNGSTRÖM) distinctly bright.

---

* In this case the position of the slit must have been outside that occupied at the commencement of totality.

About 2 minutes after totality.—" F " long and bright.

Magnesium (" *b* ") short and bright.

Call 3 minutes after totality.—Only continuous spectrum with Fraunhofer lines.

Call 5 minutes after totality.—Ordinary continuous spectrum, with dark lines " F " again coming in brightly for a short time.

Call 7 minutes after totality.—" F " bright but faint, and very short.

No other bright lines.

Only ordinary continuous spectrum with Fraunhofer lines.

Note.—The portion of the spectrum observed was from " F " to just beyond " 1474 K."

It will therefore be seen that in the spectrum of the corona the arc lines of iron were seen, and also magnesium and hydrogen, together with the coronal line 1474 K, all of them bright. On the other hand, I was unable to repeat the observations made by Sir Norman Lockyer in the eclipse of 1882. A further point which may be noticed is that I was able to observe distinctly the coronal line 1474 K bright beyond the cusp of the sun one minute after totality. No carbon bands could be detected at any time.

Remarks on the Observations.

The preceding observations of bright lines may be conveniently summarised as follows :—

At cusp before totality
$\begin{cases} \text{10 min.–4 min.} & \text{. . . . . F.} \\ \text{4 min.–2 min.} & \text{. . . . F, } b. \\ \text{2 min.–1 mm.} & \text{. . . . F, } b, \text{ Fe arc quartette.} \end{cases}$

Chromosphere . . . Beginning of totality . .   8 or 9 bright lines between F and *b*.

Inner corona (3' from eastern limb)
$\begin{cases} \text{About 120–97 secs. calls by} \\ \text{time keeper.} \end{cases}$ F, 1474 K, 4 or 5 feeble lines, possibly identical with iron quartette.

Signal.

Outer corona (16' from eastern limb)
$\begin{cases} \text{97–67 . . . .} & \text{F, 1474 K, trace of .} \\ \text{67–47 . . . .} & \text{No lines seen.} \end{cases}$

Inner corona (3' from eastern limb)
$\begin{cases} \text{32 17 . . . .} & \text{F, 1474 K.} \\ \text{17– . . . .} & \text{F, 1474 K, no Fe lines.} \end{cases}$

Cusp after totality . .
$\begin{cases} \text{0 min.–1 min. . . . . . .} & \text{F, } b. \\ \text{1 min. -2 min. . . . . .} & \text{F, } b, \text{ 1474 K, three members of iron quartette.} \\ \text{2 mm.–3 min. . . . . . .} & \text{F, } b. \end{cases}$

As regards the attempt to repeat the observations of 1882, it may be remarked that the want of complete success was probably due to faintness of the spectrum as seen with a grating, and the difficulty of observing the shorter lines in consequence of the small size of the image projected on the slit. So far as they go, however, the

observations are in accordance with those of 1882. Thus, at 7 minutes before totality in 1882, in the region of the iron quartette to which reference is frequently made, short lines were noted at 4924 and 4934, and at three minutes before totality only 4957 was added as a line of moderate length ; these may be supposed to have escaped my notice at the corresponding intervals from totality on account of their shortness. At 2 minutes before totality in 1882 the same lines were observed with the addition of long and fainter lines of the other three members of the arc quartette at 4872, 4891, 4919, and a titanium line at 4933 ; with the exception of the titanium line, therefore, my observation of the arc quartette at 2 minutes to 1 minute before totality agrees, supposing that I was only able to see the longer lines.

In the chromosphere as ordinarily observed, and as photographed with the prismatic camera, the enhanced Fe lines 4924 and 5018, and the Ba line at 4934, are more easily visible than the arc quartette, and it thus results that the arc quartette as observed before totality by me in 1898 and by Sir NORMAN LOCKYER in 1882, proceeds not from the chromosphere, but from a region above it.

In connection with the observations during totality, it must be remembered that when a slit spectroscope is employed, lines may appear in the coronal spectrum which belong only to the chromosphere and prominences, on account of the light being scattered by our atmosphere.

The brightest chromospheric lines in the F–b region are those of iron at 4924 and 5016, and the barium line at 4934. These were, no doubt, among the lines which I observed at the beginning of totality, and are the lines which I should have seen in the inner corona if the lines noted were due to glare. As I did not see these lines, it would appear that glare was ineffective except in the case of F and b, and that the lines which I saw between F and b really belong to the corona. Here, again, we have an indication that the region above the chromosphere shows cool lines of iron, and, in connection with other lines of inquiry, this points to a region above the chromosphere as the source of some of the Fraunhofer absorption.

PART IV.—THE PRISMATIC CAMERAS.

By Sir NORMAN LOCKYER, K.C.B., F.R.S.

The Instruments employed and Photographs obtained.

The Instruments employed.

The success which attended the use of the prismatic cameras in the eclipses of 1898[*] and 1896,[†] led me to attempt to obtain photographs on a still larger scale in the eclipse of 1898. Two instruments were employed on this occasion,

---

* 'Phil. Trans.,' A, vol. 187 (1896), pp. 551–618.

† 'Phil. Trans.,' A, vol. 189 (1897), pp. 259–263.

each more powerful than any previously used. One of them had an aperture of 6 inches, and a focal length of 90 inches, with two prisms of 45° each ; the other had an aperture of 9 inches, and a focal length of 120 inches, with one prism of 45°. Both were fixed, and sunlight was reflected into them by siderostats. The 6-inch instrument was placed in a horizontal position, but the 9-inch was set up at such an angle that the direction of the moon's motion across the sun was coincident with the direction of dispersion.

To obtain this result in the case of the 9-inch instrument, the position of the telescope was previously carefully calculated by Dr. LOCKYER, and it was found that to gain the desired object, the tube had to be deviated 28° to the east side of the meridian, depressed at the object glass end 24° 40′, and twisted about its optical axis 41° in the opposite direction to the hands of a watch. To secure this accurate position at the eclipse camp, suitable packing cases were chosen and special wooden tops were made, the whole being erected at Kensington previous to its departure for India. For the information of future observers, the following diagram (fig. 6), is given showing a perspective view of the method of supporting the square tube.

Fig. 6.   Illustrating position of 9-inch prismatic camera.

In each case, some of the spare light from the mirror was utilized for a small telescope adjusted to act as a finder to the prismatic camera. The one used with the 9-inch instrument was arranged to view the image directly with an eye-piece, but that employed with the 6-inch was arranged to give an enlarged image on a white screen.

Each instrument was provided with plateholders similar to those employed in Brazil in 1893.* Four slides were used with the 6-inch, two holding plates 12″ × 10″, and two holding four plates each 12″ × 2″. For the 9-inch two of the plateholders carried plates 16″ × 6½″, and three carried three plates each 6½″ × 2″.

The instruments were focussed by taking trial photographs of stellar spectra.

Mr. FOWLER and Dr. LOCKYER, who were in charge of the 6-inch and 9-inch respectively, were each assisted by a number of volunteers from H.M.S. "Melpomene," who were distributed as follows :—

	6-inch.	9-inch.
In charge of finder and chronometer	Lieutenant DE WET, R.N.	Lieutenant PERCIVAL JONES, R.N.R.
In charge of siderostat and to make exposures . .	J. TURNER, I.I.	W. BRAY, Ch. Arm.
Recorder of exposures . .	C. IRONSIDES, G.M.	A. RAMAGE, A.B., L.T.O.
To hand and receive dark slides	T. BRADING, A.B. J. INNES, A.B.	A. WILKINS, Shipwright. E. ASHFORD, A.B. F. FENTON, A.B.
In charge of lamp . . .	E. SALT, Boy, 1st Class.	A. CARR, Boy, 1st Class.

The duty of the observer in charge of the finder was to keep the image in the centre of the field of view, which corresponded by previous adjustment with the centre of the plate in the prismatic camera ; records of the moments of contact were also made.

The exposures were made by means of a piece of cardboard which was used to cover or uncover the front of the prism from directions given by Mr. FOWLER and Dr. LOCKYER respectively.

In the two prismatic cameras fifty-seven photographs were secured, the exposures varying from 1 to 50 seconds. Such a result as this could only be obtained by a minute subdivision of labour.

Local Conditions of Eclipse as affecting the Prismatic Cameras.

The small difference in the apparent diameters of the sun and moon (see Part I., p. 156) was a very favourable circumstance in connection with the work of the prismatic cameras relating to the chromospheric spectrum, for the reason that the arcs of chromosphere visible at the beginning and end of totality would be

* 'Phil. Trans.,' A, vol. 187 (1896), p. 571.

comparatively long ones. A layer 10 inches in depth, for example, would be exposed over a length of 112° at the instant of contact, and a layer 1 inch in depth over a length of 34°.

Calculations showed that, after reflection from the mirror of a siderostat along a horizontal meridian line, the north point of the sun's axis, as seen by an observer looking into the mirror, would be 7° 13' to the left of the top of the image; the east and west points were to the right and left respectively.

It was evident then that with dispersion in a horizontal direction there would be a little overlapping of adjacent chromospheric arcs, but still it was considered desirable that in one of the instruments at least the dispersion should be arranged so as to give the arcs quite symmetrical.

Photographs taken with the 6-inch Prismatic Camera.

Mr. FOWLER's programme was to begin taking a series of ten snapshot pictures 5 seconds before the commencement of totality, to obtain a record every second or thereabouts of the spectrum of the chromosphere. After this he exposed eight other plates to secure photographs of the coronal rings, the exposures being of various lengths. It was also arranged that at 5 seconds before the end of totality he should commence another series of ten snapshots, exposing the last of these some few seconds after totality. The actual times of exposure, &c., are shown in the following table :—

TABLE of Exposures for 6-inch Prismatic Camera

Number	Kind of plate.	G.M.T. of exposure, A.M.		Exposure.	Remarks.
		h. m. s.			
1. *a*	EDWARDS' snapshot, isochromatic	7 52 35		Inst.	
b	„ „ „	36		„	
c	„ „ „	37		„	
d	„ „ „	38		„	
e	„ „ „	40		„	
f	„ „ „	41		„	Beginning of totality.
g	„ „ „	42		„	
h	„ „ „	43		„	
k	„ „ „	45		„	
l	„ „ „	46		„	
2. *a*	EDWARDS' snapshot, isochromatic	{ 7 52 50 to 7 53 40		} 50 secs.	Slide changed.
b	„ „ „	7 53 41		Inst.	
c	„ „ „	{ 7 53 44 to 7 53 50		} 6 secs.	
d	„ „ „	7 53 51		Inst.	

TABLE of Exposures for 6-inch Prismatic Camera—*continued.*

Number.	Kind of plate.	G.M.T. of exposure, A.M.	Exposure.	Remarks.
		h. m. s.		
3. *a*	Lumière	7 53 58 to 7 54 10	} 12 secs.	Slide changed.
b	,, ...	7 54 11	Inst.	
c	,, ...	7 54 13 to 20	} 7 secs.	
d	,, ...	7 54 22 to 30	} 8 secs.	
4. *a*	Lumière	7 54 42	Inst.	Slide changed.
b	,, ...	43	,,	
c	,, ...	45	,,	End of totality
d	,, ...	46	,,	
e	,, ...	47	,,	
f	,, ...	48	,,	
g	,, ...	49	,,	
h	,, ...	50	,,	
i	,, ...	51	,,	
l	,, ...	52	,,	

On developing the plates it was found that everything had gone satisfactorily. The large plates containing the ten snapshots give the whole story of the chromosphere during 12 seconds, the time taken to make the ten exposures.

The last set of ten exposures did not come out quite as expected, for the reason that the duration of totality was a few seconds shorter than had been provided for in the time-table, so that only two of the exposures were made before the end of totality. The very last exposure, however, taken about 8 seconds after totality, shows many bright lines.

Some of the more important photographs are reproduced, without enlargement, in Plates 6 and 7. Plate 6 shows the series of ten exposures taken near the beginning of totality; the diagonal streak of light is due to a leak in the plate-holder, which did not exist on the day preceding the eclipse, when all the slides were tested. Plate 7 includes photographs 2*c*, and 3*d*; in each case the brightest and most conspicuous features are the images of the chromosphere and prominences, while the rings representing the spectrum of the corona are relatively faint.

The succession of changes in the spectra throughout the eclipse requires but little explanation. In the first few exposures (top of Plate 6) the bands of apparently continuous spectrum were produced by the last remaining parts of the visible crescent, which was broken up by the irregularities of the moon's limb; in these bands the Fraunhofer lines are clearly seen in the region more refrangible than K, but in the violet and blue the Fraunhofer lines are not seen, probably in consequence of over-

exposure. These bands gradually disappeared as totality approached, and the spectrum of the chromosphere, which before showed its bright lines as short arcs at the cusps, then exhibits them as complete arcs broken up by the lunar irregularities. Later, the spectrum of the upper chromosphere appears alone, the lines being then relatively few in number (bottom of Plate 6).

As the middle of totality approaches, the chromospheric arcs lose their importance, and the principal features of the spectra then are the continuous spectrum of the corona and the coronal rings; at this time relatively long exposures are required to secure adequate photographic records. The continuous spectrum is not of uniform intensity on account of the unequal distribution of coronal light round the limb (Plate 7).

As the end of totality approaches, the chromospheric arcs reappear, now in the part corresponding to the sun's south-western limb, and the phenomena shown on Plate 6 are repeated in inverse order. It has not been thought necessary to reproduce the ten photographs taken at these stages.

The prominences visible throughout the eclipse were neither large nor numerous, and their spectra show comparatively few lines. One, in the south-east quadrant, appears in all the photographs.

The principles governing the interpretation of photographs taken by the prismatic camera, both in and out of totality, are fully described in my report of the photographs obtained in 1893.*

Photographs taken with the 9-inch Prismatic Camera.

The programme adopted was similar to that of the 6-inch, there being two large plates (16 × 6½) for recording a series of ten snapshots at and near the times of second and third contacts, and nine smaller plates for exposure during totality. All the exposures were successfully made, but the lines in the spectrum are not very sharp, owing to warping of the wooden tube by the heat and the consequent disturbance of the focus.

As has been previously stated, this instrument was so mounted that the arcs should appear symmetrical about the direction of dispersion. It is satisfactory to state that the photographs showed that the experiment was very successful, the arcs coming out exactly as calculated. Although this instrument was capable of only giving about half the dispersion of the 6-inch, the optical parts were better adapted for recording the ultra-violet region of the spectrum.

Particulars as to the photographs taken are given in the following table :—

TABLE OF EXPOSURES for 9-inch Prismatic Camera.

Number.	Kind of plate.	G.M.T. of exposure, A.M.		Exposure.	Remarks.
		h. m.	s.		
I. a	Lumière	7 52	35·5	Inst.	
b	,,		37·5	,,	
c	,,		38·5	,,	
d	,,		39·5	,,	
e	,,		40·5	,,	→ Beginning of totality.
f	,,		42·5	,,	
g	,,		44·5	,,	
h	,,		45·5	,,	
i	,,		46·5	,,	
k	,,		48·5	,,	
II. a	Lumière	7 52	54·5 ⎫	10 secs.	Slide changed.
		7 53	4·5 ⎭		
b	,,	7 53	6·0 ⎫	9 ,,	
			15·0 ⎭		
c	,,	7 53	16·5	Inst.	
III. a	Lumière	7 53	21·5 ⎫	4 secs.	Slide changed.
			25·5 ⎭		
b	Edwards	7 53	28·5 ⎫	36 ,,	
		54	4·5 ⎭		
c	,,	7 54	6·5	Inst.	
IV. a	Edwards	7 54	10·5 ⎫	4 secs.	Slide changed.
			24·5 ⎭		
b	,,	7 53	28·5 ⎫	36 ,,	
		54	4·5 ⎭		
c	,,	7 54	37·5	Inst.	
V. a	Edwards	7 54	43·5	Inst.	Slide changed.
b	,,		45·5	,,	→ End of totality.
c	,,		46·5	,,	
d	,,		47·5	,,	
e	,,		48·5	,,	
f	,,		49·5	,,	
g	,,		50·5	,,	
h	,,		51·5	,,	
i	,,		52·5	,,	
k	,,		54·5	,,	

The photographs show the same succession of phenomena as those taken with the 6-inch, the chief differences being that the diameter of the rings is in greater proportion to the length of the spectrum, and that the chromospheric arcs are symmetrical. The series of ten spectra taken near the beginning of totality is reproduced without

enlargement in Plate 8. Two other spectra taken with longer exposure nearer mid-
eclipse are reproduced in Plate 7, namely, photographs 2a and 4b.

REDUCTION OF THE PHOTOGRAPHS.

The Determination of Wave-lengths of Chromospheric Lines.

When we come to determine the wave-lengths of lines in a spectrum of the
eclipsed sun, taken by means of a prismatic camera, it is found that the problem is not
quite the same as in the case of the slit spectroscope, in which care must always be
taken only to utilise the centre of each of the lines for micrometric measurements, no
matter how broad they may be.

In the prismatic camera pictures, in which arcs take the place of lines, not only
must great care be taken to measure the distance between them in the direction of
dispersion, but these measurements must always be made from the side representing
the limb of the moon. Now the question has arisen, does the inner portion of these
arcs in all cases represent the limb of the dark moon? If it does, then all measure-
ments for wave-length made from it would be correct.

I have shown previously* that there may exist arcs which would be brighter some
distance from the moon's limb, and these would decrease in intensity as this limb
was approached, such an arc being due to a shell of vapour concentric with the photo-
sphere, but some distance from it. It is quite easy to understand that a photograph
of such an arc, especially if it were under-exposed, would give us a record of the
brightest part of the arc in the line of sight, but this fainter portion extending down
to the dark moon would be missing.

In such a case the measurements for wave-length made from the inner edge would
not then be justifiable, and if the arc in question were of unknown origin, its deduced
wave-length would be, and the suggested origin might be, erroneous. Theoretically
all arcs which represent layers high up above the photosphere should be more intense
where the line of sight of the observer meets the layer in question tangentially. Thus
arcs that represent layers which are lowest should be brightest nearer the moon's
limb, while the opposite should be the case with those at higher levels.

It appears, however, from the photographs taken during the last three eclipses that
such a gradation of light has not been recorded, but this is owing probably to the
small size of the solar image used.

An idea of the sizes in the 6-inch and 9-inch prismatic cameras, and the thickness
of some of the arcs, may be of interest in this place. In these two cameras the solar
images were 0·85 and 1·12 inches in diameter respectively. Considering a layer at
the sun to extend 5000 miles above the photosphere, this is equivalent at the mean
distance of the sun to 11·1 seconds of arc; taking the sun's diameter as subtending
an angle of 32' 3"·6, then the ratio of this layer to the diameter is as 11·1 to 1923·6,

* 'Roy. Soc. Proc.,' vol. 187, p. 587.

which is as 1 to 173·2. The image of the sun in the 6-inch prismatic camera being 0·85 inch, this layer would be represented at the moment of totality by an arc equal in thickness to ·0045 inch, or nearly one two-hundredth of an inch. In the 9-inch prismatic camera its breadth would be ·006 inch. In the case of the 6-inch spectra a displacement of $\frac{1}{100}$th of an inch corresponds at Hγ to a shift of 1·2 tenth-metres, while in the case of the 9-inch spectra the corresponding shift would be smaller. As the majority of the chromosphere vapours do not reach so high as 1000 miles above the photosphere, as indicated by the lengths of the arcs, the possible displacements due to the above cause are generally too small to be taken into account. Distinct displacements, however, have been noted in some of the arcs of helium and carbon, and in these cases the known wave-lengths have been adopted.

Another possibility of error is introduced in the case of the brightest arcs by irradiation. The depth of the hydrogen stratum, for example, as indicated by the *thickness* of F(Hβ) arc is in some cases nearly three times as great as that indicated by the *length* of the arc. Accordingly if the arc be supposed to have expanded equally in both directions, the true position of the moon's edge will be nearly one-third the thickness of the arc from its inner edge. For this reason the hydrogen arcs Hβ, Hγ, Hδ, Hε, as well as the K line of calcium, have not been used as fiducial lines in reducing the chromospheric spectra. The ultra-violet series of hydrogen lines, however, are much finer, and have been used for plotting the curve for the reduction of that part of the spectrum.

The actual procedure was as follows : --

For the purpose of reduction the spectrum has been divided into two parts, one extending from Hα in the red to λ 3900 in the violet, and the other from λ 3900 to the limit of the photographic impressions in the ultra-violet, about λ 3663.

In the first part the wave-lengths were determined by direct comparison with a photograph of the solar spectrum taken with the same instrument provided with a slit and collimator, the latter being a 4-inch Cooke triplet of 6 feet focal length.

For the ultra-violet part of the spectrum, this direct method could not be used, as in consequence of the absorption of ultra-violet rays by the dense prisms, satisfactory photographs of the solar spectrum beyond λ 3900 could not be obtained. Some of the photographs of metallic arc spectra, however, show a considerable number of lines in this part of the spectrum, and these have been used to confirm wave-lengths determined in other ways. In the first instance, the wave-lengths were determined by the use of an interpolation curve, adopting well-known lines of hydrogen, titanium, magnesium, and iron as fiducial points. Besides this, Dr. HARTMANN's interpolation formula for prismatic spectra[*] has been employed for the reduction of the ultra-violet part of the spectrum. The wave-lengths adopted for the ultra-violet hydrogen lines Hζ to Hγ are those of AMES,[†] from Hξ to Hρ.

* 'Publicationen des Astrophysikalischen Observatoriums zu Potsdam,' No. 42, vol. 12, 1898.

† 'Phil. Mag.,' vol. 30, 1890, p. 33.

HALE's[*] values have been assumed. The values of the lines Hσ to Hχ are those computed from the measurements of the photographs : these agree closely with those deduced from BALMER's law. The direct measures of the lines for the use of both interpolation curve and formula were made by means of an ivory scale divided to $\frac{1}{150}$th of an inch. The positions of the fiducial lines for the interpolation formula were checked with a stage micrometer reading to $\frac{1}{10000}$th of an inch, but the majority of the lines were too broad or too faint to permit the use of this instrument with satisfactory results in all cases. The reduction of the wave-lengths of the lines and the assignment of origins were, in the region between 3900 and D, mainly done by Mr. BAXANDALL. For the region more refrangible than 3900 Mr. SHACKLETON is chiefly responsible.

Determination of the Wave-lengths of Coronal Rings.

The wave-lengths of the coronal rings have been determined by direct comparison with a photograph of the solar spectrum on the same scale, utilising the well-known lines of the chromosphere and prominences for properly superposing the photographs. In the case of the coronal rings, the wave-lengths may be affected in the same way as those of the chromospheric arcs, if all the radiations are not photographed quite down to the moon's edge, and an additional slight error may be introduced by the movement of the moon during the comparatively long exposures required to photograph some of the rings. One cannot be certain that the inner edge of the ring corresponds to the same position of the moon's edge as that adopted in the case of the chromospheric arcs used as references, since the chromospheric arcs are bright enough to register their total thickness in view, throughout the exposure, while the fainter coronal rings might only be shown in those parts having the maximum exposure. Thus, in Photograph 2a of the 6-inch series, which was exposed for 50 seconds, the movement of the moon relatively to the sun during the exposure would be 18″·5, corresponding at Hγ to a displacement in wave-length of nearly two tenth-metres. The measured wave-length of a line will be different, according as the inner edge is taken to represent the moon's position at the beginning or end of the exposure, the inner edges of the chromospheric arcs at this stage representing essentially the moon's position at the beginning of the exposure. It is probable that the inner edge of a photographed ring will correspond to the positions of the moon at some time intermediate between the beginning and end of the exposure, so that the error, even in the case of an exposure as long as 50 seconds, would not amount to two tenth-metres. The more useful exposures for coronal rings were those of 6 to 8 seconds, and in these the possible errors due to the moon's motion will be smaller than those which may be introduced by imperfect settings on such indistinct appearances as those presented by most of the coronal rings. Hence no attempt has been made to make such corrections of the wave-lengths.

Determination of Origins.

In the report on the eclipse of 1893, to which reference has already been made, I pointed out, p. 557, that the chemical part of the investigation could not be satisfactorily conducted until much more inquiry had been made, "(a) into the old observations, (b) into the spectra of stars and nebulæ, and (c) into certain questions for which new observations are necessary." Since 1896 great progress has been made in our knowledge of the spectra of the heavenly bodies, and of the changes in the spectra of chemical substances with different conditions of temperature, and the chemical part of the inquiry is now placed on a much firmer basis.

The origins of the various lines have been determined partly by direct comparison with the arc and spark spectra of the elements, photographed with the same instrument fitted with a collimator, as described above, and where such comparison spectra were not available by reference to existing records of the arc and spark lines of the elements. The list of chromospheric lines has also been compared with the lists (published and unpublished) of enhanced lines of many of the elements observed at Kensington.

THE SPECTRUM OF THE CHROMOSPHERE.

Distribution of Substances in the Chromosphere.

I first deal with the determination of the heights of the various absorbing vapours so far as they can be gathered from the photographs, which, of course, only record for us the brightest lower portions of the different arcs, and not their complete extension.

The following table shows the results obtained in the case of some of the most typical lines :—

Lines.	Length of arcs	Height.	
		In miles.	In secs. of arc.
Ca(K)	130°	6000	13·3
Hydrogen	112°	4500	10·0
He 4471·25	105°	4000	8·9
He 4026·3; Sr 4077·9, 4215·66	86°	2700	6·0
Ca 4226·9; Sc 4247	72½°	2000	4·4
Mg ultra-violet triplet			
Fe triplet (4048)			
Strongest are lines (4307·96, 4325·92, &c.)	60°	1450	3·2
Al 3944·18 and 3961·67			
Fe enhanced lines 4584, 4233	51°	1100	2·4
Mn quartet (4030·9, &c.)			
Fe enhanced quartet (4523·0, &c.) and many other lines	40°	650	1·4
Carbon fluting and many lines, including some arc lines of iron	35°	475	1·05

A very noticeable feature of the chromospheric spectra, which the photographs enable us to investigate at different elevations, is the difference in the behaviour of the gaseous and metallic lines. In the spectrum taken very near the moment of second contact, representing that of the lower strata with the spectra of higher ones superposed, the metallic arcs are relatively short and very bright, while in later photographs representing the spectra of successively higher strata free from admixture with lower ones, the metallic arcs are relatively feeble. This is also indicated in another way by the varying effects seen over the tops of lunar mountains and through indentations in the moon's limb.

Some of the lines are seen to be relatively much brighter in the upper strata than in the lower, such lines showing no notable increase of brightness at the points where lower strata are revealed through lunar valleys. Chief among these lines are those of hydrogen, helium, and calcium (H and K), but there is an additional line at wave-length 4686·2 or thereabouts, which behaves in the same way.

This line does not appear in Young's list of chromospheric lines, and all attempts to trace it in known spectra have failed. A line apparently coincident with it, however, has been found in the photographed spectrum of a tube containing helium, which is one of the series of comparison spectra taken with the 6-inch prismatic camera to facilitate the reduction of the eclipse photographs.

The only recognised impurity in the vacuum tube used is oxygen, but besides the line to which reference has been made, there are a few faint lines for which no origins can at present be assigned.

It is worthy of remark that this line falls very near to the first line of the principal series in the spectrum of hydrogen, recently calculated by Rydberg to have a wave-length of 4687·88.[*]

.Fig 7. Comparison of Chromosphere and Carbon Fluting at λ 3883.

[*] ‘Astro-Phys. Jour.,’ vol. 6, p. 237.

Among the shortest arcs are those corresponding to the carbon group at λ 3883. The perfect correspondence of the five heads of the carbon band with the chromospheric arcs is demonstrated by the comparison given in fig. 7, in which the coincidences of K and a few other lines are also shown.

Comparison of Chromospheric and Fraunhofer Lines.

As in the case of the photographs taken with the prismatic cameras in 1893 and 1896, the spectrum of the chromosphere in 1898 is very different from the Fraunhofer spectrum, so that we have not to deal with a mere reversal of the dark lines of ordinary sunlight into bright ones. (See fig. 8.)

Many very strong chromospheric lines, as the helium lines for example, are not represented among the Fraunhofer lines, while many Fraunhofer lines are absent from the chromospheric spectrum.
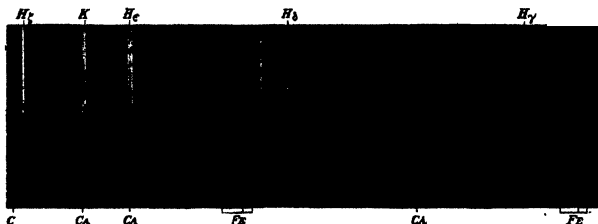
Fig. 8. Spectrum of Chromosphere as photographed during Eclipse compared with Fraunhofer lines.

The conclusion at which I had already arrived, that the chromosphere is not the seat of the Fraunhofer absorption, thus receives further confirmation.

Table of Chromospheric Radiations.

A complete list of the lines which have been measured in the spectrum of the chromosphere, with their origins, so far as they have at present been investigated, is given in the table at the end of this report.

The distribution of the lines throughout the spectrum may be summarised as follows :—

$$
\begin{aligned}
&\text{More refrangible than K} &. \quad . \quad . \quad . &\quad 265 \\
&\text{K to H}\gamma. &. \quad . \quad . \quad . \quad . \quad . \quad . &\quad 190 \\
&\text{H}\gamma \text{ to H}\beta &. \quad . \quad . \quad . \quad . \quad . \quad . &\quad 178 \\
&\text{H}\beta \text{ to D}. &. \quad . \quad . \quad . \quad . \quad . \quad . &\quad 223 \\
\hline
&\text{Total number} &. \quad . \quad . &\quad 856
\end{aligned}
$$

The intensities are on a scale such that 10 indicates the strongest and 1 the faintest lines.

The suggested origins have been tabulated in two columns, one in which there is considerable certainty as to the origins suggested, and the other in which the evidence is not so complete. The general results with regard to the origins of the chromospheric lines are indicated by the following table.

TABLE showing the probable Presence or Absence of some of the Chemical Elements in the Chromosphere.

Element.	Whether any of the strongest arc lines are represented in the chromospheric spectrum.	Whether any of the enhanced lines are represented in the chromospheric spectrum.
Aluminium	Yes	Doubtful
Antimony . . .	No	Not investigated
Arsenic	No arc lines to compare	Not investigated
Barium	Yes	Not investigated
Bismuth	No	Doubtful
Boron	No arc record	No enhanced lines to compare
Cadmium	No	Not investigated
Cæsium . . .	No	Doubtful
Calcium	Yes	Yes
Carbon	Yes	Doubtful
Cerium	Doubtful	Not investigated
Chromium . .	Yes	Yes
Cobalt	Doubtful	Doubtful
Copper	Doubtful	Doubtful
Gold . . .	No	Doubtful
Indium	Doubtful	Doubtful
Iridium	No	Not investigated
Iron . .	Yes	Yes
Lanthanum	Doubtful	No
Lead	Doubtful	Doubtful
Lithium	Doubtful	No enhanced lines to compare
Magnesium . .	Yes	Doubtful
Manganese	Yes	Yes
Mercury	No	Not investigated
Molybdenum . . .	Doubtful	Doubtful
Nickel	Yes	Yes
Osmium	No	No enhanced lines to compare
Palladium	No	No enhanced lines to compare
Platinum	No	Not investigated
Potassium	Doubtful	Doubtful
Rhodium	Doubtful	No enhanced lines to compare
Rubidium	Doubtful	Doubtful

TABLE showing the Probable Presence or Absence of some of the Chemical Elements in the Chromosphere—*continued.*

Element.	Whether any of the strongest arc lines are represented in the chromospheric spectrum.	Whether any of the enhanced lines are represented in the chromospheric spectrum.
Ruthenium	No	Doubtful
Scandium . . .	Yes	Enhanced lines not investigated
Selenium	No arc record	Enhanced lines not investigated
Silicon	No	Doubtful
Silver	No	Not investigated
Sodium	Yes	No enhanced lines to compare
Strontium	Yes	Yes
Tantalum	Doubtful	No enhanced lines to compare
Tellurium . .	No arc record	Not investigated
Thallium . . .	No	Doubtful
Thorium . . .	No	Not investigated
Tin	No	Doubtful
Titanium . .	Yes	Yes
Tungsten . . .	No	Doubtful
Uranium . .	No arc record	Not investigated
Vanadium . .	Doubtful	Yes
Yttrium	No	Doubtful
Zinc	No	Doubtful
Zirconium	No	Doubtful

The word "doubtful" signifies that though some of the lines agree approximately in position with chromospheric lines, there is considerable doubt as to whether such lines are really due to the presence of the element in question in the chromosphere. In the tables, coincidences with enhanced lines are indicated by the prefix p, signifying the presence of the corresponding "proto-metal"; thus p Ti indicates a coincidence with an enhanced line of titanium.

THE SPECTRUM OF THE CORONA.

Differentiation of Coronal Rings.

One of the chief results which, in my opinion, would be secured by the use of the prismatic camera in eclipse work was the differentiation between chromospheric and coronal phenomena. The photographs taken during the eclipses of 1893, 1896, and 1898 all enabled this distinction to be made very clearly, and various radiations formerly attributed to the corona have been shown to belong to the chromosphere alone. The photographs taken in Africa in 1893 showed eight rings in the spectrum of the corona; in Novaya Zemlya, in 1896, with a less powerful instrument, a smaller number was secured; but those taken with increased dispersion in India, in 1898, show a much greater number.

The coronal rings not only differ from the chromospheric ones in regard to the heights to which they extend above the photosphere, but also in appearance.

The outlines of these rings are distinctly not connected with the configuration of the chromosphere and prominences. In photographs taken near the beginning and end of totality, the green ring is brightest on the same side of the moon, although the chromosphere and prominences are first visible on one side and then on the other. None of the rings give any indications of increased brightness at the places occupied by prominences. The green ring, which is the brightest of the rings seen, can be traced completely round the limb, and while in some parts it is very feeble, in others it is bright enough to show the brightest projections of the inner corona as photographed with short exposures with the coronagraph. The other principal rings at 3987 and 4233 can also be traced completely round the limb, but they are fainter on the average and of much more uniform intensity than the green ring.

The Wave-length of the Chief Line in the Green.

The reasons which led to a re-examination of the wave-length of the chief coronal radiation in the green, and the result of the new measurement, have already been stated in a preliminary note communicated to the Society.[*] The main point of this communication was that two of the chief coronal lines, one in the blue about λ 4231, and the other in the green, had been supposed coincident with two bright chromospheric lines which my investigations had shown to correspond to enhanced lines of iron. The absence of other prominent enhanced iron lines from the corona spectrum, and the improbability of there being a sufficiently high temperature in the corona to produce enhanced lines, suggested a more complete investigation of the supposed coincidences of the coronal and chromospheric lines. The non-coincidence of the coronal line 4231 with the chromospheric line 4233 which was indicated by the measurement of the photographs of 1893[†] was fully confirmed, and it was also found that the green line was by no means coincident with the chromospheric line 1474 K. Its wave-length was, in fact, estimated as 5303·7, or about 13 tenth-metres more refrangible than 1474 K (5316·79). This result has since been confirmed by CAMPBELL[‡] and EVERSHED.[§]

The Coronal Radiations.

In another preliminary communication to the Royal Society,[||] I gave the results of a more general investigation of the coronal spectrum. The main points may be briefly summarised as follows :—

---

* 'Roy. Soc. Proc.,' vol. 64, p. 168.   † 'Phil. Trans.,' A, vol. 187 (1896), p. 593.
‡ 'Astro-Physical Journal,' 1899, vol. 10, p. 186.
§ "The Indian Eclipse, 1898," publication of the Brit. Astronom. Assoc., 1899, p. 79.
|| 'Roy. Soc Proc.,' vol. 66, p. 189.

(1.) The coronal rings may be divided into three groups, defined by the position angles in which they have their greatest brightness. The typical rings are at wavelengths 5303·7, 3987, and 4359·5.

(2.) The different forms of the coronal rings indicate that they are not all due to the same substance, at least three being in question.

(3.) The origins of the rings have not yet been traced.

(4.) There are possibly feeble indications of some of the chromospheric gases in the inner corona.

(5.) The chief coronal ring in the green is very closely associated with the form of the inner, and appears to have no distinct connection with the outer corona.

(6.) The outer corona gives no indications of bright rings in photographs taken with the prismatic cameras.

Further examination of the photographs has shown that some of the most minute detail of the inner corona is represented in the green ring. Near the equator, on the eastern limb, for instance, there is a well-defined loop-shaped structure in the inner corona, which presents almost the appearance of a prominence, and this is faithfully reproduced in the corresponding part of the green ring ; its spectrum, however, has nothing in common with that of the prominences. A similar relationship is shown at other position angles, but as the Viziadrug photographs were all exposed too long to clearly separate the inner from the outer corona in all parts, a complete discussion of this point must be deferred until photographs taken with shorter exposures by other observers become available for comparison.

The table of coronal radiations is reproduced at the end of this report in order to facilitate comparisons with the chromospheric radiations.

The Continuous Spectrum.

The coronal rings are superposed upon a continuous spectrum which is broken up into bands of various intensities (see Plate 7), and it is important to note that each bright part of the green ring is accompanied by one of these bands. In some of the photographs a few bands of continuous spectrum are seen apparently without corresponding brightenings of the green ring, but an examination of the other photographs generally reveals the brightening in a lower reach of the corona. One or two of the bands appear in places which correspond to the points on the sun's limb about 90° degrees from the points of second and third contact, and in these cases the corresponding brightenings of the green ring are probably hidden by the moon throughout the eclipse.

These facts indicate that the action which produces a brightening of the green ring also produces a brightening of the continuous spectrum, not only in the region where the gaseous mass is rendered more luminous, but in the region immediately overlying it. The continuous spectrum of the outer corona exhibits no definite structures in

the photographs which are traceable to individual streamers, but simply appears brightest at the edges of the spectrum where the light of the corona is grasped tangentially by the prisms.

TABLE I.—Lines in the Spectrum of the Chromosphere.

Eclipse 1898.

Wave-lengths			Intensity	Chemical origin.				Remarks
				Probable.		Possible.		
Comparison.	Curve.	Computed.		Origin.	λλ.	Origin.	λλ.	
	3663·0	3663·2	<1	—	—	Ti	3662·38	
	65·6	66·2	<1	—	—	—	—	Very hazy line
	66·3	66·9	<1	—	—	—	—	Very diffuse line
	67·0	67·6	1					The λλ of these lines for curve were deduced by extrapolation.
	68·0	68·5	1					
	69·1	69·5	<1	—	—	Ti	3669·11	
	69·6	70·0	1	—	—	Fe	3669·67	
	71·0	70·9	1	—	—	—		
	71·7	71·7	1	H	3671·7	—	—	Very hazy line
	74·2	74·2	1	H	3674·2	—	—	Hₓ Very hazy line.
	75·0	75·2	1		—		—	H₄ (fiducial line for curve).
	76·4	76·3	<1	—	—	Fe	3676·46	
	76·7	76·7	1–2	H	3676·7	—	—	Hₛ (fiducial line for curve).
	77·4	77·6	1	—	—	Fe	3677·76	
	78·0	78·1	1–2		—		—	
	79·0	79·1	1					
	79·6	79·6	1–2	H	3679·6	—		Hₛ (fiducial line for curve).
	81·6	81·6	1–2	—		—		Very hazy line.
	82·9	82·9	2	H	3682·9	—	—	Hₛ (fiducial line for curve).
3685 3	85·3	85·3	3	p Ti	3685 34	—	—	Fiducial line for curve and computation.
	86·7	86·7	2–3	H	3686·7	—	—	Hₛ (fiducial line for curve.)
	87·9	88·0	1	—	—	Fe {	3687·61 / 3687·80	
	89·0	89·1	1					
	90·2	90·4	1					
	91·0	91·1	1	—	—	Co	3690·87	
	91·5	91·7	3	H	3691·5	—	—	Hₛ (fiducial line for curve).
	92·4	92·6	1					
	93·5	93·5	1–2	—	—	Co {	3693·26 / 3693·02	
	94·0	93·9	1–2					
	95·0	94·9	1–2	—	—	Fe	3695·19	
	96·0	95·9	1	—	—	V	3696·01	
	96·6	96·6	1					
	97·4	97·4	3	H	3697·4	—	—	Hₛ (fiducial line for curve).
	98·3	98·1	1–2					
	99·3	99·3	1					
	3700·2	3700·1	2	—	—	—	—	Double ?
	02·4	02·3	1	—	—	Co	3702·38	Probably several lines
	03·2	03·1	1					

Table I.—*continued.*

Wave-lengths			Intensity.	Chemical origin.				Remarks.
				Probable.		Possible.		
Comparison.	Curve.	Computed.		Origin.	λλ.	Origin.	λλ.	
	3704·1	3703·9	3	H	3704·1	—	—	H$_\delta$ (fiducial line for curve).
	04·7	04·4	1–2	—	—	Fe	3704·60	} Hazy double.
	05·1	04·9	2	He	3705·15	—	—	
	05·7	05·7	2	—	—	Fe	3705·71	
	06·2	06·2	2	Ca	3706·18	p Ti	3706·36	
	07·2	07·0	1	—	—	Fe	3707·19	
	07·7	07·6	1	—	—	Co	3707·60	
	08·8	08·9	1–2	—	—	Co	3708·96	Double ?
	10·2	10·1	2	—	—	Ti	3710·09	
	11·5	11·4	1					
	11·9	11·9	4	H	3711·9	—	—	H$_\epsilon$ (fiducial line for curve).
	13·0	12·9	2					
	13·8	13·7	1–2					
	14·9	14·9	1–2					
	15·8	15·3	2	—	—	V	3715·70	
	16·6	16·4	1–2	—	—	Fe	3716·59	
	17·5	17·3	1–2	—	—	Ti	3717·54	Mush of lines
	18·4	18·3	1–2	—	—	Fe	3718·55	
	19·2	19·1	1–2					
3720·1	20·1	20·1	2	Fe	3720·08	—	—	
	20·6	20·6	1–2	—	—	—	—	Thick line.
	21·8	22·0	4	H	3721·8	—	—	H$_\kappa$ (fiducial line for curve).
22·7	22·7	22·6	2	Fe	3722·73	Ni	3722·64	
	23·7	23·6	2					
	24·6	24·6	2	—	—	Ti	3724·72	
	25·3	25·3	2	—	—	Ti	3725·30	
	26·1	26·0	1					
	26·6	26·6	1–2					
27·1	27·1	27·1	2–3	Fe	3727·06			
27·8	27·9	27·8	2–3	Fe	3727·78			
	28·8	28·7	1					
	29·2	29·1	1					
	30·0	29·9	1	—	—	Ti	3729·95	
	30·6	30·5	1					
	31·5	31·4	1					
	32·5	32·6	2	Fe	3732·55	Co	3732·55	
	33·5	33·5	1–2	—	—	Fe	3733·47	
	34·2	34·2	4	H	3734·15	—	—	H$_\lambda$ (fiducial line for curve).
35·0	35·0	35·1	1–2	Fe	3735·01			
	36·0	36·0	1–2	—	—	Co	3736·04	
37·0	37·0	36·9	3	Ca	3737·06	Fe	3737·28	
	38·5	38·5	1–2	Fe	3738·45	—	—	
	39·4	39·4	1–2	—	—	Ni	3739·37	
	40·3	40·3	1–2					
	41·0	41·0	1–2	—	—	Ti	3741·21	
41·8	41·8	41·8	3	p Ti	3741·78	—	—	
43·5	43·5	43·3	2	Fe	3743·59	—	—	
	43·7	43·7	2					
	45·7	45·6	3	Fe	3745·72	—	—	
	47·0	47·0	1–2	—	—	Fe	3747·07	

Table L.—*continued.*

Wave-lengths.			Intensity.	Chemical origin.				Remarks.
				Probable.		Possible.		
Comparison.	Curve.	Computed.		Origin.	λλ.	Origin.	λλ.	
3748·4	3747·7	3747·7	2					
	38·4	48·4	3	Fe	3748·41	p Ti	3748·14	
	49·6	49·6	1–2	Fe	3749·63			
	50·2	50·1	4	H	3750·2	—	—	H_ε (fiducial line for curve).
	51·5	51·5	2					
	52·5	52·5	1–2					
	53·7	53·9	1–2	—	—	Fe	3753 73	
	55·6	55·6	1–2	—	—	Co	3755·59	
	56·4	56 5	1–2					
57·8	57·7	57·8	2	p Ti	3757·82			
58·4	58·3	58·4	2	Fe	3758·38			
59·4	59·4			p Ti	3759·45			Fiducial line for curve.
		60·3	1–2	—	—	Fe	3760·20	
	60·7	60·7	1–2	—	—	Fe	3760·68	
61·5	61·5	61·5		p Ti	3761·46	—	—	Fiducial line for both curve and computation.
	62·6	62·6	1					
	63·7	63·7						
63·9	63 9	64·1		Fe	3763·95			
65·7	65·7	65·7		Fe	3765·69			
	66·8	66·8						
67·3	67·3	67·4		Fe	3767·34			
	68·4	68·4						
	69·4	69·4						
	70·7	70·8		H	3770·7	—	—	H, (fiducial line for curve).
	73·0	72·1		—	—	Ti	3771·80	
	73·2	73·4						
	74·4	74·6		—	—	Y	3774·47	Long line.
	75·3	75·4		—	—	Ni	3775·72	
	76·2	76·3		p Ti	3776·20			
	76·7	76·7		—	—	Fe	3776·60	
	77 8	78·0						
	78·6	78·8		—	—	Fe	3778·65	
	79·8	80·0						
	81·0	81						
	81·8	81						
	82·7	82						
	83·7			Ni	3783·67			
	84·1	84·1	1–2					
	85·4	85·6		—	—	Ti	3786·20	
	85·8	86·0		—	—	Fe	3786·09	
	86·6	86·8				Fe		All diffuse line
	87·2	87·4	1–2	p V	3787·34	—	—	
	88·6	88·7	2–3					
	90·0	90·1	1	—	—	Fe	3790·24	
	90·8	91·0	1–2					
	92 3	92·5	1–2					
	93·1	93·3	1–2					
	94·1	94·3	1–2	—	—	Fe	3794·49	

Table I.—*continued.*

Wave-lengths			Intensity.	Chemical origin.				Remarks.
				Probable.		Possible.		
Comparison.	Curve.	Computed.		Origin.	λλ.	Origin.	λλ.	
	3795·1	3795·2	1–2	Fe	3795·15			
	96·0	96·1	1–2					
	96·6	96·8	1–2					
	98·0	98·1	5	H	3798·0	—		Hθ (fiducial line for curve).
3799·4	99·4	99·3	1–2	Mn	3799·39			
99·7	99·7	99·8	2	Fe	3799·69			
3800·7	3801·0	3801·2	2	Mn	3800·68			
02·1	02·1	02·3	2	Mn	3802·05			
	03·6	03·7	2					
	04·7	04·8	1–2					
	05·5	05·7	1–2	—	—	Fe	3805·49	
	06·7	06·8	1–2	Mn-Fe	3806·86	Fe		Double?
	07·7	07·8	2	—	—	Fe	3807·68	
	09·9	10·0	2	—	—	Fe	3810·90	
	10·8	10·9	2	—	—	Fe	3811·95	
	12·1	12·1	2					
	13·1	13·3	2	Fe {	3813·10 3813·22			
	13·4	13·6	2	p Ti	3813·54			
	14·1	14·3	1					
14·7	14·7	14·8	3	{ p Ti Fe	3814·70 3814·67 }	p Cr — {	3814·07	Agrees with Fe line, but relatively stronger.
	15·6	15·7	2	p V	3815·50			
16·0	16·0	16·1	3	Fe	3815·99			
	17·5 } 17·8	17·8	1–2	—	—	—	—	Double?
	19·4	19·5	2					
19·3	19·8	19·9	1 2	He	3819·75			
20·6	20·6	20·7	3	Fe	3820·59			
	21·9	21·9	1					
	22·4	22·5	1					
24·6	24·7	24·7	3	Fe	3824·59			
26·0	26·3	26·4	3	Fe	3826·03			
	27·6	27·5	1–3	—	—	p V	3827·30	
28·0	28·2	28·3	3	Fe	3827·98			
	29·2	29·1	1–2					
29·5	29·5	29·5	4	Mg	3829·50	—	—	Fiducial line for curve and computation.
	30·7	30·7	2–3					
	31·2	31·0	2–3					
	31·8	31·8	1–2	—	—	Ni	3831·84	
32·5	32·5	32·3	4	Mg	3832·45	—	—	Fiducial line for curve.
	33·0	32·9	2	—	—	p Y	3833·10	
	33·6	33·6	3	—	—	Fe	3833·46	Line only low down.
34·4	34·2	34·2	3	Fe	3834·36			
	35·6	35·5	6	H	3835·6	—	—	Hη (fiducial line for curve).
38·4	38·4	38·3	4	Mg	3838·44	—	—	Fiducial line for curve.
39·4	39·2	39·1	1	Fe	3839·40	—	—	Long faint line.
	39·6	39·5	1	—	—	p Fe	3839·76	
40·6	40·6	40·6	3	Fe	3840·58			

Table I.—continued.

Wave-lengths.			Intensity.	Chemical origin.				Remarks.
				Probable.		Possible.		
Comparison.	Curve.	Computed.		Origin.	λλ	Origin.	λλ	
3841·2	3841·2	3841·1	3	Fe	3841·19	—	—	
	42·0	42·0	1	—	—	Co	3842·19	
43·4	43·5	43·5	2	Fe	3843·40	—	—	
	44·5	44·5	2	—	—	V	3844·59	
45·3	45·4	45·4	2	Fe	3845·31			
46·9	47·0	47·0	2	Fe	3846·94			
	47·9	47·9	2					
	49·0	49·1	2					
50·1	50·1	50·1	3	Fe	3850·12			
51·0	51·0	51·1	2	Fe	3850·96			
51·4	—	—	2–3	C	3851·4	—	—	5th edge of C band.
52·7	52·8	52·9	2	Fe	3852·71	Ti	3853·18	
55·0	54·8	54·8	3	C	3855·0	—	—	4th edge of C band.
56·5	56·5	56·6	4	Fe	3856·52	—	—	(Fiducial line for curve.)
	57·1	57·2	2					
	58·1	58·1	2	Ti	3858·26			
	58·4	58·5	3	Ni	3858·44			
	59·3	59·4	2					
60·1	60·1	60·2	4	Fe	3860·06	—	—	(Fiducial line for curve.)
61·8	61·4	61·5	3	C	3861·85	—	—	3rd edge of C band.
	62·9	63·0	2	—	—	Ti	3862·98	
	63·6	63·6	2	--	—	Fe	3863·89	
	64·4	64·5	2					
65·7	65·7	65·6	2	Fe p Cr	3865·67			
	65·9	66·0	1					
	66·6	66·6	2	p Cr	3866·60	Ti	3866·60	
67·4	67·3	67·3	2	Fe	3867·36			
67·6	—	—	1	He	3867·61			
	68·0	67·8	2					
	68·9	68·7	2					
	69·5	69·3	2					
	70·5	70·5	2					
71·5	71·4	71·4	4	C	3871·53	—	—	2nd edge of C band.
	72·0	72·0	2	—	—	{ Fe { p Y	3871·96 3872·30	
72·6	72·6	72·5	4	Fe	3872·64	—	—	
73·9	73·4	73·5	2	Fe	3873·90	—	—	Broad hazy line
	74·5	74·3	1					
	75·0	75·0	1	—	—	{ V { Ti	3875·22 3875·44	
	76·2	76·1	1–2	—	—	Fe	3876·19	
	76·8	76·8	2	—	—	Co	3876·98	
78·2	78·2	78·1	2–3	Fe	3878·15	p Y	3878·40	
78·7	78·7	78·7	3	Fe	3878·72	p V	3878·80	
	79·3	79·3	2					
	79·9	79·9	2					
	80·9	80·8	2					
	82·5	82·5	2					
83·6	83·4	83·4	4	C	3883·55	—	—	1st edge of C band.
	84·4	84·3	2	—	—	Fe	3884·52	

Table I.—*continued.*

Wave-lengths.			Intensity.	Chemical origin.				Remarks.
				Probable.		Possible.		
Comparison.	Curve.	Computed.		Origin.	λλ.	Origin.	λλ.	
	3885·6	3885·6	3	—	—	Fe	3885·66	
3886·4	86·4	86·5	3	Fe	3886·43			
87·2	87·2	87·2	2–3	Fe	3887·20			
	89·2	89·1	8	H	3889·15	—	—	H$_\zeta$ (fiducial line for curve).
	91·5	91·4	2					
	92·3	92·2	2	—	—	Fe	3892·07	
	94·1	94·0	2	Co	3894·24			
	94·9	95·0	3	Co	3895·12			
	95·4	95·3	3	—	—	Ti	3895·38	
	95·8	95·7	3	Fe	3895·80			
	96·6	96·5	2					
	98·0	98·0	2	Fe	3898·03			
	98·3	98·2	2					
	99·3	99·2	2	p V	3899·30			
	99·9	99·8	2	Fe	3899·85			
3900·7	3900·7	3900·7	4	p Ti	3900·68	—	—	Fiducial line for curve and computation.

Wave-lengths.	Intensity.	Chemical origin.				Remarks.
		Probable.		Possible.		
Comparison.		Origin.	λλ.	Origin.	λλ.	
3903·1	2–3	Fe	3903·09	p V	3903·42	
05·3	2	—	—	p Cr	3905·66	
06·1	2–3	p Fe	3906·04			
06·8	2	—		Fe	3906·63 / 3906·89 }	
07·4	1					
08·4	1					
09·6	<1					
10·6	<1					
13·6	6	p Ti	3913·61			
16·2	3	—	—	p V	3916·55	? double.
18·6	3	—	—	Fe	{ 3918·46 / 3918·56 / 3918·79 }	
20·4	3	Fe	3920·41			
23·1	3	Fe	3923·05			
25·9	2	—	—	Fe	{ 3925·79 / 3926·09 }	Probably double.
28·1	4	Fe	3928·06			

Table I.—*continued.*

| Wave-lengths. | | Chemical origin. | | | | Remarks. |
| | Intensity. | Probable. | | Possible. | | |
Comparison.		Origin.	λλ.	Origin.	λλ.	
3930·4	3–4	Fe	3930·45			
32·2	2	p Ti	3932·16			
33·8	10	Ca	3933·83	—	—	K.
35·4	2					
36·5	1–2					
38·2	3					
38·9	1					
40·4	1					
41·9	1–2					
44·2	5	Al	3944·16			
45·2	2	—	—	p Y	3944·94	
47·2	1–2	—	—	p Co	3947·27	
48·6	2–3	—	—	{ Ti	3948·82 }	
				{ Fe	3948·93 }	
50·3	2–3	—	—	{ Fe	3950·10 }	
				{ pY	3950·50 }	
51·8	1					
52·3	3–4	—	—	p V	3952·07	
56·6	4	{ Ti	3956·48 }			
		{ Fe	3956·82 }			
58·2	4	Ti	3958·36			
61·7	6	Al	3961·67			
63·3	1	—	—	Fe	3963·25	
64·7	·2	—	—	Ast	3964·88	
66·3	1					
68·6	10	Ca	3968·63	—	—	H.
70 2	10	H	3970·18	—	—	H.
71·4	1	—	—	Fe	3971·48	
73·5	2	—	—	{ Ni	3973·70 }	↑ double.
				{ pV	3973·85 }	
76·7	1	—	—	Fe	3976·77	
77·8	2	Fe	3977·89			
78·1	trace					
79·3	1	—	—	p Cr	3979·66	
80·4	trace					
82·0	6	Ti	3981·92	Fe	3981·87	
83·1	1	—	—	p Y	3982·79	
83·8	1					
86·3	1–2	Fe	3986·32			
88·3	1					
89·9	2–3	Ti	3989·91	Fe	3990·01	↑ double.
91·3	3	—	—	—	—	
95·2	3–4	—	—	Co	3995·46	
97·7	3	—	—	Fe	{ 3997·55 }	
					{ 3998·16 }	
98·8	4	Ti	3998·79			
4000·4	1–2	—	—	p Y	4000·70	
03·3	1	—	—	p Cr	4003·55	
05·4	5	Fe	4005·41			
06·8	1–2					

Table I.—continued.

Wave-lengths. Comparison.	Intensity.	Chemical origin.				Remarks.
		Probable.		Possible.		
		Origin.	λλ.	Origin.	λλ.	
4009·5	2–3	—	—	{ Ast / Fe	4009·42 / 4009·86	
12·5	5–6	{ p Ti / p Cr	4012·54 / 4012·63			
14·5	3	—	—	Fe	4014·68	
16·7	1					
17·5	2					
18·5	1					
20·6	<1					
21·6	3					
23·1	1					
24·7	3	—	—	Ti	4024·73	
26·3	4–5	He	4026·34			
28·5	2–3	p Ti	4028·50			
30·9	5	Mn	4030·92			
33·2	3–4	Mn	4033·22			
34·6	3–4	Mn	4034·64			
35·9	1	Mn	4035·88	pV	4035·80	
37·7	1					
40·8	4	—	—	Fe	4040·79	
42·4	1					
43·4	1					
44·4	1					
45·9	7	Fe	4045·98			
49·0	3	p Fe	4048·82	Mn	4048·91	
51·0	1					
53·8	3	{ pV / pTi	4053·80 / 4053·98			
55·6	2	pFe	4055·63	pTi	4055·19	
57·4	1–2	—		Fe	4057·50	
58·2	1–2	—		{ Pb / Fe	4058·04 / 4058·37	
59·2	1–2					
61·2	1–2					
62·7	1	—	—	Fe	4062·60	
63·7	6–7	Fe	4063·76			
67·3	3	p Ni	4067·30			
68·1	1	—	—	FeMn	4068·14	
71·8	6	Fe	4071·91			
73·9	1	—	—	Fe	4073·92	
75·3	1					
77·9	10	Sr	4077·89			
80·3	1–2	—	—	Fe	4080·37	
83·1	2	—	—	Mn	4083·10	
84·0	1	—	—	Fe	{ 4083·72 / 4083·92	
85·0	1–2	—	—	Fe	{ 4084·65 / 4085·16	
86·7	3					
89·7	1					

Table I.—continued.

Wave-lengths.		Chemical origin.				Remarks.
	Intensity.	Probable.		Possible.		
Comparison.		Origin.	λλ.	Origin.	λλ.	
4092·5	2	—	—	Fe	{ 4092·43 { 4092·67	
96·2	1–2	—	—	Fe	{ 4096·13 { 4096·26	
98·2	1	—	—	Fe	4098·34	
4101·9	10	H	4102·00	—	—	Hδ (AMES' λ 4101·85).
07·6	2	Fe	4107·65			
09·9	3	Fe	4109·95	V	4109·91	
11·9	1	—	—	V	4111·94	
14·6	1–2	—	—	Fe	4114·61	
18·7	3–4	Fe	4118·71	Co	4118·93	
21·1	3	He	4120·97			
23·0	3					
28·0	3	—	—	Fe	4127·77	
29·6	2–3					
32·4	3	Fe	4132·24	Li	4132·44	
34·8	3	Fe	4134·84			
37·5	3					
40·1	1					
42·3	1					
43·8	5–6	{ Fe 4143·57 { Ast 4143·92 { Fe 4144·04		} —	—	† double.
46·0	1	p Cr	4145·91	Fe	4146·22	
47·5	1	—	—	{ Mn 4147·65 { Fe 4147·84		
49·4	3–4	—	—	Fe	4149·53	
52·1	2	—	—	Fe	4152·34	
54·8	2–3	—	—	Fe	{ 4154·67 { 4154·98	
56·5	3					
57·8	1	--	—	Fe	4157·95	
58·9	1	—	—	Fe	4158·96	
61·7	3–4	p Ti	4161·70			
63·8	4	p Ti	4163·82			
67·5	2–3					
72·1	3–4	p Ti	4172·07	Fe	4172·30	
73·5	4–5	p Fe	4173·52	p Ti	4173·70	
75·8	1	Fe	4175·81			
77·8	5	—	—	p Y	4177·75	Straight enhanced line of yttrium.
79·0	4–5	p Fe	4178·95			
81·9	3	Fe	4181·92			
84·6	2–3	p Ti	4184·40			
87·6	4–5	Fe	{ 4187·20 { 4187·94			
89·7	1					
91·7	3–4	Fe	4191·60			
94·4	<1					
95·5	1	—	—	Fe	4195·49	
96·4	2–3	—	—	Fe	4196·37	

Table I.—*continued.*

Wave-lengths. Comparison.	Intensity.	Chemical origin. Probable. Origin.	λλ.	Possible. Origin.	λλ.	Remarks.
4198·8	4	Fe	{ 4198·49, 4199·27			
4202·2	3	Fe	4202·20	p V	4202·55	
05·1	3	—	—	{ p V, p V	4204·89, 4205·24	
07·1	1	---		Fe	4207·29	
09·6	2-3					
10·9	1					
12·4	1					
15·7	10	Sr	4215·70			
17·0	<1					
19·4	2	Fe	4219·52			
22·4	3	Fe	4222·38			
23·5	<1					
26·9	7	Ca	4226·90			
29·4	<1					
30·7	<1					
33·3	6-7	p Fe	4233·33			
35·9	4	Fe	4236·11	p V, Fe	4235·88, 4238·19	λλ difficult to determine accurately as the three lines merge into each other.
38·0	1-2		---			
39·0	2-3	Fe	4238 97		--	
40·3	1					
42·0	<1					
42·8	2-3			p Cl	4242·54	
45·0	1-2	p Ni	4244 9			
47·0	7	Sc	4247·00	--		Strongest spark line of scandium.
50·4	4-5	Fe	{ 4250·29, 4250 95			
54·5	6	Cr	4254·51			
55·6	1-2					
58·2	2					
60·6	4	Fe	4260·64			
61·6	1-2					
62·8	1					
64·6	1-2					
65·5	<1					
67·7	2-3	p C	4267 5			
69·8	1					
71·6	4-5	Fe	{ 4271·33, 4271·93	-	-	Probably compound line.
72·8	1					
75·0	5	Cr	4274·96			
80·2	1-2					
83·0	2-3	—	—	Ca	4283·17	†double.
85·6	2	—	—	Fe	4285·61	
87·6	1	—	—	Ti	4287·57	
90·2	6-7	{ Cr, p Ti	4289·89, 4290·38	—	—	Probably compound line.
94·2	5	p Ti	4294·20	Fe	4294·30	
96·7	2-3	p Fe	4296·65			

Table I.—continued.

Wave-lengths. Comparison.	Intensity.	Chemical origin. Probable. Origin.	Probable. λλ.	Possible. Origin.	Possible. λλ.	Remarks.
4300·2	5	p Ti	4300·21			
03·0	4	{Ca	4302·69			
		{p Fe	4303·34			
05·8	1-2	—		FeSr	4305·61	
08·1	5	{Fe	4306·08	Ca	4307·91	Probably compound line.
		{p Ti	4306·10			
13·0	2	p Ti	4313·03			
14·0	2	—		Sc	4314·25	
15·1	4-5	p Ti	4315·14	Fe	4315·26	
18·3	2					
21·2	5	p Ti	4321·20	Sc	4320·91	
22·4	1					
25·8	6	Fe	4325·94			
30·6	2-3	p Ti	{4330·50 / 4330·87}	—		Probably compound line.
33·9	3	—	--	La	4333·93	
38·1	5	p Ti	4338·08			
40·7	10	H	4340·63	—	—	Hγ.
44·3	3	p Ti	4344·45	p Mn	4344·19	
47·4	<1					
51·9	6	p Fe	4351·93	Mg	4352·08	
55·0	1-2					
59·2	3-4	—	—	p Y	4358·88	
62·0	1	—	--	p Ni	4362·40	
64·1	1					
67·8	2-3	p Ti	4367·84			
70·2	2-3					
74·9	7	p Ti	4374·90 {	Sc	4374·63	
				p Y	4375·10	
79·7	2					
83·7	5	Fe	4383·72			
85·5	3	p Fe	4385·55			
88·1	1-2	Ast	4388·10			
91·2	2-3	p Ti	4391·19			
95·2	7	p Ti	4395·20	p Mg	4395·0	
99·9	5·6	p Ti	4399·94			
4404·9	4	Fe	4404·93			
08·1	3					
11·2	1-2	p Ti	4411·20			
15·3	4	Fe	4415·29			
17·0	4-5	p Ti	4417·88			
22·7	3	Fe	4422·74	p Y	4422·74	
25·6	1	Ca	4425·61			
27·4	3	Fe	4427·48			
30·1	2-3	—	—	La	{4430·07 / 4430·22}	
35·5	4-5	Ca	{4435·13 / 4435·85}			
36·6	1	—	--	Mn	4436·52	
41·8	1 2					

Table I.—*continued.*

Wave-lengths.		Chemical origin.				Remarks.
	Intensity.	Probable.		Possible.		
Comparison.		Origin.	λλ.	Origin.	λλ.	
4444·0	7	p Ti	4443·98			
47·0	1					
47·9	2	Fe	4447·89			
50·6	5	p Ti	4450·65			
52·7	1					
55·0	5	{ Ca / p Fe	4454·95 / 4455·30			
58·0	<1					
59·9	1-2					
62·3	3					
64·6	2-3	p Ti	4464·62			
66·5	<1	—	—	Fe	4466·73	
68·7	6	p Ti	4468·66			
71·7	8-9	He	4471·65			
76·2	3	Fe	4476·19			
79·2	1					
80·6	1-2					
82·3	4	Fe	4482·34			
84·4	1	—	—	Fe	4484·39	
85·9	1	—	—	Fe	4485·85	
89·3	3	{ p Ti / p Fe	4488·49 / 4489·35			
91·6	2-3	p Fe	4491·57			
93·8	2					
94·3	2					
95·2	2					
96·8	3					
4501·5	7	p Ti	4501·45			
05·5	1					
08·5	5	p Fe	4508·46			
12·3	1					
15·5	4	p Fe	4515·51			
18·3	1	—	—	Ti	4518·20	
20·4	3	p Fe	4520·40			
22·7	4	p Fe	4522·69			
24·1	<1					
28·8	3	Fe	4528·80	p V	4528·66	
31·0	1	Co	4531·12 {	Cr / Fe	4530·91 / 4531·33	
34·1	7-8	p Ti	4534·14			
35·9	2	—	—	Ti	4535·74	
40·0	1-2					
41·7	3	—	—	p Fe	4541·40	
44·8	3	—	—	Ti	4544·86	
49·7	7-8	{ p Fe / p Ti	4549·64 / 4549·81			
54·2	7-8	Ba	4554·21			
56·1	3-4	p Fe	4556·10	p Cu	4556·10	
58·8	3-4	p Cr	4558·83			
61·3	<1					

Table I.—*continued.*

Wave-lengths.		Intensity.	Chemical origin.				Remarks.
			Probable.		Possible.		
Comparison.			Origin.	λλ.	Origin.	λλ.	
4563·9		7–8	p Ti	4563·94			
66·3		1					
67·4		1					
72·2		7	p Ti	4572·16			
76·5		3	p Fe	4576·51			
80·0		2	—	—	Ba	4579·86	† double.
84·0		7	p Fe	4584·02			
86·8		<1					
88·4		3	p Cr	4588·38			
90·1		3	p Ti	4590·13			
92·5		3	—	—	p Cr	4592·25	
95·1		2					
96·8		2					
4600·8		3					
03·0		2	Fe	4603·13	Li	4603·26	
05·5		2			Sr	4607·51	
07·2		2	—	—			
09·8		1					
13·3		2	—	—	Fe	4613·39	
15·4		3					
19·0		3–4	p Cr	4618·97			
22·0		1					
25·3		2–3	—	—	Fe	4625·23	† double.
29·5		5–6	p Fe	4629·50			
32·8		1–2					
34·3		1–2	p Cr	4634·25			
37·6		2–3	—	—	Fe	4637·69	
40·0		2	—	—	Ti	{ 4639·83 / 4640·11	
42·8		1					
46·3		4–5	—	—	Cr	4646·35	
48·4		2					
51·8		3–4	—	—	Cr	{ 4651·46 / 4652·34	
55·4		2					
57·4		3–4	p Ti	4657·38			
62·2		2					
64·5		2					
67·0		3–4	—	—	Ti	4667·77	
70·8		3–4	—	—	Sc	4670·4*	* THALEN's λ corrected to ROWLAND's scale.
74·0		1					
76·0		1					
79·0		2–3	Fe	4679·03			
82·5		2–3	—	—	p Y	4682·60	
87·0		2–3					
91·6		3	—	—	Fe	4691·61	
94·6		1					
97·0		1					
99·5		2–3					

Table I.—*continued.*

Wave-lengths. Comparison.	Intensity.	Chemical origin. Probable. Origin	Probable. λλ.	Possible. Origin	Possible. λλ.	Remarks.
4703·2	3–4	Mg	4703·33	—	—	? double.
07·8	2–3	—	—	—	—	} Difficult to measure.
09·6	2–3	—	—	—	—	
13·8	6	He	4713·25			
18·5	1	—	—	Cr	4718·60	
21·0	1					
23·0	1					
27·7	2	—	—	{ Fe, Mn	4727·58, 4727·68	
31·4	3–4					
33·8	1	—	—	Fe	4733·78	
37·0	3	Fe	4736·96			
40·5	2					
43·0	1–2	—	—	Ti	4742·98	
45·5	1					
48·0	1					
49·0	1					
54·2	2–3	Mn	4754·23			
57·5	3					
62·0	4					
65·5	1					
67·0	2					
69·0	1					
71·5	1					
73·0	2					
75·5	<1					
79·9	3–4	p Ti	4780·20			
83·1	2					
86·7	2–3	—	—	p Y	4786·80	
89·8	2	—	—	Fe	4789·85	
92·3	1					
98·7	2					
4805·2	5	p Ti	4805·25			
11·0	3					
15·5	<1					
20·0	<1					
24·3	6	p Cr	4824·33			
26·0	<1					
28·5	1–2					
31·3	1–2					
35·5	<1					
37·3	<1					
40·4	2–3	—	—	Co	4840·45	
43·0	2					
48·5	3	p Cr	4848·44			
52·2	1					
55·0	2–3	—	—	p Y	4855·06	
61·5	10	H	4861·53			H_β (F).
66·4	1	p Ti	4866·4			
71·8	4–5	Fe	{ 4871·51, 4872·33			

Table I.—continued.

Wave-lengths.		Chemical origin.				Remarks.
		Probable.		Possible.		
Comparison.	Intensity.	Origin.	λλ.	Origin.	λλ.	
4876·0	1					
78·4	2	Cu	4878·31	Fe	4878·41	
83·9	3–4	—	—	p Y	4883·87	
87·0	1					
91·2	4–5	Fe	{ 4890·95 / 4891·68			
4900·0	3	—	—	p Y	4900·25	
04·6	1					
09·5	1					
11·4	2	p Ti	4911·38			
13 5	1					
21·0	3	Fe	{ 4919·17 / 4920·09 }	Ast	4922·10	Broad and hazy, λ difficult to determine, possibly compounded of Ast and Fe.
24 1	6	p Fe	4924·11			
28·5	1					
34·2	5	Ba	4934·24			
38·9	2	—	—	Fe	4939·00	
46·0	1–2					
57·5 / 57·8 }	3–4	Fe	{ 4957·48 / 4957·79			
62·8	1					
66·2	1	—	—	Fe	4966·27	
68·1	1					
70·7	1					
73·3	1	—	—	Fe	4973·28	
78·4	1					
81·9	1	Ti	4981·91			
82·7 / to / 84·3 }	3	—	..	{ Fe / / Ni	{ 4982·68 / 4984·03 / 4984·30	Apparently a group of lines merging into each other.
91·5	1–2					
94·3	1–2	—	–	Fe	4994·32	
99·7	2	Ti	4999·69			
5005·9 / 06·3 }	2–3	Fe	{ 5005·90 / 5006·31			
12·9	1					
15·8	2	Ast	5015·73			
18·6	5–6	p Fe	5018·63			
22·4	<1	—	–	Fe	5022·41	
27·3	1	—	—	Fe	5027·31	
31·2	2					
35·6 / to / 36·7 }	2	{ Ni / Ti	{ 5035·54 / 5036·09 / 5036·65		—	Chromospheric line apparently covers this group of Fraunhofer lines.
41·1 / to / 41·9 }	3	{ Fe / Ca / Fe	{ 5041·07 / 5041·26 / 5041·80 / 5041·94	—	—	Ditto.
50·5	2					

Table I.—*continued.*

Wave-lengths. Comparison.	Intensity.	Chemical origin. Probable. Origin.	Chemical origin. Probable. λλ.	Chemical origin. Possible. Origin.	Chemical origin. Possible. λλ.	Remarks.
5052·8	2					
56·0	1					
60·2	1					
66·1	2	—	—	Ti	5066·174	
68·9	1	Fe	5068·94			
72·5	1	ρ Ti	5072·50			
73·7	1					
79·2 } 79·9 }	3	{Fe {Fe	5079·41} 5079·92}	-	-	Chromospheric line covers the group of Fraunhofer lines in solar spectrum.
85·5	1-2					
88·5	1 2					
97·2	3	—	—	{Ni {Fe	5097·04} 5097·18}	
98·8	2	—	-	Fe	{5098·75 {5098·89	
5105 7	2	Fe	5105·72			
07·6 } 07·8 }	2	{Fe {Fe	5107·62} 5107·82}	—	—	Broad line central with narrow Fe double.
10·6	1	Fe	5110·57			
15·5	1	—	—	Ni	5115·57	
21·8	2-3					
23·9 } 25·3 }	2	Fe	{5123·90} {5125·30}	-	-	Chromospheric line covers the position occupied by Fe lines at given λλ.
30·5	2-3					
37·4	2			{Ni {Fe	5137 25} 5137·56}	
43·5	1					
48·2	1			Fe	{5148·22 {5148·41	
51·0 } 52·0 }	1	—	—	Fe	{5151·02 {5152·09	
54·2	2-3	ρ Ti	5154·24			
56·0	1-2	—	-	Ni	5155·94	
62·4	1	Fe	5162·45			
67·6	6	{Mg {Fe	5167·50} 5167·68}	-		b4
69 1	6	ρ Fe	{5169·07} {5169·22}	- -		b3. *
73·9	6	Mg	5172·86	-		b2.
78·0	1					
83·8	8	Mg	5183·79	—	—	b1
88·9	2	ρ Ti	5188·87	Ca	5189·02	
91·6 } 92·5 }	2	Fe	{5191·63} {5192·52}	—	—.	Chromospheric line covers position occupied by Fe double.
5202·5	2	Fe	5202·52			

* This is one of the most enhanced Fe lines, but which of the two arc lines at the given λλ is the real enhanced line it is impossible to say without using much larger dispersion.

Table I.—*continued.*

Wave-lengths. Comparison.	Intensity.	Probable. Origin.	Probable. λλ.	Possible. Origin.	Possible. λλ.	Remarks.
5204·7 } 06·2 }	4–5	Cr	{ 5204·68 / 5206·22 }	Fe	5204·77	Chromospheric line single, but covers position of double Cr line.
08·6 } 08·8 }	2–3	Cr	5208·60	Fe	5208·78	
15·3 } to 17·6 }	3	Fe	{ 5215·35 / 5216·44 / 5217·55 }	—		Chromospheric line occupies the position of the three Fraunhofer lines.
27·0	5	{ p Ti / Fe }	{ 5226·70 / 5227·04 / 5227·36 }			Chromospheric line central with fine triplet in Fraunhofer spectrum.
34·8	4					
44·0	1					
47·2 } 47·7 }	1	-		Fe Cr	5247·23 5247·74	Line apparently central with Fraunhofer double.
50·4 } 50·8 }	1–2	—		Fe	{ 5250·39 / 5250·82 }	Faint line apparently central with Fe double.
54·0	1					
58·0	1					
63·0	2					
65·7 } to 66·8 }	2–3	Ca Fe	5265·73 5266·74			Broad line difficult to measure, and apparently covering solar group of Fraunhofer lines.
69·7 } to 70·6 }	3–4	Fe Ca Fe	5269·72 5270·44 5270 56			
75·1 } to 76·2 }	4·5	p Fe	5276·17	Cr	{ 5275·31 / 5276·20 }	
83·8	4	Fe	5283·80			
97·0	1	--		Cr	5296·87	
5300·0	3	—	–	—	—	Broad hazy line.
03·5	1					
07·6	1	—	---	Fe	5307·54	
09·5	1					
16·8	6	p Fe	5316·79	--	—	1474 K.
25·7	2					
28·1 } to 28·7 }	5	Fe	{ 5328·24 / 5328·70 / 5328·75 }	—		Chromospheric line central with Fe double.
33·1	1	—		Fe	5333·09	
37·0	1	p Ti	5336·96			
40·0	2–3	Fe	5340·12			
49·7	2	Ca	5349·65			
53·6	2	—	---	Fe	5353·57	
63·0	4					
65·0	1	—		Fe	5365·07	
70·2 } 71 7 }	4–5	Fe	{ 5370·17 / 5371·73 }			Chromospheric line central with Fe double.
77 0	1					
81·2	2–3	p Ti	5381·22			

Table I.—*continued.*

Wave-lengths. Comparison.	Intensity.	Chemical origin. Probable. Origin.	Probable. λλ.	Possible. Origin.	Possible. λλ.	Remarks.
5383·6	1-2	Fe	5383·58			
91·7	2	—	—	Fe	{ 5391·66 5391 82	
97·3	3	Fe	5397·34			
5404·4 } 06·0 }	4	Fe	{ 5404·36 } { 5405·99 }	—	—	{ Chromospheric line covers position occupied by Fe double.
10·0	2	Cr	5410·00			
15·4	2	Fe	5415·42			
19·0	1					
22 0	1					
25·5	2					
29·9	3	Fe	5429·91			
32·9	2-3					
34·7 }	---	Fe	5434·74	—	—	{ Diffuse line covering Fraunhofer group.
to	—	—	—	Ni	5436·07	
36·8 }	—	—	—	Fe	{ 5436 51 5436·80 }	
47·1	5-6	Fe	5447·13			
53·0	1					
55·8	5	Fe	5455·83			
59·5	1					
63·5	4	Fe	5463·49			
66 6	1-2	—	—	Fe	5466·61	
70·5	2					
74·1	3	Fe	5474·11			
76·5 } 76·8 }	4-5	—	—	Fe	{ 5476·50 5476·78 }	
83·3	1-2	—	—	Fe	5483·31	
90·4	1-2	—	—	Ti	5490·37	
94·0	1-2					
97·7	2	Fe	5497·74			
5501·7	2-3	Fe	5501·68			
07·0	2-3	Fe	5507·00			
11·6	2-3					
14·5	2	—	—	Ti	{ 5514·56 5514·75 }	
18·0	1-2					
27·6	6	---	—	--	—	Rather broad and probably double. The strong Mg spark line at λ 5528 64 falls on the less refrangible portion of the eclipse line.
35·6	5	Ba	5535·69	Fe	5535·64	
40·0	1					
43·4 } 44·2 }	2	—	—	Fe	{ 5543·41 5544·16 }	
46·7 } 47·2 }	2	—	—	Fe	{ 5546·73 5547·22 }	
55·1	1-2	—	—	Fe	5555·12	
58·2	2	—	—	Fe	5558·21	

Table I.—continued.

Wave-lengths. Comparison.	Intensity.	Chemical origin.				Remarks.
		Probable.		Possible.		
		Origin.	λλ.	Origin.	λλ.	
5560·4	1	—	—	Fe	5560·43	
62·9 } 63·8 }	1–2	—	—	Fe	{ 5562·93 { 5563·82	
65·9	2	Fe	5565·93			
69·8	2–3	Fe	5569·85			
73·1	2–3	Fe	5573·08			
76·3	2	Fe	5576·32			
82·2	1–2	Ca	5582·20			
87·0	6	Fe	5586·99			
89·5	1–2	Ca	{ 5588·99 { 5590·34			
92·0	2–3					
94·7	1	Ca	5594·69			
98·7	3	Ca	5598·71			
5601·5 } 03·1 }	3	Ca	{ 5601·51 { 5603·08			
10·0	1					
15·9	5–6	Fe	5615·88			
24·8	3–4	Fe	5624·77			
33·0	1					
37·0	2					
39·5	1					
43·0	2					
55·6	2–3	—	—	Fe	5655·72	
59·1	6	Fe	5659·05			
62·7	3	Fe	5662·74			
69·5	3					
82·9	2					
5701·8	2	Fe	5701·77			
06·2	2	—	—	Fe	5706·22	
09·7	4	Fe	5709·60			
28·0	1–2					
32·0	1–2	Fe	5731·98			
53·3	3	Fe	5753·34			
63·2	3	Fe	5763·22			
80·8	3–4					
91·2	1–2	—	—	{ Cr { Fe	5791·17 5791·24	
5809·4	2	—	—	Fe	5809·44	
53·9	1–2	Ba	5853·90			
75·9	7	He	5875·87	—	—	D₃.
90·2	3	Na	5890·19	—	—	D₂.
96·2	3	Na	5896·16	—	—	D₁.
6563·0	5	H	6563·05	—	—	Hα (C).

TABLE II.—Coronal Radiations.

Group I.   Typical ring, λ 5303·7.

Wave-length.	Brightness. Max. = 10.	Wave-length.	Brightness. Max. = 10.
3952·5	2	4518	1
4007	1	4536	1
4022	1	4588·5	1
4056	2	4657	1
4068	1	4685·5	2
4085	1	4714	1
4121	1	4727	1
4168	1	4737	1
4220	2	4768	1
4231·3	5	4808	1
4248·5	2	4922	2
4262	1	5125	1
4400	1	5137	1
4430	1	5303·7	10

Group II.   Typical ring, λ 3987·0.

Wave-length.	Brightness. Max. = 10.
3800	3
3987 0	5
4275	1
4568·5	3

Group III.   Typical ring, λ 4359·5.

Wave-length.	Brightness. Max. = 10
4030	1
4192	1
4204	1
4302	1
4323	2
4359·5	3
4485	1
4648	1
4662	1
4788	1
4890	1
5001	1
5255	1

PLAN OF FORT VIZIADRUG.
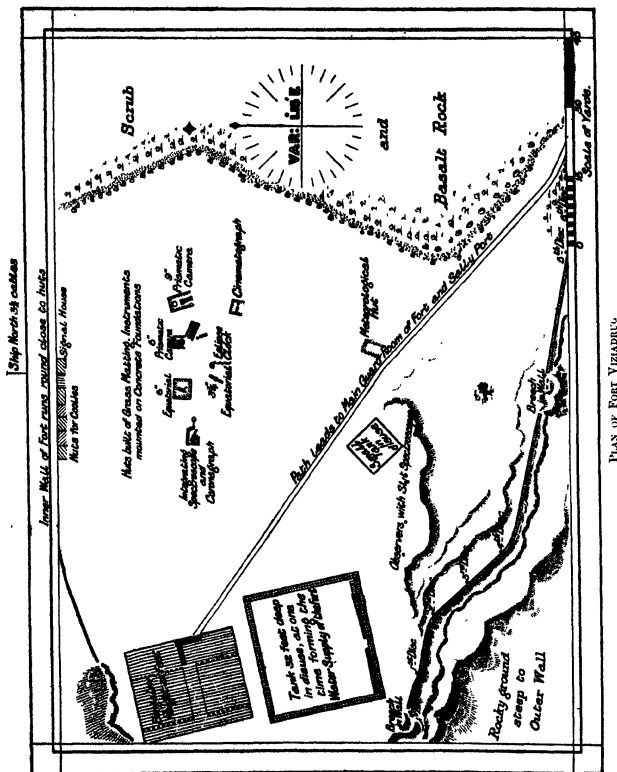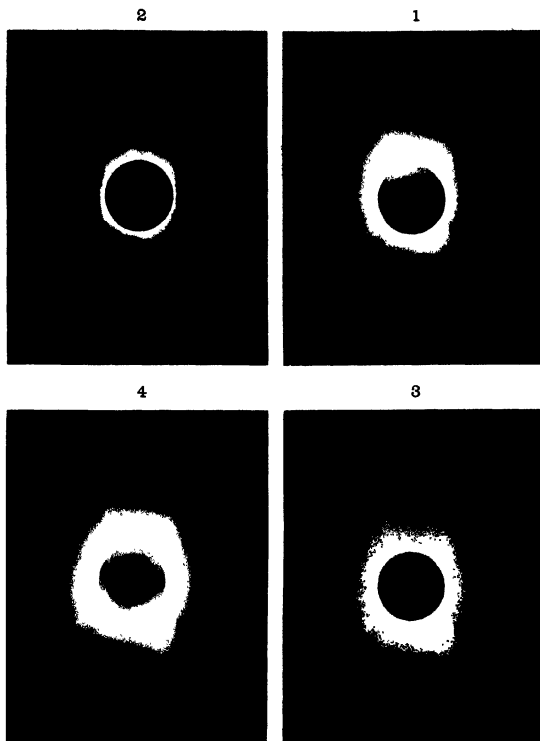
TOTAL ECLIPSE OF THE SUN, JANUARY 22, 1898.

PHOTOGRAPHS TAKEN AT VIZIADRUG.

a

b

c

d

e

f

g

h

i

j
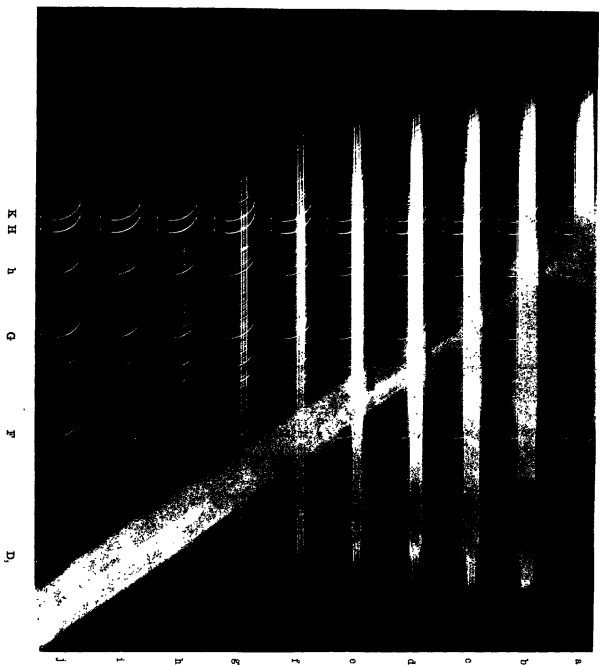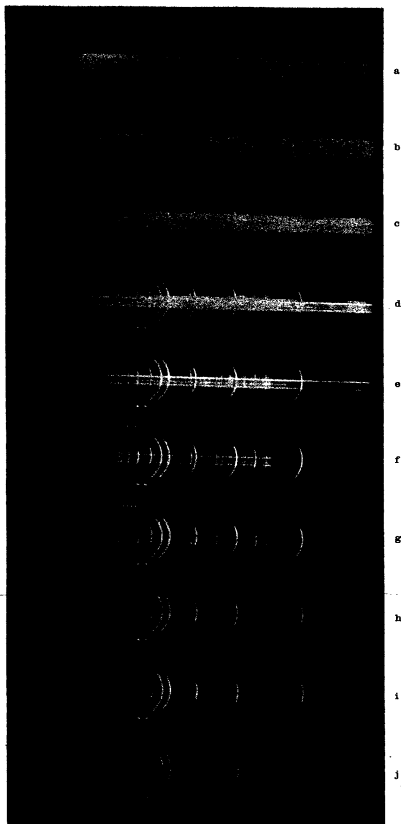
VI. *On the Heat Dissipated by a Platinum Surface at High Temperatures.*
Part IV. *Thermal Emissivity in High-pressure Gases.*

By J. E. PETAVEL, *A.M. Inst. C.E., A.M. Inst. E.E., Past* 1851 *Exhibition Scholar,*
John Harling Fellow of the Owens College, Manchester.

Communicated by ARTHUR SCHUSTER, *F.R.S.*

CONTENTS.

PART IV.*

Thermal Emissivity in High-pressure Gases.

The question of the heat dissipated by a hot body in gases at ordinary pressures has received considerable attention during recent years. The subject has been

* For Parts I., II., III. see 'Philosophical Transactions of the Royal Society,' A, vol. 191, p. 501, 1898.
    Part I. was entitled "Emissivity of a Bright Platinum Surface in Air and other Gases."
    Part II.   „   "A Bolometric Study of the Law of Thermal Radiation."
    Part III.   „   "On the Variation of the Intrinsic Brilliancy of Platinum with Temperature."

         23.10.1901

experimentally treated in many different ways. The rate of cooling of a body of known specific heat has been directly measured by DULONG and PETIT, NARR, MACFARLANE, NICHOL, STEFAN, BRUSH, BOTTOMLEY, WINKELMANN, KUNDT, and WARBURG, ECKERLEIN, GRAETZ, &c. By CHRISTIANSEN'S method, the value of the conductivity has been derived from the fall of temperature per unit length along the axis of a cylinder carrying a constant flow of heat. SCHLEIERMACHER, SALA, AYRTON, and others have preferred to measure the quantity of electrical energy dissipated per unit time. These experiments have, however, been carried out at or below the atmospheric pressure, and the question of the heat dissipated in gases at high pressures has rarely been touched upon. From the ordinarily accepted principles of the Kinetic theory of gases, it may be shown that the conductivity of any perfect gas is independent of pressure. The experimental work of STEFAN and of KUNDT and WARBURG has gone far to confirm this law as far as ordinary pressures are concerned. It will be seen, however, that at higher pressures, only a small proportion* of the loss of heat is due to conductivity, and the question as to whether the theoretical law is strictly correct, though well worth investigation, is not of primary importance.

For the above reasons the present work has been restricted to a study of the total heat dissipated at exceptionally high pressures and temperatures.

The Apparatus.

The method employed is the same as that described in 1898 in the first part of the present series. It will therefore be sufficient to recall here that the measurements are made by means of a wire, calibrated as an electric thermometer according to CALLENDAR'S system, and heated by an electric current, the readings of the current passing through the wire, and the electromotive force at the potential terminals, being made by a potentiometer. The standard resistance had a temperature coefficient of 0·000003, and as it was efficiently cooled, the small outstanding variations in temperature involved no correction.

The results of the comparison of this coil (0·01 ohm, WOLFF, No. 779) with two standards of reference (0·1 ohm, No. 709, and 0·001 ohm, No. 834) are given below :—

	Standard 0·1 ohm in terms of the working standard.	Standard 0·001 ohm in terms of the working standard.
On the 27th September, 1895 . .	10·005	0·10004
On the 3rd March, 1900	9·985	0·09991

The original resistance of the working standard was, according to the Reichsanstalt Certificate, 0·0099967 at 24°·9 C. and 0·0099965 at 16°·5 C. On the 3rd March, 1900, its resistance having increased about 0·16 per cent., was therefore

* About $\frac{1}{14}$th at 160 atmospheres.

0·010012 ohm.   From similar observations it was deduced that the electromotive force of the CLARK's cell (No 5217 A) had fallen by 0·2 per cent. during the last two years.   The variations being in opposite directions tend to cancel, and the outstanding error as far as the energy measurements are concerned is not sufficient to affect the results.   On the other hand the temperature measurements are, of course, independent of the absolute value of the standard.

The CLARK's cell was kept in the inner chamber of a Berthelot calorimeter; only very slow variations of its temperature were therefore possible, and the agreement of the electromotive force with the temperature indicated by the attached thermometer was thus ensured.

The enclosure was formed by a steel cylinder 6·0 centims. in outside diameter and 2·06 centims. bore and 45 centims. long (see fig. 1).   It was surrounded by a water jacket.   The temperature of the enclosure was estimated by a thermometer, the spherical bulb of which just fitted a hole drilled in the walls of the steel cylinder, half-way along its length ; the depth of the hole being such that the bulb was equidistant from the inside and outside face of the wall.   After the thermometer was in position the hole was filled with mercury to ensure satisfactory contact.

The ends of the steel enclosure were closed by the fibre plugs (P_1, P_2, fig. 1), held in position by the gunmetal nuts, N_1, N_2.   The current was led in and out by the copper bars, B_1, B_2 (about 1 centim. in diameter), passing through the centre of the plugs.   One end of the radiating wire R was silver-soldered to the copper terminal B_1, the other end was connected to B_2 by a flexible connection (200, No. 40, S.W.G.).   The radiating wire was kept taut by means of the light spring S.   One of the potential terminals T_1 was connected to the steel cylinder, the other kept insulated and passed through the fibre plug P_1.   The enclosure was kept at a constant temperature by means of a water circulation.

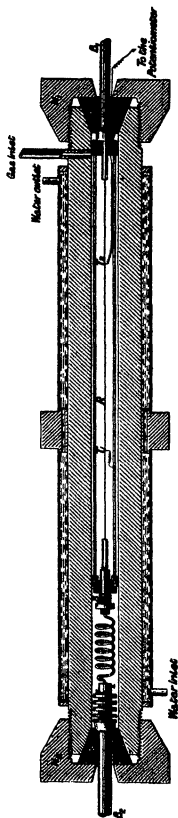The same pure platinum wire was used as a radiator

Fig. 1. Scale = about ⅓.

in all the experiments, but owing to two slight accidents the calibration is not quite the same throughout. The wire was calibrated four times during the course of the work.

It is, of course, necessary to eliminate the cooling effect of the terminals, and to ensure this, the heat dissipated by the middle third of the wire was alone measured. In the experiments made in 1898 the radiating wire was placed vertically. At the pressure of one or two atmospheres that was then used, consistent results could be obtained by this disposition, and it had the advantage that the wire was an axis of symmetry with regard to the convection currents. At high pressures, however, owing to increased convection, it is impossible to keep a sufficient length of radiating surface at a uniform temperature unless the axis of the radiator is horizontal. For the reasons given in Part I. (see page 506 and Plate 20), the absolute value of the emissivity is rather greater when the radiator is horizontal than when it is vertical.

The pressures recorded were obtained from the readings of an air manometer, which was calibrated in the ordinary way by weighing mercury. The glass mano-meter tube ended in an air bulb, contained in a strong steel vessel resembling a small Cailletet bottle. The relative volumes of the air bulbs were regulated so that one instrument began reading at 7 atmospheres and the other at 100. The tubes were so arranged that the level of the mercury in the reservoir could be estimated, and the correction for the weight of the mercury column calculated. An ordinary Bourdon gauge was attached to each manometer for convenience of reference during the experiment.

All joints and connections were made metal to metal without intermediate packing. Where high pressures are to be used, the importance of designing the connections in this manner cannot be exaggerated. A metallic joint once made will remain tight for any length of time. And on the other hand, if at any time it is necessary to change the connections, the joint can be broken and re-made in a few minutes.

Experimental Work.

The same routine was observed throughout in taking the observations. The enclosure having been filled with gas at about the required pressure, the water circulations were started. As soon as the temperature had become constant, the pressure of the gas, and the temperatures of the standard resistance, of the pressure gauge, and of the enclosure, were read, and any other observations that might be of importance, such as the barometric pressure, the temperature of the room, of the circulating water, &c., recorded.

In all cases two readings of the current and electromotive force were taken, so as to cancel any thermo-electric effects by reversing the current. The differences in the readings shown in Table I. were due in great part to inequalities in the resistance

of the two sides of the reversing switch. They were in a large degree eliminated in subsequent work. As however the mean of the two readings is always taken, variations of this kind hardly affect the results.

The method of working out the observations has in all cases been identical to that shown in Table I., but for the sake of brevity and clearness the final results have alone been given.

The emissivity expressed in C.G.S. units is equal to the number of therms (water-gramme-degrees) dissipated, per square centimetre of surface of radiator,* per second, per degree Centigrade above the temperature of the enclosure. During the experiments the temperature of the enclosure was usually about 18° C. The results of the observations are recorded in Tables II.–VI. In the case of each series the pressure indicated is the absolute pressure of the gas when no current is passing through the radiating wire. The pressure readings recorded have been corrected for the temperature of the manometer, the height of the mercury column and the departure of air from BOYLE's law, this latter correction being made according to AMAGAT's results.

The importance of having some data with regard to the total heat dissipated cannot be denied, but our knowledge of the phenomena is not at present sufficient for a general theory to be evolved. It was therefore thought preferable to reduce the results in such a manner that they should not involve any assumptions which later work might prove to have been unjustifiable. It must be admitted that the emissivity as above defined is dependent to a certain degree on experimental conditions, such as the size and shape of the radiator and enclosure. But the advantage is gained that, with these conditions specified, it is a well-defined quantity, and that when the laws which govern the transfer of heat through high-pressure gases become accurately known, the generalisation of the present results will present no difficulties. The choice of the size of the radiator and enclosure was made with a view to minimise the objection that has been pointed out.

Before passing to other considerations, two possible causes of error require investigation. If what SMOLUCHOWSKI DE SMOLAN† has called the discontinuity of temperature—the equivalent of what is known to electricians as contact resistance—exists between the enclosure and the contained gas, the observations will give results too low.

This effect, however, is inversely proportional to the pressure of the gas, and above 1 atmosphere would correspond to an effective increase in the radius of the enclosure of less than 0·00001 centim. It need not, therefore, be brought into consideration.

A more serious cause of error lies in the fact that the walls of the enclosure being 2 centims. thick, the inside surface might be considerably above the tempera-

---

* All the results are referred to the area of the radiator as measured at a temperature of about 18° C.

† 'Phil. Mag.', vol. 46, p. 192, 1898; also E. WARBURG and E. GEHRKE, 'Annalen der Physik,' 4th s., vol. 2, p. 102, 1900.

ture indicated by the thermometer ; this again would cause the emissivity to be under-estimated. Taking the maximum values of the temperature and pressure, we find that in air at 160 atmospheres and 1200° C. the emissivity is 0·01259, and the heat dissipated = E$\vartheta\pi.d$ = 5·249 C.G.S. units per second, per unit length. The inside diameter of the enclosure is 2·06 centims., its superficial area 6·472 sq. centims. per unit length. Taking the conductivity of steel as 0·11, this would correspond to a fall of temperature of 7°·3 C.

Thus the temperature interval we estimated at 1200, really was 1193. This cause of error would be very serious were it not for the fact that it decreases not only with the square of the temperature but also with the pressure. At 160 atmospheres and 500° C. the flow of heat is 1·612 C.G.S. units, making the error 2°·3 C. ; at 100 atmospheres and 500° C. the error is 1°·6 C., and for 30 atmospheres and 500° C. = 1°·0 C.

Over the greater part of the range of observation the error due to this cause is therefore below one-third per cent., but it rises to about two-thirds per cent. at the highest temperatures and pressures

Results obtained.

In all cases the heat dissipated by a hot body surrounded by gas is the sum of three distinct quantities, all three being functions of the temperature, and one at least being also a function of the pressure. The formula for the total radiation expressed in words will therefore read—

$$\text{Total heat dissipated} = \text{Convection} + \text{Conduction} + \text{Radiation}.$$

$$\text{Or Emissivity} = F_1(p, t) + F_2(t) + F_3(t) \quad . \quad . \quad . \quad . \quad \text{(i.).}$$

Of these four quantities, two, Conductivity and Radiation, have been determined by previous experimenters, and the determination of a third will enable us to solve the equation for any value of t or p. The heat carried off by convection not being directly measurable, our only resource is to determine it by difference.

The experimental results will be found plotted in figs. 2 to 6. It will be seen that in all cases above about 10 atmospheres the emissivity is practically a linear function of the temperature, from which fact we conclude that the loss by radiation must be relatively very small. We may therefore write for any given pressure

$$E = m + n\vartheta.$$

Both the values m and n increase as the pressure rises : closer observation shows that for any one gas they may be expressed as exponential functions of the pressure. We thus obtain for the total emissivity an expression of the form
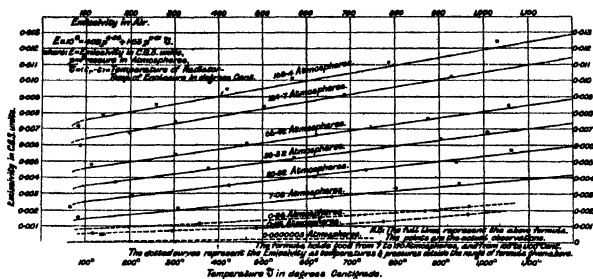
$$E = ap^a + bp^\beta\vartheta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii.),}$$
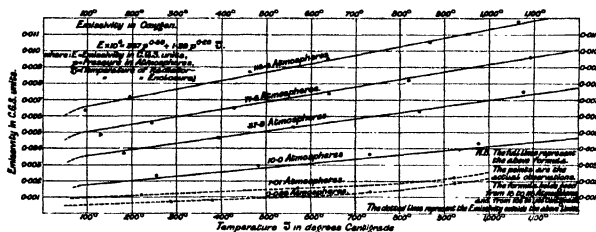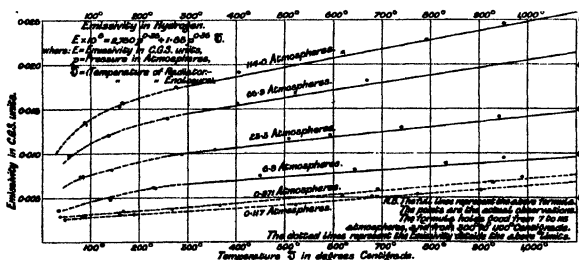
Fig. 2.

Fig. 3.

Fig. 4.
2 H 2
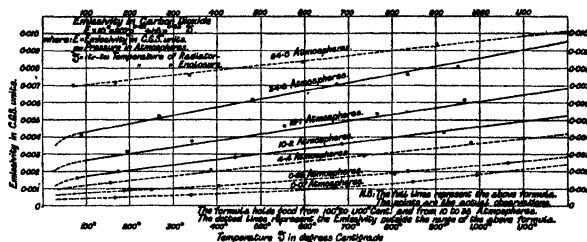
Fig. 5
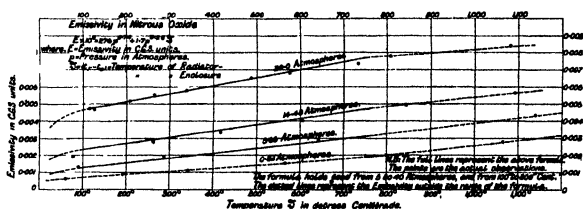
Fig. 6.

where E = emissivity in C.G.S. units, p = pressure in atmospheres, ϑ = (Temperature of the Radiator —Temperature of the Enclosure) in degrees C. The values of the constants a, b, α and β, are given in the following table :—

| | $a \times 10^6$. | $b \times 10^6$. | α. | β. | The formula holds good | | | |
					From ϑ =	To ϑ =	And from p =	To p =
Air	403	1·63	0·56	0·21	100	1100	7	170
Oxygen . . .	387	1·39	0·58	0·28	100	1100	15	115
Hydrogen . .	2750	1·88	0·35	0·36	300	1100	7	113
Nitrous oxide .	276	1·70	0·74	0·28	100	800	5	40
Carbon dioxide .	207	1·50	0·82	0·33	100	1100	10	35

In figs. 2 to 6 the full lines represent the formulæ, the constants of which have just been given ; the points are the actual observations. The agreement between the calculated and observed results will be seen to be satisfactory.

The increase of emissivity with pressure in the case of air is shown in fig. 7.
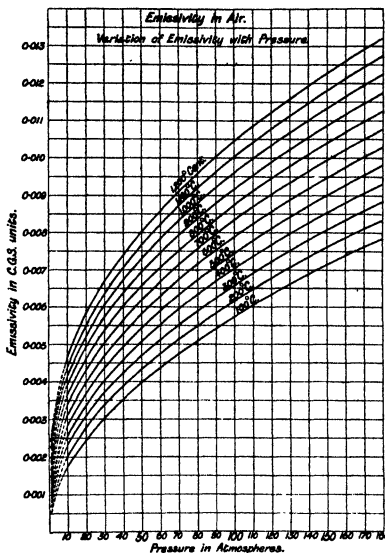

Fig. 7

On the Variation of Emissivity with Pressure.

To return to the general formula for the heat dissipated. From the results so far obtained, we may write for pressures above 10 atmospheres

$$\text{Emissivity} = ap^{\kappa} + bp^{\beta}(t_r - t_s) = F_1(p, t) + F_2(t) + F_3(t),$$

and therefore

$$\left.\begin{array}{l}\text{Heat lost by convection}\\\text{in therms per sq.}\\\text{centim., per degree}\\\text{temperature interval,}\\\text{per second}\end{array}\right\} = \left[ap^a + bp^\beta(T_r - T_e)\right] - \left[C_1k(1 + nT_r)\right] - \left[C_2\,\frac{T_r^4 - T_e^4}{T_r - T_e}\right]$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{. . . (iii.)},$$

where a, α, b, β, C_1, k, n_1C_2 are constants, p the pressure, and T_r and T_e the absolute temperatures of the radiator and the enclosure.

The numerical values of each of these three terms of the above equation* for air at temperatures between 100 and 1000 and pressures between 10 and 160 atmospheres will be found in Table VII., and the values of the emissivity for the other gases studied are recorded in Tables VIII. to XI.

If we consider the temperature constant, we obtain for the variation of emissivity with pressure :†

$$E = ap^a + b_1p^\beta,$$

and we have seen that the exponents of p are different for every gas studied. It is not easy to reconcile this fact with the theory that conductivity, viscosity, and specific heat are all three independent of pressure. It must be remembered, however, that the theory is deduced on the assumptions that the molecular paths are straight lines, that the radius of the molecular sphere of action is very small compared with the mean free path, and that the cohesion of the gas is a negligible quantity.

Neither of these hypotheses seems altogether justifiable at pressures of one or two hundred atmospheres.

* The numerical value of the heat lost by conductivity is merely given to show that, as far as our present knowledge goes, it forms a very small proportion of the total loss observed at high pressures. Though the conductivity of gases at ordinary pressures is fairly accurately known its temperature coefficient is as yet uncertain. No observations are available above 200° C., and even at lower temperatures there is much discrepancy between different observers. For air, for instance, WINKELMANN gives 0·0019, EICHHORN 0·00199, SCHLEIERMACHER 0·0028, and ECKERLEIN 0·0036. Again, the effective temperature of the gas is uncertain, but to make sure of not under-estimating the part played by conductivity we have used the maximum value T_r.

† DULONG and PETIT ('Annales de Chimie et de Physique,' 1st s., vol. 7, p. 337, 1817) found for the loss due to convection and conduction per degree temperature interval

$$Np^a\vartheta^{1\cdot 233}.$$

At constant temperature the variation with pressures would be

$$a_1p^{a_1},$$

where a_1 = a constant, p = pressure, a_1 = 0·45 for air, 0·38 for hydrogen, and 0·517 for CO_2.

In the formula (iii.) above $\dfrac{b}{a}$ is of the order $\frac{1}{100}$, and for small values of ϑ, the emissivity is approximately equal to

$$a_2p^a,$$

where a_2 = a constant, p = pressure, and α (see p. 236) is 0·56 for air, 0·35 for hydrogen and 0·82 for

The results of WARBURG and VON BABO* have shown that for gases near their critical temperature, there is a great increase of viscosity as the critical pressure is approached.

MEYER explains, and at the same time confirms this result, by a calculation based on the assumption that the gas is a mixture of double and simple molecules.

The relatively large value of the exponents of p (see page 236) in the case of carbon dioxide and nitrous oxide, may be accounted for by the above fact.

It has been shown that the conductivity of a gas is proportional to its viscosity. Taking into account that even the so-called permanent gases diverge considerably from BOYLE's law at the higher pressures, it seems probable that the constancy of this conductivity is far from absolute.

A glance at figs. 2 to 6 will show that in every case a sharp fall occurs in the curve of emissivity when ϑ decreases below a certain limit;† and from such measurements as have been made at small temperature intervals there can be no doubt that at pressures above 10 or 20 atmospheres the conductivity accounts for only a very small proportion of the total heat dissipated, but none the less the conductivity may have considerably increased in absolute value.

The present experiments do not, therefore, definitively settle the question. To eliminate convection at the higher pressures, the temperature interval between the radiator and the enclosure would have to be a small fraction of a degree. An apparatus designed to measure up to 1000 degrees is not suitable for this kind of work. It was therefore thought preferable to leave the subject for a future investigation, rather than record values which would only be a rough approximation.

On the other hand, at ordinary pressures the constancy of the conductivity has received ample experimental verification, and there is no difficulty in determining the absolute amount of heat lost by each of the three factors: radiation, convection, and conduction.

An Analytical Study of the Total Heat Dissipated.

Our first object must be to determine what proportion of the total loss is due to conduction alone.

In the present case we have two coaxial cylinders : the one forming the radiator, the other the enclosure, and we must determine what is the thermal conductance of the air filling the intervening space.

---

carbon dioxide. When it is taken into account that DULONG and PETIT's formula was derived from observations made at pressures below 1 atmosphere and under totally different experimental conditions, even this approximate agreement is not devoid of interest.

* 'Wied. Ann.,' vol. 17, p. 390, 1882.

† About 300° for hydrogen and 100° for all other gases.

Let us divide this space into a number of concentric cylinders, having a length l equal to the distance between the potential terminals of the wire and a thickness Δr.

The thermal resistance of each of these cylinders will be

$$= \frac{\Delta r}{l 2 \Pi r} \times \frac{1}{K},$$

where r is the radius of the cylinder and K the conductivity of the gas under consideration, in this case air.

The total thermal resistance will therefore be

$$= \frac{1}{2 \Pi l K} \int \frac{dr}{r}.$$

Integrating between the limits $r = R$ = radius of the enclosure and $r = r_1$ = radius of the radiator, we have

$$\text{Total resistance} = \frac{1}{2 \Pi l K} (\log_e R - \log_e r_1),$$

or the conductance $= \frac{K 2 \Pi l}{\log_e R - \log_e r_1}$ = Total flow of heat per degree Centigrade of temperature interval per second.

Now the emissivity is defined as the flow of heat per unit surface of radiator per degree Centigrade per second, and the above expression divided by the superficial area of the wire will thus be equal to the part of the emissivity due to conduction alone.

$$\mathrm{E}_c = \frac{K 2 \Pi l}{2 \Pi r_1 l (\log_e R - \log_e r_1)} = \frac{K}{(\log_e R - \log_e r_1) r_1} \quad . \quad . \quad . \quad . \quad \text{(iv.)},$$

where E_c is the part of the emissivity due to conduction alone. In the present case $r_1 = 0.0553$, $R = 1.03$, thus :

$$\mathrm{E}_c = \frac{K}{0.1617}.$$

Inversely, if the convection is zero and the radiation $= R$,

$$0.1617 \, (E - R) = K = \text{conductivity}.$$

By measuring the emissivity at low temperatures and pressures, we can reduce, though not entirely suppress convection, and, by subtracting the heat lost by radiation, obtain a comparison with the standard determinations of the conductivity of gases. The value thus calculated will always be in excess of the true conductivity of the gas by an amount proportional to the heat lost by convection.

In the following table E is the emissivity at 100° C., and 0·1 atmosphere ; R the radiation calculated from J. T. BOTTOMLEY's experiments,* and K_{100} is the conductivity at 100° C. according to WINKELMANN :—

* 'Roy. Soc. Proc.,' vol. 66, p. 276, 1900.

	Air.	Oxygen.	Hydrogen.	Carbon dioxide.
E	0·00050	0·00052	0·00285	0·00044
E—R	0·00047	0·00049	0·00282	0·00041
0·1617 (E—E) . . .	0·000076	0·000079	0·000456	0·000066
K₁₀₀	0·000067	0·000066	0·000455	0·000046
Heat lost by convec- } tion × 0·1617 . }	0·000009	0·000013	0·000001	0·000020

The fact thus brought out, that the convection is a maximum in carbon dioxide and a minimum in hydrogen,* may cause some surprise, for the coefficient of viscosity of the former is only half that of the latter. It must, however, be remembered that the force causing convection is the buoyancy of the heated gas which is proportional to its density. The heat conveyed by the stream of gas is proportional to the volume of the gas in motion, its mean rise of temperature, its mean velocity, its specific heat, and its density; thus we have for a given temperature†

$$\text{Convection} = A \frac{\rho^2 C}{\gamma}.$$

Where A = a constant, ρ = the density of the gas, C = the specific heat, γ = the viscosity.

According to the above formula, the relative convection in carbon dioxide, oxygen, air, and hydrogen is represented by the numbers 20, 8·6, 8·5, and 1·2. If we take into consideration that the experimental numbers 0·000020, 0·000013, 0·000009, and 0·00001 are obtained from the difference of two nearly equal quantities, the agreement may be considered satisfactory.

We are now in a position to express in absolute units the heat lost respectively by convection, conduction, and radiation.

* This fact is confirmed by the experiments of KUNDT and WARBURG (see 'Pogg. Ann.,' vol. 156, p. 179, 1875). They found that, in a certain apparatus, the heat dissipated was independent of pressure up to 30 millims. in the case of air, and up to 154 millims in the case of hydrogen.

† This relation between the heat lost by convection and the density, viscosity, and specific heat of a gas only holds good when the speed of the convection currents is very low ; at higher temperature and pressures the phenomenon is more complex.

At one-tenth of an atmosphere and 100° C., we have for the various gases—

Pressure, 0·1 Atmosphere.   Temperature interval, $\vartheta = 100°$ C.*

Heat lost—	Air.		Oxygen.		Hydrogen.		Carbon dioxide.	
	In C G.S. units.	In per cent.	In C.G.S. units.	In per cent.	In C.G.S. units.	In per cent.	In C.G.S. units.	In per cent.
By conduction . .	0·00041	82	0·00041	79	0·00281	98·6	0·00028	64
By convection . . .	0·00006	12	0·00008	15	0·00001	0·4	0·00013	30
By radiation . . .	0·00003	6	0·00003	6	0·00003	1·0	0·00003	6
Total heat lost = emissivity	0·00050	100	0·00052	100	0·00285	100	0·00044	100

And if, for the present, we assume the conductivity constant the values at 100 atmospheres and 100° C. would be—

Pressure, 100 Atmospheres ; Temperature interval, $\vartheta = 100°$ C.*

Heat lost—	Air.		Oxygen.		Hydrogen.	
	In C.G.S units.	In per cent.	In C.G.S. units.	In per cent.	In C.G.S. units.	In per cent.
By conduction . .	0·00041	7·1	0·00041	6·7	0·00281	21·4
By convection . .	0·00530	92·4	0 00566	92·8	0·01026	78·4
By radiation . . .	0 00003	0·5	0·00003	0·5	0·00003	0·2
Total heat lost = emissivity	0·00574	100	0·00610	100	0·01310	100

The result is remarkable ; for instance, in the case of air, at one-tenth of an atmosphere 12 per cent. of the total loss is due to convection, and at 100 atmospheres 92 per cent. is due to this same cause. The heat lost by convection is in air 100, and in hydrogen 500 times greater at a pressure of 100 atmospheres than it is at 0·1 atmosphere.

For temperature-intervals of several hundred degrees it is impossible experimentally to eliminate either of the three effects of convection, conduction, or radiation, but the first of the three can be greatly diminished. To this end the apparatus was filled with lightly packed glass-wool. The proportion of the total volume actually occupied by the glass was 2·0 per cent.

* The loss of heat is expressed in therms per square centimetre of radiating surface, per second per degree Centigrade of temperature interval.

Emissivity in Air.

Temperature ϑ in degrees Centigrade.	Gas only.			Convection diminished by filling the enclosure with lightly packed glass wool.			Ratio $\dfrac{C}{C_1}$
	Heat dissipated by conduction and radiation.	Emissivity.	Heat dissipated by convection = C.	Heat dissipated by conduction of the glass and air and by radiation.	Emissivity.	Heat dissipated by convection = C₁.	
1·01 atmo. 100	0·00045	0·00084	0·00039	0·00056	0·00071	0·00015	2·60
200	0·00054	0·00095	0 00041	0·00065	0·00082	0·00017	2·41
300	0·00064	0·00106	0·00042	0 00075	0·00093	0·00018	2 33
400	0·00075	0·00119	0·0044	0·00086	0·00105	0·00019	2·32
69·7 atmo. 100	—	0 00474	0·00429	—	0·00176	0·00120	3·58
200	—	0 00513	0·00459	—	0·00217	0·00152	3·02
300	—	0 00553	0·00489	—	0·00235	0·00160	3·06
400	—	0·00593	0·00518	—	0·00247	0·00161	3 22
169 atmo. 100	—	0·00761	0·00716	—	0·00462	0·00406	1·76
200	—	0·00808	0·00754	—	0·00544	0·00479	1 57
300	—	0·00856	0 00792	—	0·00580	0·00505	1·57
400	—	0·00904	0·00829	—	0·00605	0·00519	1·60

As may be seen in the above table the result was to reduce convection to one-half or one-third of its former value. The effect on the total heat lost is of course least at low pressures, at which conduction plays the most important part; the decrease is about 14 per cent. at 1 atmosphere, and twice or three times as much at higher pressures.

On the Influence of Experimental Conditions.

The remaining factors which influence the total amount of heat dissipated are the dimensions of the enclosure and radiator and the temperature of the gas.*

Let us first consider the effect of a change in the diameter of the radiator.

We have found above (p. 240) that the heat lost by conduction is

$$E_c = \frac{K}{(\log_e R - \log_e r)} \frac{1}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (v.),$$

where R is the radius of the enclosure, r the radius of the radiator, and K the conductivity of the gas.

If the ratio $\dfrac{R}{r}$ is very great we may, within a limited range, consider $\dfrac{K}{\log_e R - \log_e r}$ as constant and put $E_c = c_1 r^{-1}$.

* Some experiments on this question were recorded in Part I., but the following additional considerations may be found of interest.

Simultaneously with conduction (i.e., the dissipation of heat by molecular motion) we have convection, by which the heat is carried off by the general upward flow of the gas.

With an infinitely small radiator each molecule comes in contact with the radiator at the temperature of the enclosure. With a radiator of large area a certain proportion of the molecules which strike its upper surface have already become heated by previous contact. The heat lost by convection decreases therefore as the diameter of the radiator increases; it is obvious, however, that it can never become zero, but must tend towards a constant quantity.

Such experimental observations as are available point to the same conclusion, the loss by convection being best represented by an expression of the form $c_2 + c_3 r^{-1}$. The loss by radiation is independent of the size of the radiator and may be put equal to a constant, c_4. The variation of emissivity with the size of the radiator is therefore expressed by the following formula :—

$$\text{Emissivity} = c_2 + c_4 + (c_1 + c_3)\, r^{-1} = C + C' r^{-1} \quad . \quad . \quad . \quad \text{(vi.)}.$$

This is the same expression as was established empirically by AYRTON and KILGOUR.* Within the range of their experiments ($r = 0.0157$ to $r = 0.178$ millim.) both C and C' were constants.

In an investigation on the effect of the size of the enclosure, formula (iii.) is not suitable. It was established that the length of the cylinder was sufficiently great in comparison with its diameter to render the end effects negligible. We therefore cannot put $R = \infty$. If, however, we take the simple case of a spherical enclosure and radiator, and reason by analogy, we can easily obtain the information we desire.

Proceeding as on page 240, the thermal resistance of the elementary spherical shell is

$$\frac{1}{4\Pi K} \times \frac{dr}{r^2}.$$

Integrating we get for the total resistance

$$\frac{1}{4\Pi K}\left(\frac{1}{r} - \frac{1}{R}\right) = \frac{1}{4\Pi K} \times \frac{R - r}{Rr}.$$

The part of the emissivity due to conduction is therefore

$$E_c = \frac{K}{r} \times \frac{R}{R - r}.$$

* AYRTON and KILGOUR ('Phil. Trans.,' A, vol. 183, p. 371, 1892) have shown that an expression of this form holds good for the smallest diameters that can practically be used. It would, however, be unwise to extend the formula much beyond the limit within which it has been experimentally proved. It is evident that for $r = 0$ the expression is at fault, for the value of the emissivity cannot exceed $E = NC$ where N is the number of molecular impacts and C the molecular specific heat.

If R/r is made very great E_c is nearly constant with regard to R and is equal to K/r, showing that if the enclosure is above a given size, the heat lost by conduction is practically independent of its actual dimensions.

The theoretical treatment of the variations of the emissivity with the temperatures of the gas offers some difficulties.

On the one hand the conductivity increases in proportion to the viscosity, on the other the convection is a function of the density divided by the viscosity. Taking the gas at the same density, both at the high and the low temperature, we may safely predict that at low pressures, where the effect of conductivity predominates, a rise in temperature will correspond to an increased value of the emissivity. At high pressures, however, with the data at present available, a theoretical solution is not possible.

The results of two sets of experiments made with carbon dioxide are given below—

<div style="text-align:center">Emissivity in Carbon Dioxide in C.G.S. Units.</div>

Temperature* ϑ in degrees Centigrade.	Pressure 19·9 atmospheres at 16° C.		Pressure 45·5 atmospheres at 16° C.	
	Enclosure at 18° C.	Enclosure at 100° C.	Enclosure at 18° C.	Enclosure at 100° C.
200	0·0032	0·0032	0·0058	0·0053
300	0·0036	0·0038	0·0063	0·0058
400	0·0040	0·0044	0·0068	0·0063
500	0·0044	0·0048	0·0074	0·0068
600	0·0048	0·0053	0·0079	0·0075
700	0·0052	0·0057	0·0084	0·0081

We see that even at 20 atmospheres an increase of 34 per cent. in the absolute temperature causes an increase of about 10 per cent. in the value of the emissivity.

In the case of carbon dioxide, and probably of all easily condensible gases, when the pressure is near that which would bring about a change of state, the emissivity is diminished by a rise of temperature.

Finally the question of the average temperature of the gas in the enclosure in its relation to that of the radiator offers some interest. It can be determined from the variation of pressure, for the apparatus constitutes a rough form of constant volume thermometer. On the other hand, when the heat lost by convection is small, the average temperature can be calculated from the equations on page 240.

If we divide the gas into a series of concentric cylinders, the boundaries of these cylinders will be isothermal surfaces, and the fall of temperature from the inner to

* ϑ = temperature of the radiator – temperature of the enclosure in degrees Centigrade.

the outer surface of each cylinder will be proportional to the thermal resistance. By plotting the temperatures in terms of the volume of the successive cylinders, integrating the curve and dividing by the total volume, we obtain the average temperature of the gas. This works out as 15 per cent. of the temperature of the radiator if we take the temperature of the enclosure as the zero of our scale.

The same ratio experimentally determined is about 6 per cent. This relatively low value is accounted for by the fact that each gaseous particle does not only part with its heat to the succeeding one, but streaming upwards, comes sooner or later in contact with the upper wall of the enclosure.

A Numerical Comparison of the Heat dissipated by Conductivity in Solids and by Emissivity in Gases.

Let us now consider the numerical values of the total emissivity obtained at high pressures. At 160 atmospheres, for instance, in air the emissivity is (see formula (ii.))

$$E = 0.006912 + 0.000004732 \, \vartheta.$$

Thus at 160 atmospheres air dissipates heat at the same rate as a body having a conductivity

$$K = 0.0011 \, (1 + 0.00069 \, \vartheta),$$

therefore at about the same rate as glass ($K = 0.001$) or plaster of Paris ($K = 0.0013$).

The conductivity of water is about the same—

$$K = 0.0012 \, (1 - \alpha \vartheta),$$

but in this case the comparison is not a fair one, for placed in similar circumstances convection would also come into play, causing the total loss of heat to be much greater.

Hydrogen at 120 atmospheres behaves like a body having a conductivity

$$K = 0.00237 \, (1 - 0.00072 \, \vartheta).$$

Finally the emissivity in liquid carbon dioxide is about 0·2, which would correspond to a conductivity of 0·03, therefore higher than mercury or bismuth. The conductivity in liquid nitrous oxide is nearly the same. Near the critical temperature the emissivity in the liquids rises very rapidly; it falls again instantaneously to less than one-tenth of its former value when the gaseous state is reached.

A point of some interest which was observed during the course of the work was the gradual change in the phenomenon of the decomposition of nitrous oxide. At a pressure of 1 atmosphere the decomposition was very slow and more than half an hour had elapsed before the change was anything like complete.

At increasing pressures the decomposition became more rapid, and finally, in an experiment in which the apparatus had, at the start, been partially filled with liquid gas, it took the form of a sharp explosion. The temperature of decomposition is for all pressures about the same, and lies between 800 and 900 degrees Centigrade.

Some measurements taken in liquid nitrous oxide and liquid carbon dioxide are given in Table VII. (p. 251).

Conclusions.

The general result of the investigation is to show that the law which connects the thermal emissivity with the temperature of the radiator and the pressure of the gas is—

$$E = ap^a + bp^\beta \vartheta,$$

where E = Emissivity = Heat dissipated per second, per degree Centigrade temperature interval, per centimetre of area, expressed in gramme-degrees. a, α, b, β, are the constants given on page 236, p is the pressure in atmospheres, and ϑ the temperature interval.

The bearing on astro-physical problems is at once apparent, and the high effective conductivity of compressed gases must be taken into account when treating questions relative to the mean temperature of the sun, its rate of cooling, or more generally, any problem dealing with the physical state of gaseous astral bodies.

Although extrapolations are always very uncertain, in the present case it seems justifiable to conclude that under the very great pressures with which astronomical physics has to deal, all gases or vapours would transmit heat at a higher rate than the best conducting solids known.

From a practical point of view, the results obtained explain the very rapid cooling of the products of an explosion, a fact which has not only been noticed during the course of experiments on ballistics, but has long been well known to all engineers engaged in the construction of gas engines.

The relatively small cooling surface required, in the case of gases under high pressures, is a fact that had been previously noticed, and is already to some extent taken account of in the design of the modern compressors.

The laws of thermo-dynamics show that the efficiency of all heat engines is dependent on their temperature range and pressure : in compliance with these laws the pressures used are daily becoming greater. It is therefore hoped that the present investigation may be of some use to those engaged in practical engineering.

TABLE II.—Emissivity in Air.

0·113 atmosphere at 16° C.		0·969 atmosphere at 16° C.		7·08 atmospheres at 16° C.		20·92 atmospheres at 16° C.	
Temperature ϑ in degrees Centigrade.	Emissivity in C.G.S. units.	Temperature ϑ in degrees Centigrade.	Emissivity in C.G.S. units.	Temperature ϑ in degrees Centigrade.	Emissivity in C.G.S. units.	Temperature ϑ in degrees Centigrade.	Emissivity in C.G.S. units.
118·5	0·0005545	82·1	0·0007505	98·2	0·001522	65·9	0·002205
296·7	0·0007021	199·0	0·0009533	308·0	0·002084	204·3	0·002886
466·8	0·0008680	355·4	0·001139	656·5	0·002801	422·8	0·003529
771·9	0·001325	635·8	0·001518	802·5	0·003385	735·2	0·004468
962·6	0·001784	865·9	0·001957	945·0	0·003605	899·6	0·004975
		966·0	0·002213			1060·2	0·005673

39·3 atmospheres at 16° C.		65·4 atmospheres at 16° C.		124·7 atmospheres at 16° C.		168·4 atmospheres at 16° C.	
Temperature ϑ in degrees Centigrade.	Emissivity in C.G.S. units.	Temperature ϑ in degrees Centigrade.	Emissivity in C.G.S. units.	Temperature ϑ in degrees Centigrade.	Emissivity in C.G.S. units.	Temperature ϑ in degrees Centigrade.	Emissivity in C.G.S. units.
165·1	0·003724	114·3	0·004792	88·1	0·005869	82·5	0·007198
378·1	0·004544	283·2	0·005433	199·3	0·006785	150·0	0·007909
571·5	0·005191	463·5	0·006086	303·2	0·007453	258·3	0·008573
771·0	0·005856	621·7	0·006635	503·7	0·008386	418·8	0·009496
902·5	0·006363	872·6	0·007670	689·1	0·009113	567·1	0·01012
1006·7	0·006809	1055·3	0·008485	926·9	0·01022	787·5	0·01116
						915·5	0·01175
						1031·1	0·01242

TABLE III.—Emissivity in Oxygen.

0·086 atmosphere at 16° C.		1·01 atmospheres at 16° C.		10·0 atmospheres at 16° C.	
Temperature ϑ in degrees Centigrade	Emissivity in C.G.S. units	Temperature ϑ in degrees Centigrade.	Emissivity in C.G.S. units.	Temperature ϑ in degrees Centigrade.	Emissivity in C.G.S. units.
86·5	0·000513	222·7	0·001080	256·4	0·002349
288·1	0·000683	571·7	0·001515	483·0	0·002946
381·1	0·000793	921·9	0·002231	733·3	0·003557
585·0	0·001066			976·9	0·004280
736·3	0·001352				
922·6	0·001869				

TABLE I.—Emissivity in Air at 89.8 Atmospheres (18° C.).*

	I	II	III	IV	V	VI	VIII	IX	X	XI	XII	XIII	XIV	XV	XVI	XVII	XVIII	XIX	XX	XXI	XXII	XXIII	XXIV	XXV	XXVI
	Current in amperes.	Electro- motive force in volts.	Current reversed. Current in amperes. Electro- motive force in volts.	Mean current.	Mean electro- motive force.	Power dissi- pated in radiator in watts.	Resistance of radiator in ohms at 0° C.	$(R_t - R_0)$ where R_t = actual resist- ance of radiator in ohms R_0 = resist- ance of platinum at 0° C.	Tempera- ture of radiator in Centi- grade degrees.	Tempera- ture differ- ence in Centi- grade degrees.	Tempera- ture in Centi- grade Fahr.	Power in watts per degree Centi- grade.	Emis- sivity in C.G.S. units per degree Centi- grade.	Tempera- ture of water circula- ting in ohms.	Tempera- ture per degree Centi- grade.	Zero of pressure gauge.	Pressure reading of pressure gauge.	Zero of Pressure gauge.	Pressure from column XVIII temperature pressure.	P₁V₁ where V₁ = original volume radiated in column XX.	P₂ corrected from mercury column XX.	Pₓ cor- rected temper- ature.	weight of cor- rected of air radia- tion.	P₂V₂ from BOYLE'S law.	

Before the density current was started.

I	II	III	IV	V	VI	VIII	IX	X	XI	XII	XIII	XIV	XV	XVI	XVII	XVIII	XIX	XX	XXI	XXII	XXIII	XXIV	XXV	XXVI
0.3900	22.36		0.3811	22.38	8.559	0.01701	0.00664	179.0	161.0	18.3	0.06175	0.00374	18.3		—	44.72	44.70	77.08	1.07	39.10	39.03	39.35	17.03
0.7618	31.92		0.7638	31.95	25.81	0.02040	0.01082	377.6	359.6	18.0	0.07135	0.00444	18.1		—	44.70	44.64	77.04	1.96	39.03	40.24	43.67	19.0
0.9129	36.20		0.9183	36.49	33.81	0.02490	0.01443	549.6	531.6	18.5	0.08115	0.00445	18.6		—	44.64	44.95	77.10	1.80	40.24	41.28	43.64	19.0
1.023	37.88		1.220	38.90	41.11	0.02606	0.01561	624.0	606.0	19.4	0.08730	0.00543	19.4		—	44.95	44.95	77.13	1.87	41.28	41.28	43.67	19.0
1.520	42.15		1.827	42.10	48.57	0.02709	0.01751	708.7	690.7	19.9	0.09590	0.00596	18.8	44.83		44.95	45.45	77.24	1.81	41.28	42.67	43.94	19.3
1.800	45.93		1.818	46.00	79.60	0.04111	0.02865	929.40	911.40	22.5	0.10234	0.00636	17.9		—	45.45	45.28	77.34	1.83	42.67	41.57	43.18	19.3
2.035	48.40		2.055	48.45	99.01	0.04402	0.03366	1100.4	1082.4	21.5	0.10648	0.00663	18.2		—	45.28	45.45	77.40	1.99	41.57	38.94	39.09	18.3

Ten minutes after the radiator current was stopped.

* For each column in the following table a similar set of observations were taken as shown above, but to save space and avoid collision the final results are alone recorded.

† The emissivity is numerically equal to the heat dissipated per sq. centim. of surface of the radiator per degree Centigrade temperature interval, the heat being expressed in water-gramme-degrees (therms) or $E = \dfrac{V \times A}{3SJ}$ where

S = surface of radiator.

$J = r_2 - r_1$ = temperature of radiator - temperature of enclosure.

A = Current in amperes.

V = electromotive force in volts.

(Quotations.)

The diameter of the bright platinum radiating wire was 1.106 millims.

The length between potential contacts . . . 9.409 centims.

The superficial area . . . 3.300 sq. centims.

The resistance of the radiator was at 100° C. = 0.01987

at 0° C. = 0.01568

Therefore the temperature coefficient was
$$\frac{R_t - R_0}{R_0} \times \frac{1}{100}$$

Where R_t = resistance at 100° C.; R_0 = resistance at 0° C.

$$\frac{R_t - R_0}{100 R_0}$$

$$= \frac{.00419}{R_0}$$

$$R_0 = 1.5739$$

The fundamental coefficient =

$$\frac{R_t - R_0}{100 R_0} = 0.00267$$

$$/° C. - R_0 = 0.01568$$

$$/° C. = 0.0000832$$

In the formula $(t - t') = \left[\left(\frac{t}{100}\right)^2 + \frac{t}{100}\right]$ the coefficient δ was $\delta = 1.495$.

The temperature of the CLARK's cell was at 18° C.; it was balanced at 1.4377 at the potentiometer.

TABLE III.—Emissivity in Oxygen (continued).

37·8 atmospheres at 16° C		71·6 atmospheres at 16° C.		115·9 atmospheres at 16° C.	
Temperature ϑ in degrees Centigrade	Emissivity in C.G.S. units	Temperature ϑ in degrees Centigrade	Emissivity in C.G.S. units	Temperature ϑ in degrees Centigrade.	Emissivity in C.G.S units
182·5	0·003777	132·1	0·004877	96·2	0·006370
395·9	0·004660	244·9	0·005631	194·6	0·007100
563·8	0·005315	429·2	0·006512	337·7	0·008124
845·1	0·006329	639·3	0·007403	462·0	0·008687
1075·8	0·007450	820·4	0·008218	638·0	0·009452
		1091·4	0·009561	867·0	0·01056
				1064·2	0·01176

TABLE IV.—Emissivity in Hydrogen

0·117 atmosphere at 16° C.		0·971 atmosphere at 16° C.		6·9 atmospheres at 16° C	
Temperature ϑ in degrees Centigrade	Emissivity in C G S units	Temperature ϑ in degrees Centigrade	Emissivity in C.G.S. units	Temperature ϑ in degrees Centigrade	Emissivity in C G S. units
189·8	0·003062	168·8	0·003476	141·8	0·001899
523·9	0·004158	508·9	0·004920	229·8	0·006100
681·1	0·005098	692·5	0·005818	644·7	0·008038
690·1	0·005032	929·9	0·006675	831·2	0·008738
775·7	0·005372	989·4	0·007288	952·2	0·009417
903·7	0·005854	619·8	0·005150	450·2	0·007425
		367·3	0·004180	233·3	0·006030
		268·7	0·003762		
		162·2	0·003365		

23·3 atmospheres at 16° C		66·9 atmospheres at 16° C		114·0 atmospheres at 16° C.	
Temperature ϑ in degrees Centigrade.	Emissivity in C.G.S. units	Temperature ϑ in degrees Centigrade.	Emissivity in C.G S. units.	Temperature ϑ in degrees Centigrade	Emissivity in C G S units.
82·5	0·007380	136·1	0·01203	161·1	0·01533
143·9	0·008369	257·8	0·01396	404·7	0·01920
355·6	0·01032	402·1	0·01554	620·4	0·02135
594·1	0·01194	521·6	0·01670	791·1	0·02268
740·0	0·01278	669·8	0·01816	950·9	0·02454
941·4	0·01405			275·8	0·01742
79·8	0·007338			165·8	0·01558
287·9	0·009841			89·8	0·01350
508·1	0·01149				

TABLE V.—Emissivity in Carbon Dioxide.

0·07 atmosphere at 16° C.		0·99 atmosphere at 16° C.		4·4 atmospheres at 16° C		10 2 atmospheres at 16° C.	
Temperature ϑ in degrees Centigrade.	Emissivity in C.G.S. units.	Temperature ϑ in degrees Centigrade.	Emissivity in C.G.S. units.	Temperature ϑ in degrees Centigrade.	Emissivity in C.S.G. units.	Temperature ϑ in degrees Centigrade.	Emissivity in C.G.S. units.
167·6	0·0004537	187 9	0·0008695	387·3	0·002037	79·7	0·001619
340·0	0 0006094	403·7	0 001176	732·3	0·002911	173·5	0 001997
662·5	0 001018	804 5	0·001893	979·5	0·003666	441 3	0·002853
847·5	0·001410	1063·0	0·002509	156·0	0 001335	684·2	0·003588
992·1	0·001780	247·0	0 0009489			919 3	0·004310
		602·2	0·001527				
		835·9	0·002005				

18·1 atmospheres at 16° C.		34 8 atmospheres at 16° C.		54 0 atmospheres at 21°·1 C.	
Temperature ϑ in degrees Centigrade.	Emissivity in C.G.S. units.	Temperature ϑ in degrees Centigrade.	Emissivity in C.G.S. units.	Temperature ϑ in degrees Centigrade.	Emissivity in C.G.S. units.
197·4	0·003102	88·1	0·004127	68·7	0·007002
341 0	0·003752	265·8	0·005157	167·4	0·007215
554·2	0·004576	480·5	0·006132	337·4	0·007618
764·5	0·005306	671·2	0·007038	595·8	0·008414
964·7	0·006130	950·1	0·008066	901·4	0·009339
		835·2	0·007670		

TABLE VI.—Emissivity in Nitrous Oxide.

0·93 atmosphere at 16° C.		5·66 atmospheres at 16° C.		14·48 atmospheres at 16° C.		39·0 atmospheres at 16° C.	
Temperature ϑ in degrees Centigrade.	Emissivity in C.G.S. units.	Temperature ϑ in degrees Centigrade.	Emissivity in C.G.S. units.	Temperature ϑ in degrees Centigrade.	Emissivity in C G.S. units.	Temperature ϑ in degrees Centigrade.	Emissivity in C.G.S. units.
60·8	0·000647	90·6	0·001292	78·5	0·001962	124·8	0 004687
192·8	0·000898	289·7	0·001922	263·0	0 002835	207 9	0·005161
341·4	0·001126	642·0	0·002842	418·4	0 003378	337·6	0·005782
585·6	0·001546	1138·9	0·004296	605·1	0·004006	576·4	0·006839
799·2	0·002003			844·5	0·004967	732·2	0·007383
1066·5	0·002773			1093·3	0·005653	807·3	0·007796
						1080 8	0·008325
						115 3	0 004712
						264 2	0 005498
						485 9	0·006505

TABLE VII.

Emissivity in liquid carbon dioxide.		Emissivity in liquid nitrous oxide.	
Temperature ϑ in degrees Centigrade	Emissivity in C.G.S. units.	Temperature ϑ in degrees Centigrade.	Emissivity in C.G.S. units
5·9	0·27*		
7·0	0·37*		
	gas.		
176·7	0·016		
84·8	0·018	10·5	
98·8	0·017	11·1	0·20'
308·1	0·016	11·4	0·26'
463·9	0·016		

* These measurements are only given as preliminary and altogether approximate values; as already stated, the apparatus is not suitable for measurements depending on small temperature intervals.

<div align="center">TABLE VIII.—Emissivity in Air.</div>

$$E \times 10^6 = 403\,p^{0.56} + 1.63\,p^{0.921}\,\vartheta.$$

ϑ = temp of radiator above the enclosure in degrees Centigrade	0·1 atmosphere	1·0* atmosphere	10 atmospheres	20 atmospheres	30 atmospheres	40 atmospheres	50 atmospheres	60 atmospheres
100	*0·00010*	*0·00084*	0·001728	0 002461	0 003010	0·003534	0·003974	0·004376
200	*0·00050*	*0·00095*	0·001992	0 002769	0·003378	0·003888	0·004345	0·004761
300	*0·00070*	*0·00106*	0·002256	0 003074	0·003706	0·004241	0·004715	0·005146
400	*0·00078*	*0·00119*	0·002520	0 003380	0·004039	0·004593	0·005086	0·005531
500	*0·00086*	*0·00132*	0·002785	0 003686	0·004372	0·004949	0·005457	0·005916
600	*0·00095*	*0·00142*	0·003049	0 003992	0·004705	0·005302	0·005827	0·006302
700	*0·00113*	*0·00163*	0·003314	0 004298	0·005038	0·005656	0·006198	0·006687
800	*0·00133*	*0·00183*	0·003578	0 004608	0·005371	0·006010	0·006568 *	0·007072
900	*0·00157*	*0·00203*	0·003843	0 004809	0·005704	0·006363	0·006939	0·007457
1000	*0·00180*	*0·00231*	0·004107	0 005215	0·006037	0·006717	0·007310	0·007842
1100	—	—	0·004371	0 005511	0·006370	0·007071	0·007681	0·008227
1200	—	—	0 001686	0 005827	0 006702	0 007424	0 008051	0·008612

ϑ = temp of radiator above the enclosure in degrees Centigrade	80 atmospheres	100 atmospheres	120 atmospheres	140 atmospheres	160 atmospheres	Temp of radiator in degrees Centigrade above the temperature of the enclosure.	Heat lost by convection = E_c calculated, taking $K_0=0\cdot00000561$, $a = 0\cdot0019$, and $E_c = 6\cdot184 \times K$	Radiation according to J. T. Bottomley's results.	Loss of heat due to conduction + radiation.
100	0·005038	0·005741	0 006329	0 006874	0·007385	100	0 000413	0 000034	0 000447
200	0·005507	0·006170	0 006775	0 007334	0·007859	200	0 000479	0 000059	0 000538
300	0·005916	0·006599	0 007220	0 007795	0·008332	300	0 000545	0 000094	0 000639
400	0·006325	0·007028	0 007666	0 008255	0·008805	400	0 000611	0 000134	0 000745
500	0·006734	0·007456	0 008111	0 008715	0·009278	500	0 000677	0 000176	0 000853
600	0·007143	0·007885	0 008556	0 009175	0·009751	600	0 000742	0 000218	0 000960
700	0·007552	0 00·814	0 009002	0 009635	0·010225	700	0 000808	0 000259	0 001067
800	0·007961	0 008743	0 009447	0 010095	0·010698	800			
900	0·008371	0 009171	0 009893	0 010555	0 011170	900			
1000	0·008780	0 009600	0 010338	0 011016	0·011643	1000			
1100	0·009189	0 0100.9	0 010784	0 011475	0 012117	1100			
1200	0 009598	0 010457	0 011229	0 011936	0 012590	1200			

* In this and the four following tables the figures in italics, referring to pressures outside the range of the formula, are obtained from a direct interpolation of the observed values. In every case the emissivity is expressed in water-gramme-degrees per square centimetre of radiating surface per second.

TABLE IX.—Emissivity in Oxygen.

$$E \times 10^4 = 387\,p^{0.06} + 1.39\,p^{.28}\,9.$$

Temperature θ in degrees Centigrade.	0·10 atmosphere.	1·0 atmosphere.	10 atmospheres.	30 atmospheres.	50 atmospheres.	80 atmospheres.	100 atmospheres.	120 atmospheres.
100	0·00062	0·00095	0 001735	0·003142	0·004158	0·005125	0 006099	0·006749
300	0·00072	0 00118	0·002265	0·003863	0·004988	0·006337	0·007108	0·007811
500	0·00097	0 00143	0·002796	0·004583	0·005821	0·007285	0·008118	0·008874
700	0·00126	0·00168	0 003225	0·005304	0·006652	0·008334	0·009127	0·009936
900	0·00174	0·00215	0·003855	0·006025	0·007483	0·009192	0·010136	0·010998
1100	—	—	0·004885	0·006745	0·008315	0·010130	0·011146	0·012052

TABLE X.—Emissivity in Hydrogen.

$$E \times 10^4 = 2750\,p^{0.98} + 1.88\,p^{.39}\,9.$$

Temperature θ in degrees Centigrade.	0·10 atmosphere.	1·0 atmosphere.	10 atmospheres.	30 atmospheres.	50 atmospheres.	80 atmospheres.	100 atmospheres.	120 atmospheres.
100	0·01285	0·01414	0·00520	0·00860	0·01000	0 01195	0·01410	0·01416
300	0·01442	0·01304	0 00745	0·01096	0·01312	0·01548	0·01674	0·01785
500	0·01613	0·01495	0 00831	0·01224	0·01466	0 01730	0·01872	0·01996
700	0·04598	0·01657	0·00917	0·01352	0·01619	0·01912	0·02069	0·02207
900	0·04589	0·01967	0·01003	0·01480	0·01773	0·02094	4·02266	0·02417
1100	0·00673	0·01757	0·01089	0·01608	0·01997	0·02276	0·02464	0·02628

TABLE XI.—Emissivity in Carbon Dioxide.

$$E \times 10^6 = 207\, p^{0\cdot87} + 1\cdot5\, p^{0\cdot33} \vartheta.$$

Temperature ϑ in degrees Centigrade.	0·1 atmosphere.	1·0 atmosphere.	10 atmospheres.	30 atmospheres.	50 atmospheres.
100	0·00044	0·00074	0·001688	0·003828	0·00675
300	0·00060	0·00103	0·002330	0·004749	0·00735
500	0·00079	0·00134	0·002971	0·005671	0·00792
700	0·00110	0·00172	0·003613	0·006593	0·00852
900	0·00155	0·00210	0·004254	0·007515	0·00912
1100	0·00210	0·00260	0·004895	0·008436	0·00975

TABLE XII.—Emissivity in Nitrous Oxide.

$$E \times 10^6 = 276\, p^{0\cdot74} + 1\cdot7\, p^{0\cdot23} \vartheta.$$

Temperature ϑ in degrees Centigrade.	1·0 atmosphere.	10 atmospheres.	30 atmospheres.
100	0·00070	0·001841	0·003360
300	0·00105	0·002488	0·004741
500	0·00140	0·003136	0·005623
700	0·00172	0·003784	0·006504

VII. *A Comparative Crystallographical Study of the Double Selenates of the Series* $R_2M(SeO_4)_2,6H_2O$.—*Part II. Salts in which M is Magnesium.*

By A. E. TUTTON, *B.Sc., F.R.S.*

[PLATE 9.]

IN this memoir are communicated the results of an investigation of the magnesium group of double selenates, in which R is represented by potassium, rubidium, and cæsium. It is analogous and strictly comparable to that concerning the salts of the zinc group, which was presented to the Royal Society in March, 1900 ('Roy. Soc. Proc.,' vol. 67, p. 58). The potassium salt of the series was partially studied by TOPSOE and CHRISTIANSEN,* in the year 1874, with the less perfect means then available; the rubidium and cæsium salts have not hitherto been investigated.

The three salts were prepared precisely as in the case of the zinc salts, except that the normal magnesium selenate required for the production of the double salt, by addition to the calculated amount of the normal alkaline selenate, was prepared by digestion of the calculated quantity of pure diluted selenic acid with excess of pure calcined magnesia and subsequent filtration.

The same spherical projection is common to the series of double selenates and that of the double sulphates previously investigated by the author, and is given on p. 343 of the memoir describing the latter.†

POTASSIUM MAGNESIUM SELENATE, $K_2Mg(SeO_4)_2,6H_2O$.

A determination of the content of magnesium in a specimen of the crystals employed afforded the following result :—1·1430 gramme yielded 0·2623 gramme of magnesium pyrophosphate, corresponding to 4·95 per cent. of magnesium. The calculated percentage of Mg is 4·84.

Goniometry.

Twelve good crystals of small size, selected out of five crops, were used in the measurements.

* 'Ann. de Chim. et de Phys.,' vol. 1, 1874, p. 75.
† 'Journ. Chem. Soc., Trans.,' 1893.

26.10 1901.

Habit : Short prismatic to tabular.

Axial angle : $\beta = 75° 42'$.

Ratio of axes : $a : b : c = 0.7485 : 1 : 0.5031$.

Forms observed : $a = \{100\} \infty P \infty$; $b = \{010\} \infty P \infty$; $c = \{001\}oP$;
   $p = \{110\} \infty P$; $q = \{011\}P\infty$; $r' = \{\bar{2}01\} + 2P\infty$; $o' = \{\bar{1}11\} + P$.

The results of the measurements are set forth in the accompanying table of angles.

The crystals were distinguished by the preponderating development of the c and p faces. Faces of the form r' were generally present, but small, while the q faces were even smaller and frequently absent altogether. A typical crystal is shown later in fig. 1 (p. 274), in the discussion of the comparative habits of the three magnesium salts. Many of the crystals resembled fig. 21 in the double sulphate memoir (*loc. cit.*, p. 384), except that there was no trace of b or o' faces. It occasionally happened that while one r' face was very small, the other one was relatively largely developed. The orthopinacoid a was discovered fairly well formed on one of the crops investigated, but was absent on the other crops. A trace of the hemi-pyramid o' was observed on one crystal, but the signal images afforded were not adequately good for the purposes of measurement.

Morphological Angles of Potassium Magnesium Selenate.

Angle measured.	No. of measurements.	Limits.	Mean observed.	Calculated.	Difference.
		° ′ ° ′	° ′	° ′	′
$ac = 100 : 001 = \beta$	—	—	—	75 42	—
$as = 100 . 101$	—	- -	—	46 30	—
$sr = 101 · 001$	—	—	—	29 12	- -
$r'd = 001 · \bar{2}01$	21	62 55— 63 4	62 58	63 53	5
$c'd = 001 · 101$	—	- -	—	36 1	—
$s'r' = \bar{1}01 : 201$	—	—	—	24 5$\frac{1}{2}$	—
$r'a = \bar{2}01 : 100$	—	- ·	—	41 25	—
$r'c = \bar{2}01 : 00\bar{1}$	21	116 57—117 8	117 2	117 7	5
$ap = 100 : 110$	2	35 53— 35 55	35 54	35 55	1
$pp' = 110 : 120$	—	—	—	19 28	—
$p'b = 120 . 010$	—	—	—	34 37	—
$pb = 110 : 010$	2	54 5— 54 7	54 6	54 5	1
$pp = 110 · 1\bar{1}0$	17	71 42— 71 57	71 50	*	—
$pp = 110 . \bar{1}10$	18	108 0—108 21	108 10	108 10	0
$cq = 001 : 011$	28	25 53— 26 8	26 0	*	—
$qb = 011 · 010$	3	64 0— 64 6	64 2	64 0	2
$qq = 011 : 0\bar{1}0$	6	128 0—128 3	128 1	128 0	1
$ao = 100 · \bar{1}11$	—	—	—	49 43	- -
$oq = \bar{1}11 · 011$	—	—	—	27 27	—
$aq = 100 : 011$	—	- -	—	77 10	—
$qo' = 011 : \bar{1}11$	—	- ·	—	34 16	—
$o'a = \bar{1}11 : 100$	—	—	—	68 34	—

Morphological Angles of Potassium Magnesium Selenate—*continued*.

Angle measured.	No. of measurements.	Limits.	Mean observed.	Calculated.	Difference.
co = 001 : 111	—	—	—	34 54	- -
op = 111 : 110	—	—	--	43 34	---
cp = 001 : 110	48	78 20— 78 34	78 28	*	---
po' = 110 · 11$\bar{1}$	---	—	—	57 13	---
o'c = 11$\bar{1}$: 00$\bar{1}$	—	—	—	44 19	---
pc = 110 : 00$\bar{1}$	48	101 22—101 42	101 32	101 32	0
bo = 010 : 111	—	—	—	69 57	---
os = 111 : 101	—	—	—	20 3	---
bo' = 010 : $\bar{1}$11	—	—	—	65 16	.
o's' = $\bar{1}$11 . $\bar{1}$01	—	—	—	24 44	---
sq = 101 : 011	—	—	—	38 19	---
qp = 011 : $\bar{1}$10	13	85 26-- 85 46	85 35	85 34	1
ps = 110 : $\bar{1}$0$\bar{1}$	—	—	--	56 7	---
pq = 110 · 0$\bar{1}$1	13	94 12— 94 40	94 25	94 26	1
s'q = $\bar{1}$01 : 011	---	—	—	44 55	---
qp = 011 : 110	14	64 0— 64 15	64 7	64 5	2
ps' = 110 : 10$\bar{1}$	—	—	—	71 0	---
pq = 110 : 0$\bar{1}$1	14	115 45—116 5	115 53	115 55	2
r'o' = 201 : $\bar{1}$11	—	—	—	34 30	---
o'p = $\bar{1}$11 : 110	—	—	—	92 54	---
pr' = 110 : 20$\bar{1}$	28	52 21— 52 42	52 33	52 36	3
r'p = 201 : 110	28	127 18—127 37	127 27	127 24	3

Total number of measurements, 324.

From measurements made by TOPSÖE in 1870, TOPSÖE and CHRISTIANSEN (*loc. cit.*, p. 75) quote for the ratio of the axes and the axial angle $a : b : c = 0.7447 : 1 : 0.5014$, and $\beta = 75° 43'$.

The cleavage direction common to the series, namely, parallel to *r'* {$\bar{2}$01}, is well developed.

Volume.

Relative density.—The following four determinations were carried out with different specimens of the salt :—

Weight of salt employed.	Sp. gr. at 20°/4°.	
7·4818	2·3627	
6·2738	2·3630	
7·1272	2·3634	Mean, 2·3630
6·2631	2·3630	

Molecular Volume.—$M/d = 495 \cdot 6 + 2 \cdot 363 = 209 \cdot 73$.

Distance Ratios.—The molecular volume combined with the axial ratios and axial angle yield the following distance ratios :—

$$\chi : \psi : \omega = 6 \cdot 2233 : 8 \cdot 3144 : 4 \cdot 1829.$$

Optics.

Orientation of Axes of Optical Ellipsoid.—The symmetry plane is the plane of the optic axes (binormals). The sign of the double refraction is positive.

The following extinction angles were afforded by two section plates ground parallel to the symmetry plane. They are referred to the normal to the basal plane :—

Section 1, 3° 40′; section 2, 2° 20′; Mean, 3° 0′.

Topsöe and Christiansen give 2° 0′ as the mean of two determinations.

The direction is behind the normal (that is, nearer to the vertical axis *c*). This direction is the second median line, the first median line being consequently also in the obtuse angle *ac* of the morphological axes and inclined 3° 0′ to the axis *a*. The second median line is inclined 11° 18′ to the vertical axis *c*.

Refractive Indices.—The six prisms employed were ground on six different crystals. The results are given in the accompanying table.

Refractive Indices of Potassium Magnesium Selenate.

	Light.	Prism 1.	Prism 2.	Prism 3.	Prism 4.	Prism 5.	Prism 6.	Mean.	Values of TOPSÖE and CHRISTIANSEN.
a Vibrations parallel 2nd median line	Li	—	1·4933	—	1·4937	1·4935	1·4937	1·4936	—
	C	—	1·4938	—	1·4944	1·4941	1·4942	1·4941	—
	Na	—	1·4965	·—	1·4973	1·4969	1·4969	1·4969	1·4950
	Tl	—	1·4994	—	1·5001	1·5000	1·4999	1·4999	—
	F	—	1·5030	—	1·5037	1·5037	1·5036	1·5035	—
	G	—	1·5087	—	1·5096	1·5091	1·5091	1·5091	—
β Vibrations parallel symmetry axis *b*.	Li	1·4967	1·4950	1·4957	1·4957	—	·—	1·4958	—
	C	1·4974	1·4955	1·4962	1·4961	—	—	1·4963	1·4942
	Na	1·5001	1·4984	1·4989	1·4989	—	—	1·4991	1·4970
	Tl	1·5031	1·5013	1·5021	1·5021	—	—	1·5022	—
	F	1·5070	1·5050	1·5057	1·5056	—	—	1·5058	1·5039
	G	1·5127	1·5107	1·5110	1·5113	—	—	1·5114	—
γ Vibrations parallel 1st median line.	Li	1·5108	—	1·5103	—	1·5101	1·5099	1·5103	—
	C	1·5114	·—	1·5107	—	1·5106	1·5106	1·5106	—
	Na	1·5143	—	1·5137	—	1·5138	1·5139	1·5139	1·5120
	Tl	1·5178	—	1·5169	—	1·5170	1·5170	1·5172	·—
	F	1·5216	—	1·5208	—	1·5208	1·5208	1·5210	—
	G	1·5272	·—	1·5263	—	1·5266	1·5263	1·5266	—

The mean refractive index (mean of all three indices) for sodium light is 1·5033.

The values obtained by TOPSÖE and CHRISTIANSEN, which are given in the last column, and of which only the β values were obtained directly, do not show such close agreement with the author's values as was observed in the case of potassium zinc selenate.

The following general formula represents the intermediate refractive index of potassium magnesium selenate, with great accuracy, throughout the whole length of the spectrum. The index is corrected to a vacuum (by the addition of 0·0004 to the index as given in the table).

$$\beta = 1·4852 + \frac{490\,038}{\lambda^2} + \frac{222\,100\,000\,000}{\lambda^4} \cdots$$

The α indices are closely reproduced by diminishing the constant 1·4852 by 0·0022, and the γ indices by increasing the constant by 0·0148.

Alteration of Refraction by Rise of Temperature.—A series of determinations carried out at 75° indicated that the indices diminish by about 0·0020 for 60° rise of temperature.

Axes of the Optical Ellipsoids.—Following are the calculated values of the axial ratios of the two optical ellipsoids :—

Axes of optical indicatrix :

$$\alpha : \beta : \gamma = 0·9985 : 1 : 1·0099.$$

Axes of optical velocity ellipsoid :

$$\mathfrak{a} : \mathfrak{b} : \mathfrak{c} = 1·0015 : 1 : 0·9902.$$

Molecular Optical Constants.—The values of these constants, calculated from the refractive indices with the aid of the density given on a preceding page, are as follows :—

		α.	β.	γ.
Axis of optical indicatrix				
Specific refraction, $\dfrac{n^2-1}{(n^2+2)d} = \mathfrak{n}$	C	0·1232	0·1237	0·1267
	G	0·1264	0·1269	0·1300
Molecular refraction, $\dfrac{n^2-1}{n^2+2} \cdot \dfrac{M}{d} = \mathfrak{m}$	C	61·07	61·30	62·81
	G	62·64	62·87	64·44
Specific dispersion, $\mathfrak{n}_G - \mathfrak{n}_C$		0·0032	0·0032	0·0033
Molecular dispersion, $\mathfrak{m}_G - \mathfrak{m}_C$		1·57	1·57	1·63
Molecular refraction, $\dfrac{n-1}{d} \cdot M$	C	103·63	104·09	107·13

Optic Axial Angle.—Following are the results of the measurements made with three excellent pairs of section plates, ground perpendicular to the first and second median lines.

Determination of Apparent Angle in Air of Potassium Magnesium Selenate.

Light.	Section 1.		Section 2.		Section 3.		Mean 2E.	
Li	61°	12′	61°	3′	61°	18′	61°	11′
C	61	11	61	2	61	17	61	10
Na	61	7	60	59	61	14	61	7
Tl	61	2	60	55	61	11	61	3
F	60	55	60	49	61	7	60	57

Determination of True Optic Axial Angle of Potassium Magnesium Selenate.

Light.	No. of section perp. 1st median line.	Observed values of 2Ha.	No. of section perp. 2nd median line.	Observed values of 2Ho.	Calculated values of 2Va.	Mean value of 2Va.
Li . .	1	35° 54′	1a	118° 17′	39° 30′	39° 43′
	2	35 52	2a	116 33	39 48	
	3	35 58	3a	116 45	39 52	
C	1	35 52	1a	118 12	39 29	39 42
	2	35 49	2a	116 29	39 46	
	3	35 56	3a	116 42	39 50	
Na . . .	1	35 43	1a	117 41	39 26	39 38
	2	35 40	2a	116 0	39 42	
	3	35 44	3a	116 10	39 45	
Tl	1	35 32	1a	117 5	39 22	39 34
	2	35 28	2a	115 24	39 38	
	3	35 34	3a	115 30	39 42	
F	1	35 15	1a	116 20	39 14	39 25
	2	35 10	2a	114 40	39 29	
	3	35 14	3a	114 45	39 32	

TOPSOE and CHRISTIANSEN obtained for the angle in air 62° 12′, and for the true angle 40° 22′, both referring to sodium light.

The *Dispersion of the Median Lines* was investigated by immersion in benzene, whose mean refractive index is approximately the same as that of the crystals. It proved to be exceedingly small, not exceeding 5′ between F and C, and while the largely preponderating number of determinations gave the indication that the first median line lies nearer to the morphological axis a for blue than for red, the amount is really so small as to lie within the limits of experimental error.

Effect of Rise of Temperature on the Optic Axial Angle.—Measurements at 80° showed that the angle in air increases 3° 10′ for 60° rise of temperature.

Rubidium Magnesium Selenate, $Rb_2Mg(SeO_4)_2,6H_2O$.

A determination of magnesium in 0·4133 gramme of the crystals employed afforded as result 0·0821 gramme magnesium pyrophosphate, which corresponds to 4·29 per cent. of magnesium. The theoretical percentage is 4·08.

Goniometry.

Twelve highly perfect small crystals were used in the goniometrical work. They were selected from the four most suitable crops.

Habit: thick tabular to prismatic.

Axial angle: $\beta = 74° 46'$.

Ratio of axes: $a : b : c = 0·7424 : 1 : 0·5011$.

Forms observed: $b = \{010\} \infty P \infty$; $c = \{001\} oP$; $p = \{110\} \infty P$; $q = \{011\} P \infty$; $o' = \{\bar{1}11\} + P$; $r' = \{\bar{2}01\} + 2P \infty$.

The accompanying table sets forth the results of the measurements.

The crystals of this salt are not distinguished by richness in the number of faces, and are frequently of a remarkably simple character. No other faces were present in a large proportion of the crystals examined besides those of the basal plane c, ortho-dome r', and clinodome q. The prevailing type was very similar to that of potassium magnesium selenate, except that the q faces were relatively more largely developed with respect to the faces of the basal plane, in accordance with the rule which has been established in the cases of the double sulphates and selenates already studied. No faces of the orthopinacoid a were observed, but those of the clinopinacoid b were frequently present, although small. The faces of the hemi-pyramid o' were but rarely observed, and were only measurable in one instance, the reflections in this case being good.

Morphological Angles of Rubidium Magnesium Selenate.

Angle measured.	No. of measurements.	Limits.	Mean observed.	Calculated.	Difference.
ac = 100 : 001 = β	—	—	—	74 46	—
as = 100 : 101	—	—	—	45 50	—
sc = 101 : 001	—	—	—	28 56	—
cr' = 001 : 201	14	63 25— 63 41	63 35	63 37	2
ce' = 001 . 101	—	—	—	38 20	—
$s'r'$ = 101 : 201	—	—	—	25 17	—
$r'a$ = 201 . 100	—	—	—	41 37	—
$r'c$ = 201 . 001	13	116 19—116 32	116 25	116 23	2
up = 100 : 110	—	—	—	35 38	—
pp' = 110 . 120	—	—	—	19 28	—
$p'b$ = 120 : 010	—	—	—	34 54	—
pb = 110 010	6	54 20— 54 26	54 23	54 22	1
pp = 110 . 1̄10	20	71 4— 71 30	71 17	*	—
pp = 110 . 110	20	108 27—108 54	108 43	108 43	0
cq = 001 : 011	54	25 39— 25 52	25 47	*	—
qb = 011 : 010	10	64 11— 64 14	64 13	64 13	0
qq = 011 : 01̄1	27	128 15—128 36	128 25	128 26	1
ao = 100 : 111	—	—	—	49 3	—
oq = 111 : 011	—	—	—	27 16	—
uq = 100 : 011	—	—	—	76 19	—
qo' = 011 : 1̄11	—	—	—	34 34	—
$o'a$ = 1̄11 : 1̄00	—	—	—	69 7	—
co = 001 : 111	—	—	—	34 33	—
op = 111 : 110	—	—	—	43 7	—
cp = 001 : 110	38	77 34— 77 47	77 40	*	—
po' = 110 . 1̄11	1	—	57 42	57 45	3
$o'c$ = 11ı . 00̄1	1	—	44 35	44 35	0
pc = 110 : 00̄1	38	102 1—102 30	102 20	102 20	0
bo = 010 : 111	—	—	—	70 14	—
os = 111 . 101	—	—	—	19 46	—
bo' = 010 . 1̄11	—	—	—	65 15	—
$o's'$ = 1̄11 . 1̄01	—	—	—	24 45	—
sq = 101 : 011	—	—	—	38 0	—
qp = 011 : 1̄10	40	86 20— 86 41	86 30	86 29	1
ps = 1̄10 . 1̄01	—	—	—	55 31	—
pq = 110 . 01̄1	40	93 21— 93 40	93 30	93 31	1
$s'q$ = 1̄01 : 011	—	—	—	45 4	—
qp = 011 : 110	39	63 21— 63 39	63 31	63 32	1
ps' = 110 . 1̄01	—	—	—	71 24	—
pq = 110 . 01̄1	39	116 19—116 41	116 30	116 28	2
$r'o'$ = 201 . 1̄11	—	—	—	34 48	—
$o'p$ = 1̄11 . 110	—	—	—	92 37	—
pr' = 110 . 201̄	28	52 28— 52 48	52 37	52 35	2
$r'p$ = 201 : 110	26	127 12—127 38	127 23	127 25	2

Total number of measurements, 474.

A typical crystal of this salt is represented later in fig. 2 (p. 274).

The common cleavage of the series is well marked in this salt, namely, parallel $r'\{\bar{2}01\}$.

Volume.

Relative Density.—The following results were obtained from four determinations with independent material :—

Weight of salt employed.	Sp. gr. at 20°/4°.	
6·4573	2·6803	
6·1688	2·6808	Mean, 2·6805
6·9637	2·6806	
6·7175	2·6803	

Molecular Volume.—M/d = 588 ÷ 2·6805 = 219·36.

Distance Ratios.—The following are the values of these ratios, obtained by combination of the above molecular volume with the axial ratios and axial angle already given :—

$$\chi : \psi : \omega = 6\cdot3001 : 8\cdot4861 : 4\cdot2524.$$

Optics.

Orientation of Axes of Optical Ellipsoid.—The plane of the optic axes (bi-normals) is the symmetry plane.

The sign of the double refraction is positive.

The extinction angle was found to be as follows, with the aid of two excellent section plates ground parallel to the plane of symmetry :—

Section 1 . . 1° 3′; Section 2 . . 1° 18′; Mean . . 1° 10′.

The direction is in front of the normal, that is, nearer to the inclined axis a. This extinction direction is the second median line. The first median line lies in the acute morphological axial angle ac, and is inclined 1° 10′ to the axis a. The second median line is inclined 16° 24′ to the vertical axis c.

Refractive Indices.—Six prisms, ground on separate crystals derived from four different crops, were employed in the determinations, which afforded the following results :—

Refractive Indices of Rubidium Magnesium Selenate.

	Light.	Prism 1.	Prism 2.	Prism 3.	Prism 4.	Prism 5.	Prism 6.	Mean.
	Li	1·4975	—	··	1·4979	1·4981	1·4977	1·4978
	C	1·4981	—	—	1·4984	1·4985	1·4984	1·4983
Vibrations parallel	Na	1·5008	—	—	1·5011	1·5013	1·5011	1·5011
2nd median line	Tl	1·5039	—	··	1·5041	1·5043	1·5042	1·5041
	F	1·5076	—	—	1·5077	1·5079	1·5076	1·5077
	G	1·5131	—	—	1 5132	1·5135	1·5132	1·5133
	Li	1·4994	1·4997	1·4997	—	1·5000	—	1·4997
	C	1·4999	1·5002	1·5002	—	1·5003	··	1·5002
Vibrations parallel	Na	1·5029	1·5032	1·5031	—	1 5033	—	1·5031
symmetry axis b	Tl	1·5059	1·5059	1·5061	—	1·5062	—	1·5060
	F	1·5097	1·5096	1·5102	—	1·5096	—	1·5098
	G	1·5151	1·5150	1·5155	—	1·5153	—	1·5152
	Li	—	1·5098	1 5101	1·5100	—	1·5100	1·5100
	C	—	1·5103	1·5107	1·5105	—	1·5106	1·5105
Vibrations parallel	Na	—	1·5133	1·5135	1 5135	—	1·5137	1·5135
1st median line	Tl	—	1·5165	1·5168	1·5167	—	1·5167	1·5167
	F	—	1·5203	1·5207	1·5204	—	1·5204	1 5205
	G	—	1·5262	1·5266	1·5265	—	1·5263	1·5264

The mean refractive index (mean of all three indices) for sodium light is 1·5059.

The intermediate refractive index β is represented accurately to near F of the spectrum by the following formula :—

$$\beta :$$

The α indices are nearly reproduced by the formula if the constant 1·4857 is diminished by 0·0019, and the γ indices if the constant is increased by 0·0105. The indices thus yielded by the formulæ are corrected for a vacuum, being 0·0004 higher than those given in the table, which are not so corrected.

Alteration of Refraction by Rise of Temperature.—Determinations of refractive index at 75° indicated that the indices become reduced by 0·0019 by a rise of 60° in temperature.

Axes of the Optical Ellipsoid.—These are as follows :—

Axes of optical indicatrix :

$$a : \beta : \gamma = 0.9987 : 1 : 1.0069.$$

Axes of optical velocity ellipsoid :

$$\mathfrak{a} : \mathfrak{b} : \mathfrak{c} = 1.0013 : 1 : 0.9931.$$

Molecular Optical Constants.—The calculated values of these constants are the following :—

Axis of optical indicatrix		$\alpha.$	$\beta.$	$\gamma.$
Specific refraction, $\dfrac{n^2-1}{(n^2+2)\,d}=n$ . .	{ C	0·1094	0·1098	0·1117
	{ G	0·1122	0·1125	0·1146
Molecular refraction, $\dfrac{n^2-1}{n^2+2}\cdot\dfrac{M}{d}=m$.	{ C	64·33	64·54	65·66
	{ G	65·96	66·17	67·38
Specific dispersion, n_G-n_C		0·0028	0·0027	0·0029
Molecular dispersion, m_G-m_C		1·63	1·63	1·72
Molecular refraction, $\dfrac{a-1}{d}\,M$ . . . C		109·31	109·72	111·99

Optic Axial Angle.—Three pairs of excellent section plates, ground parallel to the first and second median lines, afforded the following values :—

Apparent Angle in Air of Rubidium Magnesium Selenate.

Light.	Section 1	Section 2	Section 3.	Mean 2E.
Li . . .	74° 3′	74° 15′	74° 1′	74° 6′
C . . .	73 58	74 9	73 52	74 0
Na . .	73 34	73 40	73 29	73 34
Tl .	73 9	73 10	73 0	73 6
F .	72 22	72 34	72 27	72 28

Determination of true Optic Axial Angle of Rubidium Magnesium Selenate.

Light	No. of section perp. 1st median line.	Observed values of 2Ha.	No. of section perp. 2nd median line.	Observed values of 2Ha.	Calculated values of 2Va.	Mean value of 2Va.
Li . . .	1	42° 55′	1a	112° 29′	47 30′	
	2	42 50	2a	112 29	47 26	47° 26′
	3	42 47	3a	112 33	47 22	
C . . .	1	42 49	1a	112 25	47 26	
	2	42 47	2a	112 26	47 24	47 24
	3	42 46	3a	112 32	47 21	
Na . . .	1	42 18	1a	112 6	47 1	
	2	42 19	2a	112 6	47 2	47 3
	3	42 22	3a	112 4	47 6	
Tl	1	41 46	1a	111 40	46 37	
	2	41 49	2a	111 36	46 40	46 37
	3	41 44	3a	111 41	46 35	
F	1	41 0	1a	111 5	46 2	
	2	41 7	2a	111 1	46 9	46 6
	3	41 5	3a	111 7	46 6	

An examination of the optic figures in white light when the section perpendicular to the first median line is immersed in benzene, whose refraction is almost identical

with that of this salt, shows the brushes fringed according to the following colour scheme :—

<div align="center">blue | red   1st M.L.   red | blue.</div>

The obtuse morphological axial angle ac is to the left in this scheme.

It is thus evident that the optic axial angle is greater for red than for blue, and determinations for C-light and F-light gave the angles for these respective wave-lengths 47° 34′ and 46° 3′, values so close to those derived from the determinations of 2Ha and 2Ho as seen in bromnaphthalene (*vide* the table) as to confirm the accuracy of the amount of dispersion of the axes shown in the table.

Dispersion of the Median Lines.—The actual circle readings obtained during the benzene immersion observations indicated that the first median line is so dispersed that it lies nearer to the morphological axis a for blue than for red by about 7′–8′ (between C- and F-light).

Effect of Rise of Temperature on the Optic Axial Angle.—Determinations of the apparent angle in air at 75° indicated that the 2E increases 25′ for 60° rise of temperature.

<div align="center">CÆSIUM MAGNESIUM SELENATE, Cs$_2$Mg(SeO$_4$)$_2$,6H$_2$O.</div>

A determination of magnesium in 1·2312 gramme of crystals gave 0·2077 gramme of magnesium pyrophosphate, which corresponds to 3·64 per cent. of magnesium. The theoretical percentage of this metal is 3·51.

<div align="center">*Goniometry.*</div>

Ten excellent crystals of this salt, selected from three of the best crops, were used in the goniometrical measurements.

Habit: flattened prismatic.

Axial angle : $\beta = 73° 43′$.

Ratio of axes : $a : b : c = 0·7314 : 1 : 0·4960$.

Forms observed : $b = \{010\} \infty P \infty$; $c = \{001\}oP$; $p = \{110\} \infty P$;
  $q = \{011\} P\infty$; $o′ = \{\bar{1}11\} + P$; $r′ = \{\bar{2}01\} + 2P\infty$.

The accompanying table represents the results of the measurements.

The cæsium salt of the magnesium group exhibits precisely the habitual configuration which has been shown to be characteristic of all the cæsium salts of the sulphate group of this series and of cæsium zinc selenate. It invariably presents large q (clinodome) faces, giving the prismatic appearance to the crystals, narrower (often very narrow) c basal plane faces, and for the end faces those of the primary prism p and of the hemi-pyramid $o′$, sometimes one and sometimes the other predominating. The clinopinacoid b is generally present as a strip, but the orthopinacoid a has not been observed. The faces of the orthodome $r′$ were usually small, but in one of the best crops were frequently found much more prominently developed.

Morphological Angles of Cæsium Magnesium Selenate.

Angle measured.	No. of measurements.	Limits.	Mean observed.	Calculated.	Difference.
ac = 100 : 001 = β	—	—	—	73 43	—
as = 100 : 101	—	—	—	45 3	—
sc = 101 : 001	—	—	—	28 40	—
cr' = 001 : $\bar{2}$01	14	64 30— 64 47	64 39	64 31	8
cd = 001 : $\bar{1}$01	—	—	—	38 46	—
$d'r'$ = $\bar{1}$01 : $\bar{2}$01	—	—	—	25 45	—
$r'a$ = $\bar{2}$01 : 100	—	—	—	41 46	—
$r'c$ = $\bar{2}$01 : 00$\bar{1}$	14	115 11—115 31	115 20	115 29	9
ap = 100 : 110	—	—	—	35 6	—
pp' = 110 : 120	—	—	—	19 28	—
$j'b$ = 120 : 010	—	—	—	35 26	—
pb = 110 : 010	14	54 49— 55 2	54 54	54 54	0
pp = 110 : $\bar{1}$10	19	70 5— 70 19	70 12	*	—
pp = 110 : $\bar{1}$10	19	109 42—109 58	109 49	109 48	1
cq = 001 : 011	42	25 15— 25 34	25 29	*	—
$q'b$ = 01$\bar{1}$: 010	14	64 23— 64 39	64 31	64 31	0
qq = 011 : 01$\bar{1}$	21	128 56—129 13	129 1	129 2	1
ao = 100 : 111	—	—	—	48 11	—
oq = 111 : 011	—	—	—	27 9	—
aq = 100 : 011	—	—	—	75 20	—
qo' = 011 : $\bar{1}$11	—	—	—	35 0	—
$o'a$ = $\bar{1}$11 : 100	—	—	—	69 40	—
ro = 001 : 111	—	—	—	34 7	—
op = 111 : 110	—	—	—	42 37	—
cp = 001 : 110	39	76 36— 76 54	76 44	*	—
po' = 110 : 11$\bar{1}$	24	58 9— 58 23	58 16	58 24	8
$o'c$ = 11$\bar{1}$: 00$\bar{1}$	24	44 51— 45 7	45 0	44 52	8
pr = 110 : 00$\bar{1}$	39	103 7—103 30	103 16	103 16	0
bo = 010 : 111	—	—	—	70 39	—
ox = 111 : 101	—	—	—	19 21	—
$b'x'$ = 010 : $\bar{1}$11	20	65 13— 65 31	65 22	65 22	0
$o'x'$ = $\bar{1}$11 : $\bar{1}$01	—	—	—	24 38	—
sq = 101 : 011	—	—	—	37 37	—
qp = 011 : $\bar{1}$10	39	87 34— 87 45	87 40	87 42	2
pn = 110 : $\bar{1}$01	—	—	—	54 41	—
pq = 110 : 01$\bar{1}$	39	92 9— 92 28	92 20	92 18	2
$s'q$ = 10$\bar{1}$: 011	—	—	—	45 16	—
qp = 011 : 110	40	62 55— 63 8	63 1	62 58	3
ps' = 110 : $\bar{1}$0$\bar{1}$	—	—	—	71 46	—
pq = 110 : 01$\bar{1}$	40	116 47—117 6	116 59	117 2	3
$r'o'$ = $\bar{2}$01 : $\bar{1}$11	20	34 53— 35 8	35 1	35 3	2
$o'p$ = $\bar{1}$11 : 110	23	92 31— 92 56	92 43	92 34	9
pr' = 110 : $\bar{2}$01	23	52 12— 52 22	52 16	52 23	7
$r'p$ = $\bar{2}$01 : 110	23	127 37—127 54	127 44	127 37	7

Total number of measurements, 550.

Fig. 4 in the double sulphate memoir, representing cæsium magnesium sulphate, is equally applicable to the commonest type of cæsium magnesium selenate. It is reproduced in fig. 3 (p. 274) for the purpose of the comparison of the habits of the three magnesium selenates

There is a good cleavage parallel ⟩′ {$\bar{2}01$}.

Volume.

Relative Density.—The following four determinations with separate quantities of material were made :—

Weight of salt employed.	Sp. gr. at 20°/4′.	
5·8081	2·9385	
5·7961	2·9391	Mean, 2·9388
6·2659	2·9389	
5·9848	2·9387	

Molecular Volume.—M/d = 683 ÷ 2·9388 = 232·41.

Distance Ratios.—The following distance ratios are afforded by combination of the molecular volume with the axial angle and ratios already given :—

$$\chi \cdot \psi \quad \omega = 6\cdot3918 \cdot 8\cdot7390 \cdot 4\cdot3345.$$

Optics.

Cæsium magnesium selenate exhibits extraordinary optical characters, including crossed axial plane dispersion of the optic axes and great sensitiveness of the optic axial angle to change of temperature, together with corresponding apparently abnormal refraction phenomena. In this respect it is surprisingly similar to cæsium magnesium sulphate, the change of selenium for sulphur simply advancing all the optical constants without materially altering their mutual relations. Moreover, the abnormalities will be shown to be the direct result of the progression in optical properties which has so far throughout both sulphate and selenate series been found to accompany progress in the atomic weight of the alkali metal.

Orientation of Axes of Optical Ellipsoid.—The extinction angle in the symmetry plane, with respect to the normal to the basal plane, was determined with the aid of two sections parallel to the symmetry plane as usual, and afforded the following results :—

Section 1, 20° 35′; Section 2, 20° 25′. Mean, 20° 30′.

The direction is in front of the normal to (001), that is, nearer to the morphological axis *a*. This direction is the second median line for all wave-lengths of light

from the red end of the spectrum as far as wave-length 466 in the blue; beyond this the symmetry axis b becomes the second median line. The first median line lies in the symmetry plane for all wave-lengths, in the acute angle of the morphological axes ac, and is inclined to the axis a 20° 30'. The second median line so long as it remains in the symmetry plane lies in the obtuse angle of the axes ac, and for sodium light (to which the measurements of extinction refer) is inclined 36° 47' to the vertical axis c.

The *double refraction* is of positive sign.

Refractive Indices.—These were determined with six excellent prisms, ground on six different crystals selected from three different crops. Each of the two prisms prepared to furnish α and β, and whose refracting edge was in each case parallel to the second median line (for wave-lengths as far as 466) and whose bisecting plane was that containing this edge and the symmetry axis, appeared to exhibit only one image of the Websky slit of the refractometer instead of the usual two; and this image had the further peculiarity of remaining permanent at all positions of the nicol, and for all wave-lengths, although its position naturally varied with the wave-length. On using the high-power eyepiece, the single image was clearly resolved into two images separated by 1' for red light, approximating again to a single image as the greenish-blue was approached. One of the two images corresponded to α, and extinguished with the nicol at 90°; the other, corresponding to β, extinguished when the nicol was rotated to 0°. For F-light the positions of the two images were identical, and for G-light the two images were found to have passed each other, the one which was formerly right being now left and *vice versâ*, the amount of separation being about 1°. Hence for G-light the directions in the crystal parallel to which the α and β vibrations occur are interchanged, compared with those corresponding to wave-lengths on the red side of the crossing wave-length.

The actual results of the refractive index determinations are set forth in the accompanying table. It will be observed that the difference between α and β works out to 0·0002 for Li-light. Considering the minuteness of this quantity, it is as well to have an independent check upon its accuracy, and this is afforded by calculation from the optic axial angle for this wave-length, assuming also the accuracy of one of the two indices, either α or β, and of course of γ, with the aid of the formula

$$\text{Cos } Va = \sqrt{\left(\frac{1/\beta^2 - 1/\gamma^2}{1/\alpha^2 - 1/\gamma^2}\right)}.$$

The value of α thus obtained is 1·51433, or 0·00017 less than β_{Li}, a value for the difference which satisfactorily confirms that (0·0002) given in the table as the result of the determinations.

The wave-length for which the indices α and β are truly identical is 466 in the blue, as will subsequently be shown in considering the optic axial angle.

Refractive Indices of Cæsium Magnesium Selenate.

Direction of vibrations	Light.	Prism 1.	Prism 2.	Prism 3.	Prism 4.	Prism 5.	Prism 6.
Parallel 2nd median line lying in symmetry plane.		1·5142	·5143	1·5144	1·5143	1·5143	1·5143
		1·5147	·5148	1·5149	1·5148	1·5148	1·5148
		1·5177	·5179	1·5179	1·5179	1·5177	1·5178
		1·5209	·5212	1·5211	1·5210	1·5209	1·5210
		1·5246	·5250	1·5246	1·5250	1·5247	1·5248
Parallel symmetry axis, b, now 2nd median line. }	G						
Parallel symmetry axis, b.		1·5144	—	—	—	1·5145	1·5146	1·5145
		1·5149	—	—	—	1·5150	1·5151	1·5150
		1·5178	—	—	—	1·5178	1·5181	1·5179
		1·5211	—	—	—	1·5210	1·5211	1·5211
		1·5247	—	—	—	1·5247	1 5249	1·5248
In symmetry plane at right angles to 1st median line			1·5303	1·5305	1·5303	1·5308	1·5306
Parallel 1st median line.	Li		·5200	1·5200			1·5201
	C		·5205	1·5205	·5206		1 5206
	Na		·5237	1·5235	·5237		1·5236
	Tl		·5270	1·5267	·5270		1·5269
	F		·5306	1·5307	·5310		1·5308
	G		·5363	1·5363			1·5364

The mean refractive index (mean of all three indices) for sodium light is 1·5198.

The intermediate index β, corrected to a vacuum, is very accurately expressed throughout the whole length of the spectrum by the following formula :—

$$\beta = 1·5033 + \frac{522\ 267}{\lambda^2} - \frac{52\ 000\ 000\ 000}{\lambda^4} + \ldots$$

The a index is given by diminishing the constant 1·5033 by 0·0002 as far as C of the spectrum, by 0·0001 between that and Tl-light, it is afforded exactly by the formula for wave-lengths between that of Tl-light and that of blue light, while for G-light 0·0001 requires again to be subtracted from the constant. The γ index is afforded by the formula fairly accurately if the constant is increased by 0·0057.

Alteration of Refraction by Increase of Temperature.—Measurements were carried out at 70° in Na-light with prisms 3 and 5, affording respectively a and γ, and a and β. The latter prism exhibited only a single inextinguishable image of the slit at 70°, consisting of two identically situated (overlapping) images. It will be subsequently shown that at this temperature a uniaxial interference figure is also observed. Both prisms afforded values which indicated that the refractive indices are reduced by 0·0014 for 55° rise of temperature. For wave-lengths beyond the

neighbourhood of wave-length 560 the α vibrations occur parallel to the symmetry axis b, and the β vibrations take place in the symmetry plane.

Ratio of Axes of Optical Ellipsoid.—These ratios are as follows :—

Axes of optical indicatrix :

$$\alpha : \beta : \gamma = 0 \cdot 9999 : 1 : 1 \cdot 0038.$$

Axes of optical velocity ellipsoid :

$$\mathfrak{a} : \mathfrak{b} : \mathfrak{c} = 1 \cdot 0001 : 1 : 0 \cdot 9963.$$

Molecular Optical Constants.—The calculated values of these constants are as under :—

Axis of optical indicatrix	α.	β.	γ.
Specific refraction, $\dfrac{n^2-1}{(n^2+2)d} = \mathrm{n}$. $\begin{cases} \text{C} \\ \text{G} \end{cases}$	0·1026 0·1052	0·1026 0·1052	0·1035 0·1062
Molecular refraction, $\dfrac{n^2-1}{n^2+2} \cdot \dfrac{M}{d} = \mathrm{m}$. $\begin{cases} \text{C} \\ \text{G} \end{cases}$	70·06 71·84	70·08 71·85	70·72 72·51
Specific dispersion, $\mathrm{n}_G - \mathrm{n}_C$	0·0026	0·0026	0·0027
Molecular dispersion, $\mathrm{m}_G - \mathrm{m}_C$	1·78	1·77	1·79
Molecular refraction, $\dfrac{n-1}{d} M$ . . . C	119·64	119·69	120·99

Optic Axial Angle.—Three capital section-plates of this salt were obtained by grinding, out of large but very perfect crystals, perpendicular to the first median line. Such sections require to be very thick, 5 millims. or more, in order to exhibit clear interference figures, on account of the extraordinarily small amount of double refraction. Sections perpendicular to the second median line show no axial brushes even in bromnaphthalene, the obtuse angle of the optic axes being too large for measurement. The plan has accordingly been adopted which was employed in the case of cæsium magnesium sulphate, for the determination of the true optic axial angle, namely, measurement while the sections perpendicular to the first median line were immersed in a liquid whose refractive index was identical with the mean index of the crystal. Cedar oil answers the purpose admirably, its refractive index for sodium light being 1·520.

The optic axial interference figures afforded by thick section-plates are both exceptional and very beautiful. In white light a deeply coloured figure is presented, in which the hyperbolic brushes are broad spectra. In monochromatic light a rapidly changing series of figures is presented, commencing with the brushes separated by 30° for red light, and approximating closer and closer to each other through the yellow and green, until for F-light the separation is only 13°. Passing into the blue, the hyperbolic brushes eventually coalesce, for light of wave-length 466, into the uniaxial rectangular cross, and the lemniscates become circles. The exact wave-length was readily determined by taking the circle reading of the monochromatic illuminator corresponding to the production of the uniaxial figure, and ascertaining the wave-

length to which this reading corresponds from the curve prepared in the calibration of the instrument. (*Vide* 'Phil. Trans.,' A, 1894, p. 925.) When the prism-circle of the illuminator is further rotated towards the violet, the brushes open out again, and separate along the vertical diameter of the field of the polariscope, the plane of the optic axes having crossed from the symmetry plane to a plane at right angles to the symmetry plane. These beautiful changes are illustrated by the first six photographic reproductions given in the accompanying Plate; they represent the interference figures observed at the ordinary temperature in Li-, Na-, Tl- and F-light, light of wave-length 466, and G-light respectively.

The actual measurements are recorded in the following two tables. The phenomena when the section-plate is immersed in the cedar-oil cell are precisely similar to those in air, the angles afforded being merely smaller, namely, the true angles of inclination of the optic axes within the crystal.

In both cases the angle for G-light was determined with the section rotated 90° in its own plane, so as to bring the new direction of separation horizontal, and therefore convenient for the measurement of the angle.

The determinations of the circle reading corresponding to uniaxiality were made repeatedly for each of the two positions of the section, and the mean of the whole is the value recorded in the table.

Determination of Apparent Angle in Air of Cæsium Magnesium Selenate.

Plane of optic axes.	Light.	Section 1		Section 2.		Section 3.		Mean 2E.	
In symmetry plane	Li	30	7'	31°	43'	30°	39'	30°	50'
	C	29	32	31	19	30	20	30	24
	Na	26	31	28	30	27	20	27	27
	Tl	21	4	23	33	22	11	22	16
	F	12	7	14	53	13	23	13	28
	Wave-length in blue	0	0	0	0	0	0	0	0
Perpendicular to symmetry plane.	G	21	13	18	8	18	59	19	27

Determination of True Optic Axial Angle of Cæsium Magnesium Selenate by Immersion in Cedar Oil.

Plane of optic axes.	Light	Section 1.		Section 2.		Section 3.		Mean 2Vₐ.	
In symmetry plane.	Li	21°	4'	20°	53'	20°	31'	20°	49'
	C	20	49·	20	39	20	14	20	34
	Na	18	47	18	36	18	22	18	35
	Tl	15	40	15	25	15	10	15	25
	F	10	38	10	23	10	29	10	30
	Wave-length in blue	0	0	0	0	0	0	0	0
Perpendicular to symmetry plane.	G	12	51	11	57	13	38	12	49

<div align="center">

Circle Readings for production of Uniaxial Figure.

</div>

	Section 1.	Section 2.	Section 3.	Mean reading.
In air	4° 59'	5° 27'	5° 16'	5° 14'
In cedar oil . . .	5 23	5 27	5 18	5 23

<div align="center">

Mean of two series 5 19

</div>

The reading 5° 19' corresponds to the passage of blue light of wave-length 466 through the exit slit of the monochromatic illuminator.

Dispersion of the Median Lines.—Although there is such large dispersion of the optic axes, the median lines remain fairly constant. The first median line lies nearer to the axis *a* for red C-light than for greenish-blue F-light by about 15'.

Effect of Rise of Temperature on the Optic Axial Angle.—Sections 1 and 3 were studied at temperatures up to 97°. The phenomena presented were highly interesting, for within this comparatively small range of temperature (the most that can be employed as regards the upward direction on account of the presence of water of crystallisation), the uniaxial cross and circles are produced for all wave-lengths of light in turn, from 466 in the blue to the extreme red of the spectrum. The axes are observed to begin to approach each other as soon as the temperature commences to rise appreciably.

The following table represents the temperatures (corrected for the slight conduction of the crystal holder as described in the memoir concerning cæsium selenate, 'Journ. Chem. Soc., Trans.,' 1897, 895) at which the uniaxial figure is produced for different wave-lengths of light :—

<div align="center">

Mean Corrected Temperatures for Production of Cross.

For F-light at 34°
„ Tl „ 60
„ Na „ 78
„ C „ 91
„ Li „ 94

</div>

When the cross is produced for thallium light at 60° the axes are still separated 13° 0' for sodium light and 20° 30' for lithium light ; and when the temperature attains 78° and the uniaxial figure is formed for sodium light, the axes remain separated 13° 30' for lithium light.

Repeated heating of the same section would appear to slightly lower permanently the temperatures at which the cross is produced.

The second series of six figures in the Plate represent the phenomena observed at 78°, in Li-, C-, Na-, Tl-, and F-light, and in light of wave-length 466.

COMPARISON OF THE THREE MAGNESIUM SALTS.

The *Morphological Angles* are compared in the accompanying table. The results deducible are as follows :—

The axial angle β of rubidium magnesium selenate is nearly the mean of the axial angles of potassium magnesium and cæsium magnesium selenates.

With only one exception all the morphological angles of the rubidium salt are intermediate in value between those of the potassium and cæsium salts. The exception, the angle *bo'*, only escapes following the rule by 2′, an amount within the limits of experimental error.

The change in the exterior angles on passing from one salt to another of the triplet is not, as a rule, directly proportional to the change in the atomic weight of the alkali metal. The primary prism zone shows the greatest divergence from direct proportion, the change in *ap* being as 1 to 2.

Comparison of the Axial Ratios.

For potassium magnesium selenate . . $a : b : c = 0\cdot7485 : 1 : 0\cdot5031$
„ rubidium   „   „   . . $a : b : c = 0\cdot7424 : 1 : 0\cdot5011$
„ cæsium   „   „   . . $a : b : c = 0\cdot7314 : 1 : 0\cdot4960$

From this table it is clear that the morphological axial ratios of rubidium magnesium selenate are intermediate between those of potassium magnesium and cæsium magnesium selenates.

The general *Habit* of the crystals of the three salts exhibits very markedly the progression of type which has been established throughout the double sulphate series, and in the case of the zinc group of double selenates.

The potassium salt is characterised by a stout prismatic habit, the prism zone forming the prism, and by relatively large end basal plane faces. The cæsium salt is characterised by another prismatic habit, the prism faces being those of the clinodome *q* {011}, and the faces of the basal plane are generally reduced to a strip. The rubidium salt has been observed to exhibit every gradation between these two quite different habits, and a very large proportion of the crystals examined were of a clearly intermediate type, showing moderate-sized faces of the basal plane, and the prism and clinodome faces more or less equally developed. The three accompanying figures, representing typical crystals of the three salts, will render this progression of habit perfectly plain.

Fig. 1.   Fig. 2.   Fig. 3.
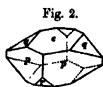
KMg selenate.   RbMg selenate.   CsMg selenate.

Comparison of the Morphological Angles of the Three Magnesium Salts.

Angle			Potassium salt.	Difference.	Rubidium salt.	Difference.	Caesium salt.
			° '	'	° '	'	° '
ac	= 100 : 001	= β	75 42	−56	74 46	−63	73 43
aa	= 100 : 10$\bar{1}$		46 30	−40	45 50	−47	45 3
ac	= 101 : 001		29 12	—	28 56	—	28 40
cr'	= 001 : 201		62 53	+44	63 37	+54	64 31
ca'	= 001 : 10$\bar{1}$		38 1	—	38 20	—	38 46
a'r'	= 10$\bar{1}$: 301		24 52	+25	25 17	+28	25 45
r'a	= 201 : $\bar{1}$00		41 25	—	41 37	—	41 46
ap	= 100 : 110		35 55	−17	35 38	−32	35 6
pp'	= 110 · 120		19 28	—	19 28	—	19 28
p'b	= 120 : 010		34 37	+17	34 54	+32	35 26
pb	= 110 · 010		54 5	—	54 22	—	54 54
cq	= 001 : 011		26 0	−13	25 47	−18	25 29
qb	= 011 : 010		64 0	—	64 13	—	64 31
ao	= 100 : 111		49 43	−40	49 3	−52	48 11
oq	= 111 : 011		27 27	−11	27 16	− 7	27 9
aq	= 100 : 011		77 10	−51	76 19	−59	75 20
qo'	= 011 : $\bar{1}$11		34 16	+18	34 34	+26	35 0
o'a	= $\bar{1}$11 : 100		68 34	+33	69 7	+33	69 40
co	= 001 : 111		34 54	−21	34 33	−26	34 7
op	= 111 : 110		43 34	−27	43 7	−30	42 37
cp	= 001 · 110		78 28	−48	77 40	−56	76 44
po'	= 110 : 11$\bar{1}$		57 13	+32	57 45	+39	58 24
o'c	= 11$\bar{1}$ · 00$\bar{1}$		44 19	+16	44 35	+17	44 52
bo	= 010 : 111		69 57	+17	70 14	+25	70 39
os	= 111 · 101		20 3	—	19 46	—	19 21
b'o'	= 010 : 11$\bar{1}$		65 16	− 1	65 15	+ 7	65 22
o'd	= 11$\bar{1}$. 101		24 44	—	24 45	—	24 38
sq	= 101 : 011		38 19	−19	38 0	−23	37 37
qp	= 011 : $\bar{1}$10		85 34	+55	86 29	+73	87 42
ps	= $\bar{1}$10 : 10$\bar{1}$		56 7	−36	55 31	−50	54 41
s'q	= $\bar{1}$01 : 011		44 55	+ 9	45 4	+12	45 16
qp	= 011 : 110		64 5	−33	63 32	−34	62 58
ps	= 110 : 10$\bar{1}$		71 0	+24	71 24	+22	71 46
r'o'	= 20$\bar{1}$: 11$\bar{1}$		34 30	+18	34 48	+15	35 3
o'p	= $\bar{1}$11 : 110		92 54	−17	92 37	− 3	92 34
pr'	= 110 : 20$\bar{1}$		52 36	− 1	52 35	−12	52 23

The *Cleavage Direction* of all three salts is identical, namely, parallel to the orthodome r'{$\bar{2}$01}, which is also common to the whole of the salts of the series yet studied.

Comparison of the Relative Densities.

Potassium magnesium selenate . . 2·3630 } Diff. 0·3175
Rubidium   „   „   . . 2·6805
Cæsium   „   „   . . 2·9388 } Diff. 0·2583

The density increases with the atomic weight of the alkali metal, and is greater for the replacement of potassium by rubidium than for rubidium by cæsium in the proportion of 5 : 4. This proportion is the same as for the corresponding magnesium double sulphates, but the actual amounts of the differences are smaller ; the difference between potassium magnesium sulphate and its rubidium analogue is 0·354, and between the latter and cæsium magnesium sulphate 0·288.

Comparison of the Molecular Volumes.

Potassium magnesium selenate . . . 209·73 } Diff. 9·63
Rubidium   „   „   . . . 219·36
Cæsium   „   „   . . . 232·41 } Diff. 13·05

The molecular volumes show progression with the atomic weight of the alkali metal, but the replacement of rubidium by cæsium is accompanied by the greater change. The proportion is similar to that found for the double sulphates ; the differences observed between the magnesium double sulphates were 9·33 and 13·19.

Comparison of the Distance Ratios.

	χ.	Diff.	ψ.	Diff.	ω.	Diff.
KMg selenate. . . .	6·2233		8·3144		4·1829	
		768		1717		695
RbMg „ . . .	6·3001		8·4861		4·2524	
		917		2529		821
CsMg „ . . .	6·3918		8·7390		4·3345	
		1685		4246		1516

The simplified ratios, taking ψ for KMg selenate as unity, are as follow :—

	χ.	Diff.	ψ.	Diff.	ω.	Diff.
KMg selenate. . . .	0·7485		1·0000		0·5031	
		92		207		83
RbMg „	0·7577		1·0207		0·5114	
		111		304		99
CsMg „	0·7688		1·0511		0·5213	
		203		511		182

These ratios (topic axes) indicate that there is an extension of the distance separating the structural units in all three axial directions, the maximum being along

the symmetry axis and the minimum along the vertical axis. Also the change is greater for the replacement of rubidium by cæsium than for that of potassium by rubidium.

### Comparison of Orientations of the Optical Indicatrix.

Inclination of Axis of Indicatrix to Vertical Axis c.

For potassium magnesium selenate . . 11° 18′ } Diff. 5° 6′
  „   rubidium   „   „   . . 16 24 }
  „   cæsium   „   „   . . 36 47 } Diff. 20 23

The optical indicatrix rotates about the symmetry axis when one alkali metal is replaced by another, and by an amount which is very much greater when cæsium replaces rubidium than when rubidium replaces potassium.

This rotation of the optical ellipsoid is graphically represented in fig. 4, the dark lines representing the axes of the ellipsoid lying in the symmetry plane, namely, the first and second median lines.

Fig. 4.
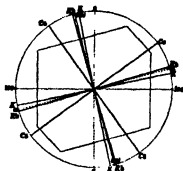
The *Refractive Indices* are compared in the accompanying table.

It will be observed that the a and β indices of rubidium magnesium selenate lie between those of the other two salts. The γ indices of the potassium and rubidium salts are almost identical, and, indeed, those of the rubidium salt are slightly the lower. The reason for this apparent anomaly in the case of the γ indices is connected with a progressive change of dispersion, as will be shown under the next heading. That there is, however, a true progression of refraction is best exhibited by comparison of the values of the mean refractive index of each of the three salts, that is, the mean of the three indices of each salt for any one and the same wave-length of light.

Such mean indices for Na-light are compared in the table following that of the actual indices.

Comparative Table of Refractive Indices.

Index.	Light.	KMg selenate.	RbMg selenate.	CsMg selenate.
α	Li	1·4936	1·4978	1·5143
Vibrations	C	1·4941	1·4983	1·5148
parallel	Na	1·4969	1·5011	1·5178
second median	Tl	1·4999	1·5041	1·5210
line.	F	1·5035	1·5077	1·5248
	G	1·5091	1·5133	1·5304
β	Li	1·4958	1·4997	1·5145
Vibrations	C	1·4963	1·5002	1·5150
parallel	Na	1·4991	1·5031	1·5179
symmetry	Tl	1·5022	1·5060	1·5211
axis b.	F	1·5058	1·5098	1·5248
	G	1·5114	1·5152	1·5305*
γ	Li	1·5103	1·5100	1·5201
Vibrations	C	1·5108	1·5105	1·5206
parallel	Na	1·5139	1·5135	1·5236
first median	Tl	1·5172	1·5167	1·5269
line.	F	1·5210	1·5205	1·5308
	G	1·5266	1·5264	1·5364

Mean Refractive Indices of the Three Salts.

$$\tfrac{1}{3}\,(\alpha + \beta + \gamma)\ \text{for Na.}$$

KMg selenate . . . . 1·5033 ⎫
⎬ Diff. 26
RbMg   ,, ... 1·5059 ⎬
⎬ Diff. 139
CsMg   ,, ... 1·5198 ⎭

The mean refractive indices of KMg, RbMg, and CsMg sulphates are respectively 1·4664, 1·4713, and 1·4877. The differences are 49 and 164, greater than in the selenate group.

The mean refractive index of rubidium magnesium selenate is thus seen to be intermediate between the mean indices of potassium and cæsium magnesium selenates; it lies, however, much nearer to that of the potassium salt than to that of cæsium magnesium selenate, as has been found general in the double sulphate series. The proportion of the differences for the two replacements is 1 : 5 for this group of double selenates, as against 1 : 4 for the double sulphates containing magnesium.

* The vibrations for G-light in the case of CsMg selenate occur in the symmetry plane.

Comparison of the Double Refraction, Na$_{\gamma-\alpha}$.

For KMg selenate . . . . 0·0170 ⎫
„ RbMg „ . . . . 0·0124 ⎬ Diff. 46
„ CsMg „ . . . . 0·0058 ⎭ Diff. 66

The double refraction is shown by the above table to decrease at an accelerating rate as the atomic weight of the alkali metal is increased. The amount of diminution for the first replacement is nearly double as much as the increase of mean refraction, which accounts for the fact already alluded to that the γ indices of rubidium magnesium selenate are slightly lower than those of potassium magnesium selenate. This latter apparent anomaly, therefore, is the direct outcome of the general rule so far established for the series as regards progressive diminution of double refraction.

The instructive axial ratios of the optical ellipsoid (calculated for Na-light) are compared in the following tables, and they are also graphically expressed in the curves of fig. 5. The table representing the values when β_{KMg} is taken as unity, and the series of dotted curves corresponding, exhibit the total change of the ellipsoid on passing from one salt to another, as distinguished from the change in the relations of the ellipsoidal axes of any one salt.

Comparison of the Optical Ellipsoids.

Optical Indicatrix.

	α.		β.		γ.	Double refraction.
KMg selenate	0·9985	:	1	:	1·0099	114
RbMg selenate	0·9987	:	1	:	1·0069	82
CsMg selenate	0·9999	:	1	:	1·0038	39

Optical Velocity Ellipsoid.

	$\mathfrak{a}$.		$\mathfrak{b}$.		$\mathfrak{c}$.	
KMg selenate	1·0015	:	1	:	0·9902	113
RbMg selenate	1·0013	:	1	:	0·9931	82
CsMg selenate	1·0001	:	1	:	0·9963	38

Indicatrix when $\beta_{KMg} = 1$.

	α.		β.		γ.	
KMg selenate .	0·9985	:	1	:	1·0099	
	28		· 27			3
RbMg selenate. . . .	1·0013	:	1·0027	:	1·0096	
	112		99			67
CsMg selenate	1·0125	:	1·0126	:	1·0163	

Fig. 5.

Axial ratios of optical indicatrix.
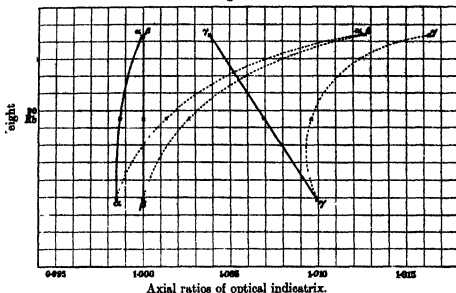
The axial ratios of the optical indicatrix of rubidium magnesium selenate are intermediate between those corresponding to the potassium and cæsium magnesium salts. The total change on passing from one salt to another is much the greater for the passage from the rubidium to the cæsium salt than for the passage from the potassium to the rubidium salt. The apparent anomaly as regards the latter change in the case of the γ ratio has already been fully explained. As regards total change, it will also be observed that the amount is considerably less along the direction of the axis γ than along the other two axes, along which the change is about equal.

The most striking circumstance about the magnesium series of curves, and in which they differ remarkably from those given in the author's last communication on the zinc group of double selenates, is the closeness of the α curve to the β straight line; and that the convergence of the α and γ curves, which graphically represents the diminution of double refraction with increase of atomic weight of the alkali metal, results in consequence in actual contact (intersection) of the α and β curves. It is thus at once made clear that the extraordinary optical properties of cæsium magnesium selenate, involving apparent uniaxial refraction phenomena and inter-ference figures in convergent polarised light, are the direct results of the operation of the rule now established, so far as the series has yet been studied, that the double refraction decreases at an accelerating rate with increase of the atomic weight of the alkali metal. The curves are drawn from the values for sodium light at the ordinary temperature, and, although on the scale employed contact appears to be just attained, this is not really so for ·sodium light, although it requires a high-power eyepiece to separate the two images of the slit of the spectrometer in Na-light afforded by a 60°-prism ground to yield α and β. It would be true, however, for sodium light at the temperature of 78°, and even for lithium light at 94°, while it is absolutely true

at the ordinary temperature itself for blue light of wave-length 466. For G-light intersection has occurred, and the curve hitherto called α has passed to the γ side of the β straight line.

This unusual optical character of cæsium magnesium selenate is similar to a remarkable extent to that of cæsium magnesium sulphate (p. 475 of the double sulphate memoir[*]); in the case of the latter salt, the wave-length for uniaxiality is 450.

<div align="center">Comparison of the Molecular Optical Constants.</div>

$$\text{Specific Refraction, } \frac{n^2 - 1}{(n^2 + 2)d} = \mathfrak{n}.$$

	For ray C (Hα).			For ray Hγ near G.		
	α.	β.	γ.	α.	β.	γ.
KMg sel.	0·1233	0·1237	0·1267	0·1264	0·1269	0 1300
	138	139	150	142	144	154
RbMg sel.	0·1094	0·1098	0·1117	0·1123	0 1125	0·1146
	68	72	82	70	73	84
CsMg sel.	0·1026	0·1026	0·1035	0·1052	0·1052	0 1062

$$\text{Molecular Refraction, } \frac{n^2 - 1}{n^2 + 2} \cdot \frac{M}{d} = \mathfrak{m}.$$

	For ray C (Hα).			For ray Hγ near G.		
	α.	β.	γ.	α.	β.	γ.
KMg sel.	61·07	61·30	62·81	62·64	62 87	64 44
	3·26	3 24	2·85	3·32	3·30	2·94
RbMg sel.	64·33	61·54	65·66	65·96	66·17	67·38
	5·73	5·54	5·06	5·88	5·68	5 13
CsMg sel.	70·06	70 08	70·72	71·84	71·85	72·51

$$\text{Specific Dispersion, } \mathfrak{n}_G - \mathfrak{n}_C.$$

	α.	β.	γ.
KMg sel.	0·0032	0·0032	0·0033
RbMg sel.	0·0028	0·0027	0·0029
CsMg sel.	0·0026	0·0026	0·0027

$$\text{Molecular Dispersion, } \mathfrak{m}_G - \mathfrak{m}_C.$$

	α.	β.	γ.
KMg sel.	1·57	1·57	1·63
RbMg sel.	1·63	1·63	1·72
CsMg sel.	1·78	1·77	1·79 .

[*] 'Journ. Chem. Soc., 'Trans.,' 1896.

Molecular Refraction (GLADSTONE), $\dfrac{n-1}{d}$ M.

	α.	β.	γ.
KMg sel.	103·63	104·09	107·13
	5·68	5·63	4·86
RbMg sel.	109·31	109·72	111·99
	10·33	9·97	9·00
CsMg sel.	119·64	119·69	120·99

The whole of the specific and molecular optical constants of rubidium magnesium selenate are intermediate between those of potassium magnesium selenate and of cæsium magnesium selenate. The molecular refraction increases considerably more when rubidium is replaced by cæsium than when potassium is replaced by rubidium, the proportion being as 5 : 3. The amount of change is less along the axis γ than along the two other directions, along which the amounts are nearly identical.

These results are independent of the temperature, for it has been shown that the refraction diminishes when the temperature is raised, and the density naturally varies in the same direction. They are also independent of the formula employed, whether that of LORENZ or of GLADSTONE and DALE.

On comparing the molecular refraction constants of the magnesium double selenates with those previously given by the author for the magnesium double sulphates, in order to arrive at the effect of replacing sulphur by selenium, it is found that this replacement is accompanied by an increase of 6·9 to 7·1 Lorenz units or 12·7 to 13·3 Gladstone units. Allowing for the presence of two atoms, this gives for the increase per atom an average of 3·5 Lorenz and 6·5 Gladstone units. Both values refer to·light of the wave-length of the C hydrogen line.

The specific and molecular dispersions of the magnesium double selenates are higher than those of the magnesium double sulphates, as was also observed with regard to the two zinc groups.

Comparison of the Optic Axial Angles.—The magnitudes of the optic axial angles of the three magnesium double selenates are not strictly comparable, on account of the extraordinary phenomena presented by the cæsium salt, involving the crossing of the plane of the optic axes. It has already been shown, in discussing the refraction phenomena, that this is the result of the operation of the rule of progression of the double refraction with the atomic weight of the alkali metal.

The closeness of the similarity between the optical behaviour of this salt and cæsium magnesium sulphate, already alluded to as regards the refraction, is very apparent as regards the optic axial angle phenomena. Cæsium magnesium sulphate and selenate exhibit respectively an angle for lithium light in the plane of symmetry of 18° 10′ and 20° 49′. In both cases the angle diminishes with diminishing wave-length until the axes unite, for wave-lengths 450 and 466 respectively, in the centre

of the field of the polariscope, to produce the uniaxial figure consisting of rectangular diagonal cross and circular rings. For still shorter wave-lengths the axial brushes separate in the vertical field of the instrument at right angles to the symmetry plane, and for G-light the separations are respectively 7° 0' and 12° 49'. The total dispersion between Li and G amounts to 25° 10' and 33° 38' respectively, the selenate thus being distinguished by considerably greater dispersion. All these figures refer to the true angles within the crystals.

The *Effect of Rise of Temperature on the Optic Axial Angle* in the case of cæsium magnesium selenate is likewise remarkably similar to that on the corresponding sulphate. In both cases the optic axial angle for wave-lengths on the red side of the crossing wave-length rapidly contracts as the temperature is raised, until the uniaxial figure is produced for each wave-length towards the red in turn. The temperatures at which the uniaxial figure is produced for the different wave-lengths are slightly lower in the case of cæsium magnesium selenate than in that of the sulphate. The temperatures for the latter salt given in the memoir concerning the double sulphates (*loc. cit.*, p. 371) were the actual temperatures read off on the thermometers of the heating apparatus of the larger Fuess polariscopical goniometer. In order to render them strictly comparable with those given in this memoir for cæsium magnesium selenate, they require to be corrected for the slight conduction of the crystal holder, as has been done in the case of the latter salt. These corrected temperatures are set forth below, and alongside them are given for comparison those of the selenate :—

Temperatures at which the Uniaxial Figure is produced.

Light.	CsMg selenate.	CsMg sulphate.
F	34°	47°
Tl	60	66
Na	78	80
C	91	93
Li	94	96

It is thus a fact that within the life-range of temperature of these dissociable water-containing salts the uniaxial figure is produced for every wave-length of the visible spectrum, and for each wave-length there is a definite temperature at which these interesting monoclinic crystals simulate uniaxial symmetry as regards their optical properties while retaining morphologically their exterior monoclinic symmetry.

The interference figures of cæsium magnesium selenate given in the Plate opposite p. 272 are almost equally applicable to cæsium magnesium sulphate.

With regard to the effect of rise of temperature on potassium magnesium and rubidium magnesium selenates, it was observed that the optic axial angle of the

former increased 3° 10′ for 60° of rise, and that the angle of the latter was but remarkably slightly affected by change of temperature, being increased by only 25′ for 60° of rise. Even in this magnesium group, therefore, in which the cæsium salt is abnormal, the character of progression which has been found to obtain throughout the double sulphate series, and in the zinc group of double selenates already studied, is still found to hold good, namely, a decided increase of the optic axial angle in the case of the potassium salt, a decided decrease in the case of the cæsium salt, and a comparative indifference to change of temperature in the case of the rubidium salt.

It is somewhat remarkable that four cases of large dispersion in crossed axial planes have been observed in the course of the author's work on the simple and double sulphates, namely, the rhombic sulphate of rubidium and selenate of cæsium and the monoclinic double sulphate and double selenate containing cæsium and magnesium. In each case the phenomenon has been rendered possible by the concurrence of two conditions, namely, extremely small double refraction (closeness of the α and γ indices of refraction) and the approximation to identity of the intermediate index of refraction β to either the α or γ index. The latter condition appears to be necessary for crossing of the axial planes to be possible, and the former condition for magnitude of separation of the optic axes in the two perpendicularly crossed planes. These two conditions are adequate to render the substance highly sensitive to slight differences in dispersion (that is, differences between α_{a-Li}, β_{a-Li}, and γ_{a-Li}), a dispersion difference of 0·0003 having been shown to be ample to cause reversal of the relations of the two nearly identical indices. Such substances are bound also to be highly sensitive to change of temperature, which usually provokes minute but influential changes in dispersion as well as producing different amounts of change of refraction along each of the three principal axes of the optical indicatrix.

As the occurrence of these conditions has been general for all the four cases of crossed axial plane dispersion studied in detail by the author, it would appear probable that they afford a general explanation of this interesting phenomenon.

In the next communication the iron group of double selenates will be described.

Series 1. *Ordinary Temperature.*

For Li-light.

For Na-light

For Tl-light.

For F-light

For light of wave-length 466.

For G-light

Series 2. *Temperature of* ?

For Li-light.

For C-light

For Na light

For Tl-light.

For F-light.

For light of wave-length 4

INTERFERENCE FIGURES AFFORDED BY CÆSIUM MAGNESIUM SELENATE IN CONVERGENT
POLARISED LIGHT.

VIII. *Mathematical Contributions to the Theory of Evolution.—IX. On the Principle of Homotyposis and its Relation to Heredity, to the Variability of the Individual, and to that of the Race. Part I.—Homotyposis in the Vegetable Kingdom.*

By KARL PEARSON, *F.R.S.,*

With the assistance of ALICE LEE, *D.Sc.,* ERNEST WARREN, *D.Sc.,* AGNES FRY, CICELY D. FAWCETT, *B.Sc., and others.*

CONTENTS.

I. INTRODUCTORY.

(1.) THE present paper endeavours to deal with a problem upon which I have long been occupied, adopting the widest basis compatible with the time and means at my disposal. In the first place, I have often been impressed with the small reduction in variability which can be produced by selection. The offspring of a single parent while diverging in character, possibly very widely from the average character of the race, will still have a variability in that character only slightly reduced, say at most 10 per cent. below the racial variability. Even if we select the ancestry for an indefinite number of generations, the offspring will have a variability upwards of 89 per cent. of that of the original race.* Now this capacity in the parent for producing variable offspring must be in some manner related to the degree of resemblance in those offspring. We have thus the two fundamental divisions of our subject :
  (i.) What is the ratio of individual to racial variability ?
  (ii.) How is the variability in the individual related to inheritance within the race ?
  I must endeavour to explain my meaning a little more fully and clearly. The individual puts forth a number of like organs, corpuscles in the blood, petals of the flower, leaves of the trees, scales on the wing. These may or may not be divided up into differentiated groups. Special forms of leaves occur in the neighbourhood of the fruit ; florets may be differentiated according to their position on the flower, scales

* "On the Law of Ancestral Heredity," 'Roy. Soc. Proc.,' vol. 62, p. 400.

according to their position on the wing; there may be two or more classes of blood-corpuscles. But if we take organs which are like, and so far as we can judge undifferentiated, there is still a difference between them. The individual in putting forth what I may perhaps term " undifferentiated like organs,"* does so with a certain measure of variability. I am not concerned at present with the source of this variability. It may be due to the individual environment, to the physique of the individual, or to some inherent or bathmic tendency in the individual due to its ancestral history. All I am concerned with at present is, that the undifferentiated like organs of an individual possess a certain variability, and that this variability is somewhat less than that of all like organs in the race. If, however, the variability were entirely due to *external* circumstances attending growth, we should hardly expect to find leaves gathered from different branches and on different sides of a tree more alike to each other than to leaves of other trees of the same race. But this is indeed the case; there is a considerable correlation among the undifferentiated like organs of an individual notwithstanding the small reduction on racial variability. Among the trees and plants considered in this paper we have data enough to enable us to determine whether, say, a hundred leaves placed before us were gathered from a single tree, or collected at random from a number of trees. Here, of course, I mean by to " determine" to state the odds for or against these two assumptions.

Just as we can find by the methods already discussed in earlier memoirs of this series, the degree of correlation between brothers and the variability of an array of brothers due to the same parentage, so we can determine the correlation, *i.e.*, the degree of resemblance between the undifferentiated like organs of the individual and the degree of variability within the individual. This determination is the answer to our first fundamental problem, that of the ratio of individual to racial variability. But turning to the process of reproduction, the offspring depend upon the parental germs, and it would thus seem that the degree of resemblance between offspring

* [I am fully conscious of verbal difficulty in the phrase " undifferentiated like organs," but I believe that the distinction between the differentiated and the undifferentiated is quite clear either from the standpoint of observation or from that of the frequency distribution itself. Differentiation, whether due to function, position on the individual, season of production, &c., is usually connected with one or two well-marked dominating factors; it is statistically discoverable by testing the frequency distribution for heterogeneity. On the other hand, variability in " undifferentiated like organs" is not to be associated with any one or two dominating factors which can be isolated; it is due to that combination of many small causes, inherent and environmental, which leads to what is familiar in both theory and observation as a homogeneous *chance* distribution. A diversity due to differentiation and a variability due to chance are quite distinct things. The one is the result of dominating factors which can be isolated and described; the other of a great number of small factors, varying from organ to organ, and incapable of being defined or specified. Indeed, upon each dominating factor of differentiation is superposed such a chance variability. Of course all things which differ even by chance variation are in a certain sense differentiated. But the term differentiation is throughout this paper reserved for the differences which arise, not from a multiplicity of small causes, but from dominating and usually easily recognizable single influences.—*July*, 1901.]

must depend on the variability of the sperm cells and the ova which may each be fairly considered as "undifferentiated like organs." Here again we are not compelled to assert that much or little is due to environment and little or much is due to inherent ancestral influence. All we assume is that such causes as produce the likeness between leaves of the same tree, or florets on the same flower, produce the likeness between spermatozoa or ova of the same individual, and that on this likeness the ultimate resemblance of offspring from the same parent depends. We have then to investigate how the quantitative resemblance between offspring of the same parents is related to the quantitative resemblance between the undifferentiated like organs in the individual; and then we must test on as wide a range of data as possible this theoretical relationship.

Now the reader will perceive at once that if we can throw back the resemblance of offspring of the same parents upon the resemblance between the undifferentiated like organs of the individual, we shall have largely simplified the whole problem of inheritance. Inheritance will not be a peculiar feature of the reproductive cells. One frog, let us say, differs from another in that it produces blood corpuscles more or less alike and unlike those of another frog. In the simplest forms of reproduction, budding and parthenogenesis, the offspring will not be absolutely alike, for buds and ova are undifferentiated like organs, and such organs have only a limited degree of resemblance. If this view be correct, variability is not a peculiarity of sexual reproduction, it is something peculiar to the production of undifferentiated like organs in the individual, and the problems of heredity must largely turn on how the resemblance between such organs is modified, if modified at all, by the conditions of nurture, growth, and environment generally. Our discussion of the subject will naturally break up into the following divisions :

(a.) On the theoretical relationship between the correlation of offspring and the correlation of undifferentiated like organs in the individual.

(b.) A determination of the correlation of undifferentiated like organs for as wide a range of life as possible.

The present paper deals only with variation and correlation in the vegetable kingdom, but the data for variation and correlation in the animal kingdom are being collected at the same time.

(c.) A comparison of the degree to which the results obtained from (b) satisfies the observed values of fraternal correlation already found when we use the relation determined in (a).

II. THEORETICAL DISCUSSION OF THE RELATION OF FRATERNAL CORRELATION TO
THE CORRELATION OF UNDIFFERENTIATED LIKE ORGANS.

(2.) Let z be the deviation from the mean of the general population of any individual with regard to any character. Let us suppose z to depend upon certain

characters in the spermatozoon and certain characters in the ovum from which the individual has developed. These characters cannot of course be determined, still less measured, but we have no reason to doubt their existence. In the particular spermatozoon from which the individual has developed, let them have deviations x_1, x_2, x_3 . . . from their mean values for all the spermatozoa of the race, and let y_1, y_2, y_3 . . . be the corresponding deviations for the ovum characters. Then

$$z = f(x_1, x_2, x_3 \ldots y_1, y_2, y_3 \ldots)$$

where f is a quite unknown function.

The mean of the z-character will, however, correspond to the mean values of the spermatozoon and ovum characters, and if we suppose the variation of these characters small as compared with their mean value, we assume as usual for such deviations :

$$z = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \ldots + \beta_1 y_1 + \beta_2 y_2 + \beta_3 y_3 + \ldots \quad \text{(i.),}$$

where the α's and β's are independent of the x's and y's, and define the male and female inheritance.

Now let σ be the standard deviation of the character z in the population ; σ_q the standard deviation of x_q, σ'_q of y_q. Let r_{pq} be the correlation of x_p and x_q, r'_{pq} of y_p and y_q. Then we will suppose that there is no selection of particular ova by particular spermatozoa, or that x_p and y_q are not correlated. Then if $n =$ number of individuals in the population :

$$\sigma^2 = \frac{S(z^2)}{n} = \Sigma \alpha_p{}^2 \frac{S(x_p{}^2)}{n} + 2\Sigma (\alpha_p \alpha_q) \frac{S(x_p x_q)}{n}$$
$$+ \Sigma \beta_p{}^2 \frac{S(y_p{}^2)}{n} + 2\Sigma (\beta_p \beta_q) \frac{S(y_p y_q)}{n},$$

where S is the sum for all individuals of any x or y for constant subscript, and Σ is the sum of α and β for every possible subscript. This follows by simple squaring and remembering that $S(x_p y_q) = 0$. We thus reach :

$$\sigma^2 = \Sigma (\alpha_p{}^2 \sigma_p{}^2) + \Sigma (\beta_p{}^2 \sigma'_p{}^2) + 2\Sigma (\alpha_p \alpha_q \sigma_p \sigma_q r_{pq}) + 2\Sigma (\beta_p \beta_q \sigma'_p \sigma'_q r'_{pq}). \quad \text{(ii.).}$$

Now let us consider the correlation of two individuals due to the spermatozoa and ova put forth by the same two individuals. Let z_1 and z_2 be the values of their characters, and x', x'', y', y'' represent the fundamental characters in the two spermatozoa, and two ova on which they depend.

Then we have

$$z_1 = \Sigma (\alpha_p x'_p) + \Sigma (\beta_p y'_p)$$
$$z_2 = \Sigma (\alpha_p x''_p) + \Sigma (\beta_p y''_p).$$

Now let us multiply z_1 by z_2 and sum for every fraternal pair ; then if R be the

fraternal correlation we have, since one set of brethren will have the same standard deviation as the other, if they be taken of the same sex,

$$S(z_1 \times z_2) = n \times \sigma \times \sigma \times R,$$

where n is the total number of pairs of brethren.

Hence

$$n \times \sigma^3 \times R = \Sigma(\alpha_p^2(x'_p x''_p)) + \Sigma(\alpha_p \alpha_q S(x'_p x'' + x'' _p x'_q))$$
$$+ \Sigma(\beta_p^2 S(y'_p y''_p)) + \Sigma(\beta_p \beta_q S(y'_p y''_q + y''_p y'_q))$$

with the same notation as before and the same assumption as to no selective process between the female and male reproductive cells.

Now x'_p, x''_p are undifferentiated like organs put forth by an individual. If ρ_{pp} be the correlation of such :

$$S(x'_p x''_p) = n\sigma_p^2 \rho_{pp},$$

Similarly $$S(y'_p y''_p) = n\sigma_p'^2 \rho'_{pp}.$$

In the next place we should have *all* pairs of brothers :

$$S(x'_p x''_q) = S(x''_p x'_q) = \sigma_p \sigma_q \rho_{pq},$$
$$S(y'_p y''_q) = S(y''_p y'_q) = \sigma'_p \sigma'_q \rho'_{pq}.$$

We thus deduce

$$\sigma^3 \times R = \Sigma(\alpha_p^2 \sigma_p^2 \rho_{pp}) + \Sigma(\beta_p^2 \sigma'^2_p \rho'_{pp})$$
$$+ 2\Sigma(\alpha_p \alpha_q \sigma_p \sigma_q \rho_{pq}) + 2\Sigma(\beta_p \beta_q \sigma'_p \sigma'_q \rho'_{pq}). \quad . \quad . \quad . \quad . \quad \text{(iii.)}.$$

Now if the degree of resemblance between undifferentiated like organs vary largely from organ to organ, we could proceed no further with this investigation. We should have shown that there was a relation, namely (iii.) between the R and the ρ's, but as the characters in the spermatozoa and ova are unknown and unmeasurable, we could not proceed further. On the other hand, if the correlation between undifferentiated like organs is approximately constant, then we may replace correlation coefficients like ρ_{pp} and ρ'_{pq} by their *mean* value ρ, say.

We have next to consider such correlations as ρ_{pq} or ρ'_{pq}. These are the two correlations between two different characters in two spermatozoa or ova from the same individual. Such correlations must vanish (i.) if there be no organic correlation between characters of the same spermatozoon or ovum, (ii.) if there be no correlation between the reproductive cells put forth by the same individual. In other words, ρ_{pq} must contain ρ and r_{pq} as factors, and since if these two correlations are perfect ρ_{pq} must be perfect, it is a reasonable hypothesis to assume

$$\rho_{pq} = \rho \times r_{pq} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iv.)}.$$

This, it will be seen, is identical in form with the result I have supposed to hold

for cross-heredity,* and which appears to be approximately true for that case. We shall consider later statistics bearing on this result.

Assuming accordingly that (iv.) holds, we find from (iii.)

$$\sigma^2 \times R = \rho\{\Sigma\,(\alpha_p{}^2\sigma_p{}^2) + \Sigma\,(\beta_p{}^2\sigma'_p{}^2) + 2\Sigma\,(\alpha_p\alpha_q\sigma_p\sigma_q r_{pq}) + 2\Sigma\,(\beta_p\beta_q\sigma'_p\sigma'_q r'_{pq})\}$$
$$= \rho \times \sigma^2 \text{ by (ii.).}$$

Hence we conclude that

$$R = \rho\ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ \text{(v.).}$$

Or: *The correlation between brothers will be equal to the mean correlation between the undifferentiated like organs put forth by an individual.*

Now, if this result be true, it is very remarkable and very fundamental. We should hardly expect it to be absolutely true, for it is very unlikely that the coefficient of correlation between undifferentiated like organs is the same whatever the organs may be. This equality may rather be spoken of as belonging to an ideal theoretical vital state approaching the actual state, perhaps, as BOYLE'S Law or the perfect fluid approach phenomena observed in physical nature. What we should expect would be a general approximation between the values of R and ρ, and a tendency to equality when large series are averaged. This is the point which we shall investigate in the sixth section of this paper, after placing before the reader in the fourth section a fairly wide† range of actual statistics.

It will be seen at once that if (v.) represents an approximate or average truth of living nature, then the problem of inheritance is to a large extent the same as the problem of variability in the individual. We have not answered, of course, even in part, the problem of why two brothers resemble each other, but we see that it is part of a much more general problem having nothing to do with sexual reproduction. It is one with the problem of the likeness in leaves of the same tree, or the likeness in scales on the same spot of a moth's wing. It brings the problem of heredity into closer touch with the problem of variability. When we ascertain the sources of variation in the individual, then we shall have light on the problem of fraternal resemblance.

III. ON THE VARIABILITY AND CORRELATION OF UNDIFFERENTIATED LIKE ORGANS IN THE INDIVIDUAL.

(3.) I must frankly admit that I have collected my material from the standpoint of the mathematical statistician and not of the trained field naturalist. I have sought things which were easy to count and measure, and endeavoured to avoid "differentia-

---

* "The Law of Ancestral Heredity," 'Roy. Soc. Proc.,' vol. 62, p. 411.

† Not wide when we regard the natural range of living forms, but from the standpoint of the labour that has been spent on the collection.

tion" only by the roughest tests and by largely untrained powers of observation. I have examined my material to see if it gave obvious signs of heterogeneity in the tabulated bulk. I have examined my individuals and their organs for obvious outward signs of differentiation. But I have not studied the morphological evolution of the organs considered, or questioned whether the parts counted were all due to the same source. Hence it is quite possible that the botanist may reject at once some of my series. What I have endeavoured to do is to take as wide a range of as different organs as possible in different types of life and trust to the bulk of my statistics to give me a substantially accurate average value of ρ to compare with the values of R we have determined on other occasions. At the same time the material here presented does not by any means exhibit all the trials made; we often enough at the first attempt did not get a suitable character to measure or count, or again the individuals collected were occasionally insufficient. Nothing, however, has been omitted which was unfavourable to the conclusions ultimately drawn. Thus the *Nigella Hispanica* statistics have been retained, although there is not the slightest doubt of a very considerable differentiation in the flowers growing at different parts of the plant.

When such a differentiation takes place the result will generally be a great reduction in the correlation; for "like" organs on the same individual, say A and *a* if differentiated, will really be less closely related to each other than to B and *b* respectively, the corresponding organs in a second individual. On the other hand a heterogeneity of material, say a mixture of two different local races, will tend as a rule to raise the correlation,* for it generally amounts to compounding two very like correlation surfaces with the mean of one approximately shifted along the regression line of the second. If we bear these two opposing influences in mind, differentiation tending to reduce, heterogeneity to increase the actual degree of correlation of undifferentiated like organs, it will not seem incompatible with an actual approximate constancy of such correlation to find a fairly wide range of values in our statistics. We set them forth as the first rough attempt to appreciate the resemblance of like parts within the individual. To the specialist in the future must be left the work of selecting, with special knowledge, truly homogeneous material and absolutely undifferentiated characters, and thus obtaining the required correlation to a much higher degree of accuracy.

(4.) It remains to explain the process by which the correlation was deduced. Let us take as an illustration beech-leaves. One hundred trees fairly of the same age and belonging to the same district, were selected, and twenty-six leaves specified by the letters of the alphabet were gathered from each of these. The leaves were gathered so far as possible all round the outside of the tree, roughly about the same height from the ground, and scattered over different parts of the individual boughs. Thus each tree was supposed to be individualised by twenty-six leaves. The veins on these leaves were then counted, and varied for beech-leaves in general between ten and twenty-two.

* See 'Phil. Trans.,' A, vol. 192, p. 277.

All the possible pairs were now taken, *i.e.*, $\frac{1}{2}(26 \times 25) = 325$ in number, and entered on a correlation table in the usual manner, the two variables being the number of veins in the first leaf and the number of veins in the second leaf. But as either member of the pair might be a "first" leaf, the table so formed was rendered symmetrical by starting with either leaf in the pair as first or second. Thus a single tree led to 650 entries in the correlation table, or with 100 trees there were 65,000 entries. This large number must not lead the reader to overweight the importance of the constants calculated upon it. There were only 100 trees leading to 32,500 pairs of leaves, each pair coming from the same tree. I should have much preferred a thousand trees, but the great labour of collecting, counting, and calculating precluded any such number. In many cases also it would have been practically impossible to have obtained 100 individuals growing under fairly like environment. For example, I only succeeded in approximating to 100 Spanish chestnuts from one district and near one age. When I took a second hundred, gathered from several districts, partly old trees and partly others of a pollard growth of fifteen to twenty years, I found the heterogeneity at once increased the correlation (see below, p. 301) In many cases, of course, it was impossible to obtain twenty-six undifferentiated like organs from the same individual. In these cases the pairs were formed in the same manner, but in some series the total number due to each individual varied very considerably, and accordingly the work of verifying the tables was much increased. When but few pairs could be obtained from each individual, we have sometimes increased the number of individuals dealt with up to a couple of hundred. But the labour of dealing even with a hundred individuals is often—for example, in the case of mushrooms and onions—very serious. Had it not been for the generous help of a number not only of willing but of competent collectors and calculators, the material here dealt with would have taken not eighteen months but years of my own unaided efforts.

The calculation of the means, standard deviations, probable errors, and correlations of each table was carried out in the manner sufficiently discussed in earlier papers of this series.* The symmetry of the tables leads to slight simplifications in calculating the product moment which will readily suggest themselves to the reader, and of course only a single mean and standard deviation is required for each table. Tests for the accuracy of the last two constants are at once provided in the case in which the same number of organs are taken from each individual, for their values must be identical with those obtained for the whole series of organs entered only once and not for each possible pair.

(5.) The quantitative measurement of the degree of resemblance between undifferentiated-like organs being, so far as I am aware, a quite novel branch of investigation, I venture, with some hesitation, to introduce certain terms to describe oft-

---

* See especially Memoir III., 'Phil. Trans.,' A, vol. 187, pp. 253–318, and Memoir IV., 'Phil. Trans.,' A, vol. 191, pp. 229–311.

recurring ideas. I shall call undifferentiated like organs *homotypes*, indicating that they are types produced by the same mould or individual. Thus two leaves of the same tree, or two blood-corpuscles from the same frog are *homotypes*; their resemblance will be homotypic, and the character by which their resemblance is quantitatively measured will be the *homotypic* character. I shall speak of *homotypic* correlation and distinguish it from *organic* correlation—although, of course, the former is in a certain sense organic. By organic correlation I refer to the correlation between two different characters in the same organ; by homotypic correlation to the correlation between the same or different characters in a pair of homotypes. If the same character, then the correlation is *direct homotypic correlation*; if different characters, then the correlation is *cross homotypic correlation*. Lastly, the principle that homotypes are correlated, *i.e.*, that variation within the individual is less than that of the race, or that undifferentiated like organs have a certain degree of resemblance, I shall speak of as *homotyposis*. Thus homotyposis denotes not only likeness of the homotypes, but that this likeness has probably definite quantitative limits. If my view be correct. heredity is only a special case of homotyposis, any multiplication of cells is homotypical, and denotes a given degree of variation and a given degree of likeness. This does not " explain " heredity, but shows it merely as a phase of a much wider natural process.

IV. ACTUAL DATA.

SECTION I.—*Leaves of Trees.*

(6.) The leaves of trees provide material for fairly easy computation without measurement. Thus we may count the veins on the leaf or the leaflets on the compound leaf, and ascertain the degree of resemblance between leaves of the same tree. The variation is, however, in some cases very considerable, and the labour of forming the tables involving thousands of entries very great. Still the leaf-series are some of my best, and considered as a whole, perhaps the most satisfactory.

(7.) A. *Ash* (Fraxinus excelsior). *Number of Pinnæ on Leaf.*—My first series (i.) consists of 26 leaves taken from each of 109 trees by Dr. ALICE LEE or myself. The trees were nearly all large old trees, growing on the commons of Great and Little Hampden, Buckinghamshire. The leaves were taken as far as possible from a variety of branches all round the tree, and from different points on these branches. The labour of tabling and deducing the constants is due to Dr. LEE. The number of pairs = 70,850.

My second series (ii.) consist of 26 leaves taken from each of 120 trees by Miss C D. FAWCETT, B.Sc. These trees, many of which were young trees growing in the hedgerows, were taken in the neighbourhood of Lyme Regis in Dorsetshire. The same rules as to gathering were observed. Dr. LEE again undertook the labour of tabling and the deduction of the constants. The number of pairs = 78,000.

My third series consists of 26 leaves from each of 100 trees in Monmouthshire, young trees being included, but the same rules as to gathering being observed. The whole of the labour of gathering, counting and deducing the constants is due to Miss MILDRED E. BARWELL, of Girton College.*

The general numerical results are given in the table below :—

Resemblance of Ash Leaves from same Tree.

Series.	Number.			Mean No. of leaflets	S. D. of leaflets.	Correlation.
	Trees.	Leaves.	Pairs.			
Buckinghamshire	109	2834	70850	10·1295 ± ·0214	1·6891 ± ·0151	·3743[± 0109]
Dorsetshire . .	120	3120	78000	9·7260 ± ·0239	1·9759 ± ·0169	·3964[± ·0102]
Monmouthshire .	100	2600	65000	9·8766 ± ·0265	2·0058 ± ·0188	·4047[± ·0111]
Mean of series .	—	—	—	9·9107	1·8903	·3918

Thus a total of 329 trees, giving 8,554 compound leaves counted and yielding 213,850 pairs on the symmetrical tables, were dealt with in all.

It will be seen at once that the mean number of *pinnæ* to the leaf, and their variability differs considerably with the locality, the age and the environment of the tree, but the correlation of leaves from the same tree is fairly alike for the three series, and almost identical with the theoretical value ·4000 given by the Law of Ancestral Heredity for the degree of resemblance between pairs of brothers.

The following is the actual distribution of the frequency of the leaflets :—

Number of Pinnæ on Leaves.

Series.	3.	4.	5.	6.	7.	8.	9.	10.	11.	12.	13	14.	15.	16.	Total.
Buckingh'shire	3	0	16	21	201	67	879	156	1140	68	257	9	17	0	2834
Dorsetshire. .	4	0	84	30	396	115	959	228	911	72	280	11	29	1	3120
Monmouthshire	1	5	42	24	279	55	836	143	896	83	216	6	13	1	2600
Total . .	8	5	142	75	876	237	2674	527	2947	223	753	26	59	2	8554

* This was a remarkably satisfactory piece of work, carried through from the collecting of the raw material down to the calculation of the constants, the statistical methods having to be mastered and applied during the course of the investigation.

† In determining the probable errors of the correlation coefficients, there is some question as to whether we should use the number of individuals, which seems too small, the number of pairs, which seems too large, or the number of individual organs dealt with. I have used the latter, but enclose the probable errors in brackets to mark the doubt.

It will be remarked at once that the leaves with an even number of leaflets are relatively infrequent; there is usually a single leaflet at the end of the compound leaf. I was able at Hampden, however, to obtain leaves illustrating almost the whole evolution of this single leaflet. Sometimes this leaflet had the slightest division at its point; this division was found in most stages of double tongued leaflet down to a complete double leaf, which would be reckoned as two *pinnæ*. In the same way the centre leaflet was found slightly trisected, three tongued and finally a triple leaflet scarcely distinguishable from the usual single leaf at the end and the two nearest side leaflets. Very occasionally the double leaflet seemed to give one side leaflet and the final leaflet. Of course, all these anomalies were rare and had to be sought for, but they would form in themselves a suggestive study.

The Dorsetshire series gives 9 *pinnæ* instead of 11 as the mode, and 7 are more frequent than 13. This latter result is also true for the Monmouthshire series, which again has nothing like the same preponderance on its modal 11 as the Buckinghamshire series. I take it that the greater variability of both Dorsetshire and Monmouthshire series, together with their greater relative proportion of compound leaves with 9 and 7 *pinnæ*, is due to those series containing a much larger proportion than the Buckinghamshire series of small trees.

For ashes in general, I think, we may safely take 10 as the average (but *not the modal*) number of *pinnæ*, the standard deviation is 1·9 *pinnæ*, and the leaf resemblance is measured by a correlation of ·4.

If we compare racial and individual variabilities we have :

Series.	S. D.	S. D. of array.	Coefficient of variation.	Percentage variability.
Buckinghamshire . .	1·6891	1 5663	15·46	92 73
Dorsetshire . . .	1·9759	1·8140	18·65	91·81
Monmouthshire . .	2·0058	1 8342	18 57	91 44
Mean . . .	1·8903	1·7382	17·56	91·99

Thus 92 per cent. of the total racial variability of this character of the ash is to be found in the individual tree, and here, as in all our other series, it is impossible to assert that variation is a result of sexual reproduction ; it is clearly an essential factor of the individual growth. I conclude with the actual tables of data for the three series.

TABLE I.—(i.) Buckinghamshire Ashes.

Number of Pinnæ on First Leaf

	3.	4.	5.	6.	7.	8.	9.	10.	11.	12.	13.	14.	15.	16.	Totals.
3	6	0	6	6	15	0	6	0	33	3					75
4	0	0	0	0	0	0	0	0	0	0					0
5	6	0	18	16	74	14	155	18	94	3	2				400
6	6	0	16	24	96	25	213	29	113	3	0				535
7	15	0	74	96	716	211	2168	235	1385	34	87	0	4		5025
8	0	0	14	25	211	50	732	82	485	17	56	0	3		1675
9	6	0	155	213	2168	732	8970	1305	7361	278	756	11	20		21975
10	0	0	18	29	235	83	1305	278	1648	69	213	6	17		3900
11	33	0	94	113	1385	485	7361	1648	13558	838	2825	57	103		28500
12	3	0	3	3	34	17	278	69	838	90	326	19	20		1700
13			2	0	87	56	756	213	2825	326	1896	88	176		6425
14					0	0	11	6	57	19	88	16	28		225
15					4	3	20	17	103	20	176	28	54		425
16															0
Totals	75	0	400	535	5025	1675	21975	3900	28500	1700	6425	225	425	0	70850

TABLE II.—(ii.) Dorsetshire Ashes.

Number of Pairs on First Leaf.

	Totals.	16.	15.	14.	13.	12.	11.	10.	9.	8.	7.	6.	5.	4.	3.
3	100	—	—	—	4	5	24	2	37	1	23	0	2	0	2
4	0	—	—	—	0	0	0	0	0	0	0	0	0	0	0
5	2100	—	2	3	28	6	324	108	786	129	506	36	170	0	2
6	750	—	4	0	31	11	186	62	245	39	114	22	36	0	0
7	9900	—	12	6	210	83	1816	641	3736	465	2298	114	506	0	23
8	2875	—	10	2	88	31	536	338	782	454	465	39	129	0	1
9	23975	—	23	13	793	308	6172	1662	9428	789	3736	245	786	0	37
10	5700	1	40	9	360	103	1554	830	1662	538	641	62	108	0	2
11	22775	6	192	62	2421	676	8806	1554	6172	536	1816	186	324	0	24
12	1800	2	28	16	417	114	676	103	308	31	83	11	6	0	5
13	7000	14	238	130	2276	417	2421	360	793	88	210	31	28	—	4
14	275	0	34	34	130	16	63	9	13	2	6	0	3	—	—
15	725	2	140	0	238	28	192	40	23	10	12	4	2	—	—
16	25	0	2	—	14	2	6	1	—	—	—	—	—	—	—
Totals...	78000	25	725	275	7000	1800	22775	5700	23975	2875	9900	750	2100	0	100

Number of Pairs on Second Leaf.

TABLE III.—(iii.) Monmouthshire Ashes.

Number of Pinnæ on First Leaf.

Number of Pinnæ on Second Leaf	3	4	5	6	7	8	9	10	11	12	13	14	15	16	Totals.
3	—	—	4	2	9	1	8	0	1	—	—	—	—	—	25
4	—	2	0	5	19	3	66	8	21	1	—	—	—	—	125
5	4	0	34	18	208	36	366	47	269	24	38	6	—	—	1050
6	2	5	18	10	106	21	251	32	138	2	15	0	—	—	600
7	9	19	208	106	1450	280	2594	283	1593	132	277	9	14	1	6975
8	1	3	36	21	280	60	464	86	365	19	38	1	1	0	1375
9	8	66	366	251	2594	464	8052	1227	6468	430	950	24	28	2	20900
10	0	8	47	32	283	86	1227	222	1344	94	216	2	13	1	3575
11	1	21	269	138	1593	365	6468	1344	9299	841	1924	46	93	5	22400
12	—	1	24	2	132	19	430	94	841	138	366	12	16	0	2075
13	—	—	38	15	277	38	950	216	1924	366	1388	45	131	12	5400
14	—	—	6	0	9	1	24	2	46	12	45	2	3	0	150
15	—	—	—	—	14	1	28	13	93	16	131	3	22	4	325
16	—	—	—	—	1	0	2	1	5	0	12	0	4	0	25
Totals...	25	125	1050	600	6975	1375	20900	3575	22400	2075	5400	150	325	25	65000

(8.) B. *Spanish Chestnut* (Castanea vulgaris). *Veins in the Leaf.*—A character which at first sight appears easy to determine is the number of veins running from the main rib to the edge of a leaf. The leaves of many trees, however, will on examination be found quite unsuitable, the veining being far too complex, too much indeed of a network. Among fairly practicable leaves are those of the beech and the Spanish chestnut. Here two fundamental difficulties occur : (i.) near the tip of the leaf the number of veins becomes rather difficult to register. In the case of the beech by holding the leaf in various lights or even using a lens, it was possible to be fairly sure of the separate veins. In the case of the chestnut, the total number of veins is so great that the addition or omission of a single vein is hardly likely on the average to lead to any serious error. (ii.) The occasional branching or bifurcation of the vein formed a difficulty, especially if it occurred near the edge of the leaf. If the bifurcation occurred close up to the main rib of the leaf, the vein system was counted as two ; if close to the edge of the leaf as one ; intermediate bifurcations had to be left to the judgment of the recorder, having regard to the general appearance of the leaf and its system of subsidiary veining. The doubtful cases were not very many, and occurred more frequently with the chestnut than with the beech leaves. In neither case was the main rib or vein included in the total number for the leaf, the record was of the side veins flowing from this. The numbers right and left in the beech leaf were found to be far more nearly equal than in the case of the chestnut, which thus possessed considerably greater asymmetry. In these cases as in those of the ash, the leaves were gathered at random round the tree, and at random on the branches, care being, however, taken in the case of the chestnut to avoid leaves in the neighbourhood of the fruit.*

<center>Series (i.). *Hampden Spanish Chestnuts.*</center>

I had hoped to get records for 26 leaves from each of 100 trees from the splendid chestnut avenue at Hampden House, but it provided records for some 60 trees only. These I owe to Miss LINA ECKENSTEIN. I was able to add by diligent search about 25 additional trees from the same neighbourhood. All these were large forest trees of splendid growth. In order to make the number up to 100, some 15 additional trees were added, these being all large old trees from other neighbourhoods. I look upon this series as a fairly homogeneous one, representing the degree of resemblance between leaves on fully developed forest trees. The counting of the veins on this series was undertaken by Miss C. D. FAWCETT, B.Sc., and the tabulating and numerical reduction by Dr. LEE. Both these processes involved an immense amount of labour, in fact days and days of careful and fatiguing work.

---

* The leaves were all gathered in the autumn, towards the fall.

Table IV.—Hampton Chestnut Trees.

Number of Veins on First Leaf.

[To face p. 300.]

Number of Veins on Second Leaf.

TABLE V.—Mixed Red Spanish Chestnut Leaves.

Section (ii.)

Number of Veins on First Leaf.

(A large correlation table cross-tabulating "Number of Veins on First Leaf" against "Number of Veins on Second Leaf," with row and column totals. The dense numeric cell values are not legibly reproducible.)

Series (ii.). *Mixed Spanish Chestnuts.*

My second series consists of 26 leaves from each of 104 trees. These were gathered in a variety of neighbourhoods. Many of these were young trees. In the case of nearly 50 sets, the trees were pollards from old stumps with about 15 years' growth ; some of the others were from fine old trees, and as a result we have many districts and many ages combined. I have to thank Miss C. D. FAWCETT for sets from the neighbourhood of Hampstead, Miss GRACE LEE for sets from that of St. Albans, Miss CHARLES for sets from the neighbourhood of Dorking, while about half the series was gathered by Mr. ROBERT J. PARKER from his chestnut woods at Haslemere. The counting of the veins was undertaken by Miss M. NOTCUTT, while the tabulation and numerical reduction are the work of Dr. W. R. MACDONELL. The series being mixed, the variability is higher than that of the first series and thus the labour spent on it was even greater.

The following table gives the general results :—

Resemblance of Spanish Chestnut Leaves from same Tree.

Series.	Number.			Mean number of veins.	S. D. of veins.	Correlation.
	Trees.	Leaves.	Pairs.			
Hampden .	100	2600	65,000	41·0735 ± ·0775	5·8780 ± ·0550	·4655[± ·0104]
Mixed . .	104	2704	67,600	41·2115 ± ·0840	6·4780 ± ·0594	·5913[± ·0084]
Mean . .	—	—	—	41·1440	6·1780	·5284

We note at once that the mean number of veins for the forest trees and for the mixed group is sensibly the same. On the other hand, the variability and the correlation differ very considerably. The latter is sensibly higher than the ·4 obtained for the leaflets of the ash. I should be inclined to consider that the heterogeneity of the mixed series is at the bottom of the difference between the two series, but there cannot, I think, be any doubt of the greater resemblance of chestnut leaves for this character than of ash leaves in the number of *pinnæ*.

The actual distribution of veins in the 5304 leaves examined was as follows :—

No. of veins.	19.	20.	21.	22.	23.	24.	25.	26.	27.	28.	29.	30.	31.	32.	33.
1st series	—	1	2	0	0	5	4	6	14	14	19	42	43	49	60
2nd series	1	1	1	2	4	6	9	10	23	19	40	41	47	54	68
Total	1	2	3	2	4	11	13	16	37	33	59	83	90	103	128

No. of veins.	34.	35.	36.	37.	38.	39.	40.	41.	42.	43.	44.	45.	46.	47.	48.
1st series.	89	102	120	125	149	165	198	155	179	153	142	149	141	111	107
2nd series	88	94	116	111	147	150	176	160	162	169	175	136	137	114	97
Total	177	196	236	236	296	315	374	315	341	322	317	285	278	225	204

No. of veins.	49.	50.	51.	52.	53.	54.	55.	56.	57.	58.	59.	60.	61.	62.	Total.
1st series.	59	62	39	48	15	19	4	4	3	0	3	—	—	—	2600
2nd series	89	68	38	48	32	25	14	12	9	9	1	0	0	1	2704
Total	148	130	77	96	47	44	18	16	12	9	4	0	0	1	5304

The total gives a fairly smooth series with the mode at 40, and a secondary mode at 42. In the forest trees we have the same result as in the total, but in the mixed group the secondary mode is at 44, and almost equally important with the primary. The influence of symmetry makes itself felt in the preponderance of the even numbers, although not in such a marked degree as in the case of the ash, where the counting of the final leaflet gave, of course, preponderance to the odd numbers.

For the Spanish chestnut we may accordingly take 41 veins as the average, 40 as the modal number to the leaf; 6·2 as the S.D. of the veins; and ·5 (slightly weighting the homogeneous series) as the degree of resemblance between the leaves of the same tree.

It remains to consider the comparative individual and racial variabilities :—

	S. D.	S. D. of array.	Coefficient of variation.	Percentage variability.
Hampden	5·8780	5·2024	14·31	88·51
Mixed .	6·4780	5·2242	15·72	80·65
	6·1780	5·2133	15·01	84·58

We see that the variability of the chestnut leaf as measured by the coefficient of variation is somewhat less than that of the ash. The individual as compared with the racial variability is also somewhat less than in the case of the ash, but the individual exhibits still some 85 per cent. of the racial variability. Tables IV. and V. give the data for the chestnut series.

(9.) C. *Beech* (Fagus sylvatica). *Number of Veins on the Leaf.*—I have here only one series of 26 leaves from each of 100 trees. These were gathered and counted by Mrs. KARL PEARSON and myself in the neighbourhood of Great Hampden. The trees were in part fine isolated specimens growing on the commons, but as there were not sufficient of these we were compelled to gather from the somewhat smaller trees abounding in the beechwoods of this part of Buckinghamshire. The leaves were gathered about 4 to 5 feet from the ground on the outside of the tree.* The tabulation of the results and the calculation of the constants are again the work of Dr. LEE.

Hampden Beech Leaves.

Number.			Mean number of veins.	S.D. of veins.	Correlation of pairs.
Trees.	Leaves.	Pairs.			
100	2600	65,000	16·1062 ± ·0230	1·7351 ± ·0162	·5699 [± ·0087]

Distribution of Number of Veins.

Number of veins ..	10	11	12	13	14	15	16	17	18	19	20	21	22	Total.
Frequency	1	7	34	110	318	479	595	516	307	181	36	15	1	2600

Individual and Racial Variation.

S.D.	S.D. of array.	Coefficient of variation.	Percentage variability.
1·7351	1·4258	10·77	82·17

Accordingly, we see that the modal value of the beech leaf veins is 16, which is very near the mean, 16·11. The standard deviation about 1·7, and the degree of resemblance among its leaves ·57. Measured by its coefficient of variation, it is the least variable of the trees so far dealt with, and the leaves of the same tree more closely resemble each other. Notwithstanding its relatively small variability, the individual on an average exhibits about 82 per cent. of the racial variability.

The table of reduced data is given below.

* I did not observe among the leaves gathered anything of the differentiation indicated by Professor MACLEOD between "licht- en schadernbladen." The material, so far as this character (number of veins) goes, seemed singularly homo eneous and of small variability.

TABLE VI.—Hampden Beech Trees.

Number of Veins on First Leaf.

	10.	11.	12.	13.	14.	15.	16.	17.	18.	19.	20.	21.	22.	Totals.
10	0	2	7	7	6	3	—	—	—	—	—	—	—	25
11	2	8	35	45	59	23	3	—	—	—	—	—	—	175
12	7	35	92	145	261	156	92	47	15	—	—	—	—	850
13	7	45	145	394	836	689	387	199	46	1	1	—	—	2750
14	6	59	261	836	2124	2021	1616	723	243	54	4	3	—	7950
15	3	23	156	689	2021	3256	3122	1856	605	221	20	3	—	11975
16	—	3	92	387	1616	3122	4576	3205	1380	437	51	6	—	14875
17	—	—	47	199	723	1856	3205	3914	1928	860	120	46	2	12900
18	—	—	15	46	243	605	1380	1928	1882	1278	213	80	5	7675
19	—	—	—	1	54	221	437	860	1278	1248	282	137	7	4525
20	—	—	—	1	4	20	51	120	213	282	138	64	7	900
21	—	—	—	—·	3	3	6	46	80	137	64	32	4	375
22	—	—	—	—	—	—	—	2	5	7	7	4	0	25
Totals	25	175	850	2750	7950	11975	14875	12900	7675	4525	900	375	25	65000

Number of Veins on Second Leaf.

(10.) D. *Holly* (Ilex Aquifolium). *Number of Prickles on the Leaf.*—This character seemed an easy one so far as mere counting goes, although the leaves were troublesome to gather. The leaves were gathered, so far as possible, all round the tree or bush, about the height at which cattle would browse, and from the outside of the bush. Internal and high branches (often differentiated by being almost prickleless) were to be avoided. Flowering trees, and, above all, leaves near the flowering buds or fruit were to be rejected. These conditions could only be approximately followed in practice when a large number of trees had to be dealt with from one neighbourhood, and these series, I fancy, allow more to the personal equation of the gatherer than is wholly satisfactory.

Series (i.). *Somersetshire Hollies.*

I owe this series of 26 leaves from each of 100 trees to Miss AGNES FRY. Of these trees, 56 were hedge-row hollies, and 44 wood trees or bushes growing in moist places.* The counting of the prickles was in some part undertaken by Mr. MACLEOD,

---

* Miss FRY writes to me: "The texture of the leaves seems rather different,— the wood hollies are much less stout and harsh and more often have few prickles. In fact, I rather wonder if *shade* does not affect the number. In the few cases where hedge-row hollies have very few prickles the leaves come from the inside of the bush. I have carefully avoided the neighbourhood of berries,—and what is more difficult of flower buds. But the holly has two flowers, one with abortive pistil and one with abortive stamens. The latter I suppose would drop away and leave no trace of their presence. But, on the whole, I have avoided flower and berry-bearing bushes—it is easy to find bushes without either. And, so far as my observation goes, I am not convinced that the neighbourhood of flowers or the top of the tree causes modification . . . . It is very difficult to keep to a standard of counting for the tip, but I have tried to do so. Doubtless the hollies too are of rather different ages, but I have avoided patriarchal trees, though doubtful if this makes much difference in this case." It will be seen that in this point, as in others, much had to be left to the judgment of a careful observer.

but for more than three-quarters of the trees by Miss Fry herself. The labour of tabulation and the deduction of the constants was undertaken by Dr. Lee.

<p style="text-align:center">Series (ii.). <i>Dorsetshire Hollies.</i></p>

This series of 26 leaves from each of 100 trees was gathered and counted by Miss C. D. Fawcett, B.Sc. It differs in several important respects from Miss Fry's series. As it came from the neighbourhood of the sea, by Lyme Regis, we might expect a considerable difference of environment; the trees were chiefly hedgerow trees. They show, however, a *lower* mean number of prickles than the Somersetshire series, which embraced a large element of wood trees. I think the series is less homogeneous than the Somersetshire series. In particular, Tree 91 contributes the almost abnormal series of 24 leaves with 1 and 2 leaves with 2 prickles. The lump at the end of the frequency distribution is practically due to this one tree. I do not feel able to reject it entirely, but I give the numerical constants as determined with and without it. I take it that of the two observers the personal equation of one tended rather to emphasise the need for a mathematically "random selection" of trees, and the personal equation of the other to emphasise the importance of a botanically "homogeneous group" of trees. The result has been the increase of correlation and variability in the first, and their decrease in the second case. Probably their mean is not far from the truth. It is noteworthy that the ash trees from Dorsetshire and Monmouthshire, although exhibiting a like substantial difference of environment, were far more alike.

The tabulation and reduction of this series is also due to Dr. Lee. I now summarise the results as before :—

<p style="text-align:center">Resemblance of Holly Leaves from the same Tree.</p>

Series.	Number.			Mean number of prickles.	S. D. of prickles.	Correlation.
	Trees.	Leaves.	Pairs.			
Somersetshire (i.)	100	2600	65,000	15·2935 ± ·0373	2·8655 ± ·0268	·3648 [± ·0115]
Dorsetshire (ii.)a.	100	2600	65,000	14·1373 ± ·0497	3·7595 ± ·0352	·6423 [± ·0078]
„ (ii.)b.	99	2574	64,350	14·3622 · ·0474	3·5661 ± ·0335	·5985 [± ·0085]
Mean (i.) and (ii.)a	—	—	—	14·3654	3·3125	·5035
Mean (i.) and (ii.)b	—	—	—	14·4278	3·2158	·4816

<p style="text-align:center">(ii.)b is the Dorsetshire series, excluding tree (91).</p>

In round numbers accordingly the number of prickles on the holly leaf is 14·4, the standard deviation 3·3 prickles, and the degree of resemblance ·5. The holly thus falls between our results for the ash and the chestnut, one series approaching our Buckinghamshire ashes, the other the group of mixed Spanish chestnuts, being somewhat in excess of the Buckinghamshire beeches.

The actual frequency of prickles is given by :—

Series.	0	1.			5.	6.		9.		11.	12.
Somersetshire (i.) . . .	1	2	2	3	6	6	10	8	10	24	43	74	135	276
Dorsetshire (ii.)a . . .	0	40	21	14	18	27	31	45	62	84	86	152	219	336
„ (ii.)b . . .	0	16	19	14	18	27	31	45	62	84	86	152	219	336
Total (i.) and (ii.)a .	1	42	23	17	24	33	41	53	72	108	129	226	354	612
Total (i.) and (ii.)b .				17	24	33	41	53	72	108	129	226	354	612

	15.	16.	17.	18.	19.	20.	21.	22.	23.	24.	25.	26.	Total.
Somersetshire (i.) . . .	335	417	407	328	235					3	1	1	2600
Dorsetshire (ii.)a . . .	350	337	311	217	128					0	0	0	2600
„ (ii.)b . . .	350	337	311	217	128					0	0	0	2574
Total (i.) and (ii.)a .	685	754	718							3	1	1	5200
Total (i.) and (ii.)b .	685	754	718									1	5174

The series is thus seen to be fairly uniform, the mode corresponding in the total result as well as the Somersetshire series to 15 prickles, while that of the Dorsetshire series is at 14, an even number of prickles. The holly leaf has clearly not the marked symmetry of ash or chestnut leaves. Thus, 16 prickles in the Somersetshire series are almost as frequent as 15, or since there is almost always a prickle at the tip, asymmetrical leaves must be very nearly as frequent as symmetrical.

Lastly, turning to the comparison of racial and individual variabilities, we have the table :—

Series.	S. D.	S. D. of array.	Coefficient of variation.	Percentage variability.
Somersetshire (i.) . .	2·8655	2 6680	18·74	93·12
Dorsetshire (ii.)a . .	3 7595	2·8815	27·98	76 65
„ (ii.)b . .	3·5661	2 8282	26·29	80·11
Mean (i) and (ii.)a . .	3·3125	2 7747	23·36	84·88
Mean (i) and (ii.)b . .	3·2158	2 7482	22 52	86·61

Here again we see how great is the variability within the individual. Looking upon Tree (91) as so largely biasing the complete Dorsetshire series that we must exclude it, we see that the individual possesses 80 to 90 per cent. of the variability of the race. The variability of the offspring of the individual will, of course, be larger still, even if we had merely uniparental reproduction. Thus we have still further evidence, if more were necessary, that variability is not a factor dependent upon sexual reproduction. For this character of the prickles it will be seen that the holly leaf is more variable than those of the ash, chestnut, or beech. The following are the fundamental tables of data :—

TABLE VII.—Somersetshire Hollies.

Number of Prickles on First Leaf

Number of Prickles on First Leaf ↓ / Number of Prickles on Second Leaf →	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	Totals
Totals	25	50	50	75	140	130	250	200	150	600	1075	3375	6900	8875	10425	10175	8200	5675	3050	1975	1100	550	150	130	76	25	65000
25	—	—	—	—	—	—	—	—	—	1	—	—	1	0	0	0	—	0	2	5	8	7	6	—	1	1	25
24	—	—	—	—	—	—	—	—	—	—	0	0	1	4	0	4	0	2	4	12	20	11	5	1	0	—	76
23	—	—	—	—	—	—	—	—	—	0	0	1	4	8	8	33	24	8	1	—	—	—	—	—	—	—	130
22	—	—	—	—	—	1	0	1	0	3	4	6	16	44	78	186	169	175	99	78	32	34	24	12	5	2	540
21	—	—	—	—	1	0	1	—	3	4	8	11	34	76	118	225	294	273	324	202	186	99	78	24	11	4	1100
20	1	1	2	0	2	0	10	2	11	23	32	58	100	141	237	415	617	512	484	283	202	98	40	17	11	6	1975
19	2	3	2	2	7	15	15	8	21	28	52	100	980	491	738	950	1025	944	324	185	76	33	21	9	6	—	3050
18	0	3	7	6	5	8	10	17	16	43	135	502	662	885	1348	1459	1834	512	272	175	71	18	18	12	7	1	5675
17	4	8	5	10	12	8	28	16	24	101	167	947	1825	1811	1852	960	617	294	160	80	20	8	5	3	—	—	8200
16	3	5	8	9	12	18	39	32	84	76	139	532	1181	1970	1811	1348	235	165	65	46	4	4	2	0	5	—	10175
15	2	4	5	8	12	18	29	22	80	146	286	532	1109	1181	1471	1825	885	491	118	22	9	8	6	4	0	4	10425
14	3	7	1	10	17	28	29	27	36	166	327	410	533	1194	1109	947	580	141	76	24	20	4	0	0	4	—	8875
13	1	2	3	7	17	12	19	17	23	139	237	296	533	532	498	337	207	140	32	34	6	4	4	0	4	—	6900
12	0	0	0	3	18	22	12	10	21	45	106	103	337	327	212	167	52	36	11	5	2	0	1	0	1	—	3375
11	2	4	2	6	14	4	14	13	21	47	70	85	189	166	120	101	43	28	8	3	3	0	0	0	—	—	1075
10	2	4	1	6	8	7	17	6	10	47	45	78	65	60	80	53	36	12	4	0	0	0	—	—	—	1	600
9	1	1	0	8	8	7	2	11	84	81	21	31	30	32	23	24	15	3	1	0	6	0	—	—	—	—	150
8	0	0	1	2	1	1	0	12	51	18	10	17	27	27	22	25	17	16	6	1	—	—	—	—	—	—	200
7	3	0	1	1	4	8	14	13	22	22	35	37	25	38	17	15	3	1	1	—	—	—	—	—	—	—	130
6	0	0	1	0	3	8	7	11	19	12	18	18	18	18	3	1	0	0	—	—	—	—	—	—	—	—	140
5	1	0	0	8	3	0	10	12	12	7	12	12	5	7	2	1	0	—	—	—	—	—	—	—	—	—	75
4	0	1	2	2	1	0	0	9	9	10	2	5	5	5	7	2	1	—	—	—	—	—	—	—	—	—	50
3	0	1	0	2	0	1	2	4	0	8	7	10	5	5	5	7	2	1	—	—	—	—	—	—	—	—	50
2	1	0	2	0	1	0	1	2	4	0	2	7	4	6	5	3	2	—	—	—	—	—	—	—	—	—	75
1	—	1	0	0	1	0	3	0	1	2	4	0	3	1	0	—	—	—	—	—	—	—	—	—	—	—	25
0	—	1	0	0	1	0	3	0	2	0	1	—	3	1	0	—	—	—	—	—	—	—	—	—	—	—	25
Totals	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	Totals

Number of Prickles on Second Leaf.

2 R 2

TABLE VIII.—Dorsetshire Hollies.

Number of Prickles on First Leaf.

(rows below are "Number of Prickles on Second Leaf")

Second Leaf	1.	3.	4.	5.	6.	7.	8.	9.	10.	11.	12.	13.	14.	15.	16.	17.	18.	19.	20.	21.	22.	23.	Totals
1	590 [598] [106] [37]	36	19	40	35	8	41	16	19	27	43	14	4	3	1	2	2	8	0	1	1	1	1000 [400] [525] [475]
2	40	49	31	51	37	24	55	29	13	18	34	11	5	3	5	6	3	0	0	3	1	1	350
3	35	26	17	31	96	14	34	15	18	19	33	12	8	10	8	0	0	2	2	4	1	0	460
4	6	17	16	46	36	60	53	43	30	21	24	18	16	11	4	0	6	3	5	2	0	0	675
5	41	31	46	63	40	51	70	47	30	52	59	32	13	18	10	18	8	3	4	2	0	1	775
6	16	26	96	54	75	142	93	76	72	52	70	36	53	18	10	16	24	15	8	9	1	8	1135
7	27	14	60	91	71	144	144	129	182	69	71	67	64	41	38	69	37	21	18	11	8	8	1550
8	43	8	43	47	63	129	188	203	176	224	129	97	173	87	115	54	90	46	19	21	6	5	2100
9	4	15	21	30	70	71	203	280	180	268	185	109	205	148	156	127	104	48	40	32	24	4	2150
10	3	13	13	33	58	69	132	176	358	404	268	296	382	156	286	190	198	80	74	33	17	2	3800
11	1	19	18	59	28	71	156	190	268	358	636	581	463	382	456	391	386	198	90	56	51	1	5475
12		32	16	59	28	67	129	185	205	296	784	1089	755	625	725	685	460	233	147	33	17	1	8400
13		11	11	18	10	41	64	173	146	388	1089	1622	1497	1234	1129	814	882	348	157	13	24	0	8760
14		5	4	18	8	29	37	116	113	353	655	1497	1594	1893	1403	1025	882	307	70	6	26	0	8485
15		1	5	0	3	18	38	69	54	127	466	1234	1234	1893	1462	910	574	157	22	0	4	1	7775
16			3	0	2	6	16	18	27	45	199	735	686	1462	1682	574	340	136	13		2		6485
17			0	0	1	3	2	15	21	8	48	391	385	1228	1025	807	222	83	4		0		3800
18			1	0		5	3	2	4	9	19	80	163	814	682	157	99	25	0		1		1750
19						4	3		1	6	8	11	40	223	147	56	35						775
20						1	0				6	27	21	82	58	17	24						300
21												3	24	12	61								200
22													2	5	4								
23																							25
Totals	1000 [400] [525] [475]	350	460	675	775	1135	1550	2100	2150	3800	5475	8400	8760	8485	7775	6485	3800	1750	775	300	200	25	65000 [64860]

Number of Prickles on Second Leaf.

The lower numbers in brackets give the results when Tree 91 is excluded.

SECTION II.—*Seed Vessels of Flowers.* A. *Poppies.*

(11.) Having dealt with several characters in a variety of trees, I thought it desirable to deal with a single character in a variety of races in a very different branch of the vegetable kingdom. After consultation with my colleague, Professor F. O. OLIVER, the stigmatic bands on the seed capsules of poppies were selected, and they have proved an easily recorded character, which could be considered for a variety of races. Here again there was some diversity in the method of collection. The rules given being (i.) that the bands were to be counted at the periphery of the capsule and not at the centre; thus was counted as 7, not 6, bands; and

(ii.) every possible bud on the plant was to be counted. The work went on from spring to autumn of the year 1899; it was impossible to collect, dry, and preserve for counting all the wild poppies gathered as soon as they had ceased flowering. Thus some poppies were taken when they had completely done flowering, others with flowers on them, and others with many buds. Few, I think, would be included which had not formed all their buds. In my own series, I counted the bands on every capsule, in every flower, and in nearly every bud, opening the bud and using a microscope when necessary. The labour of this was very great, and was not carried out by all my helpers. I doubt, indeed, whether any advantage was derived from it, except in so far as there may be correlation between the number of stigmatic bands and the early or late development of the seed-vessel.

(12.) The following series were obtained :—

Series (i.). *Buckinghamshire Wild Poppy* (Papaver Rhœas).—I gathered 200 specimens from as wide a range as possible—roadside, field-hedgerow, and cornfield—avoiding *Papaver dubium*, and counting every set of stigmatic bands right down to the smallest bud I could manage. These specimens were gathered on the very top of the Chilterns; the soil and climate is not a very favourable one, and my mean number of buds was a small one. I myself tabled the data and calculated the constants.

Series (ii.). *Buckinghamshire Wild Poppy* (Papaver Rhœas).—Miss C. D. FAWCETT gathered 236 poppy plants on the southern slope of the Chilterns, towards their foot. They came nearly all from a single cornfield, and thus their environment was much more uniform, and the climate more favourable than in Series (i.). It is quite possible that these poppies were largely the offspring of a comparatively few poppies, as they were gathered from a small area. Dr. LEE tabled the data and calculated the constants.

Series (iii.). *Somersetshire Wild Poppy* (Papaver Rhœas).—The bulk of this series were gathered by Miss AGNES FRY in the Quantocks, but being rather a short series, 86 plants, I added to it several other short series, 42 plants in all, gathered and

counted by various less active helpers. Thus the series is really an omnibus series, although mainly from the Quantocks. Dr. LEE tabulated the data and calculated the constants.

My other two series are of the Shirley Poppy, a well-known and beautiful garden variety.

Series (iv.). *Chelsea Shirley Poppy.*—These plants were grown by my colleague, Professor F. O. OLIVER, in his garden at Chelsea. The plants had scarcely been thinned, and so, being very crowded, grew tall, and with few buds on the plant. The plants were taken up and forwarded to me by rail, but being delayed in transit suffered a great deal, so that a good many capsules were broken off, or could not be counted. These circumstances account for the smallness of the number of pairs obtained from each plant. Professor OLIVER kindly came and assisted me in the counting. The data were tabulated and reduced by myself.

Series (v.). *Buckinghamshire Shirley Poppy.*—These plants were grown at Hampden Farm House, and covered a large border. They were extremely healthy, large plants, and I have counted as many as 120 capsules on a single plant ! The plants were taken up singly as they had done flowering and the bands counted. The labour of counting was severe, as there were 4443 capsules on 176 plants as compared with 1020 capsules on 325 Chelsea poppies. The tabulation involved the entry of 197,478 pairs, perhaps the largest number we have yet dealt with in a single correlation table ; this and the determination of the constants is the work of Dr. LEE.

The reader will notice that the investigations on poppies differ largely from those on the leaves of trees. In the latter case, 26 leaves were taken at random from the trees and the character on each measured. In the former case, every available capsule was dealt with. These, of course, are not *all*, the capsules put forth by the plant, and the proportion of these two sets would vary considerably from one series to another, approaching fairly close to equality in my wild poppies from the top of the Chilterns, and differing very considerably in the Shirley poppies from Chelsea. In the leaves of the trees no tree was weighted by its size or vigour ; in the case of a vigorous poppy, however, it may contribute ten times as many pairs to our table as a feeble plant. There is a good deal to be said for both methods of dealing with the problem, and even for a third method to which I shall refer immediately. If we pick up two leaves at random and wish to consider the probability that they belong to one tree, we ought certainly to allow for the larger number of pairs of leaves on the larger trees ; if we wish to discover to what extent two randomly chosen men have characters like to those of brothers, we ought certainly to allow for the large families, even if we do not deal with every possible pair of brothers in those families. These two different methods of dealing with resemblance due either to the production of like organs by the individual or to heredity must be carefully borne in mind. In dealing with Mr. GALTON's data for stature* I have taken every possible pair of

* 'Phil. Trans.,' A, vol. 187, pp. 253-318.

brothers; in my own measurements of 1000 families I have taken only two brothers from each family. We cannot *a priori* expect the results to come out exactly the same. Indeed, we might *a priori* expect the latter method to give rather higher results than the former, for it introduces more individual families, and if we tabulated all the brothers of one family only or all the like organs of one organism, we should find nothing but a spurious correlation growing indefinitely small as the number of brothers or organs was indefinitely increased.* Thus I anticipated that the Hampden Shirleys would give a lower correlation than the Chelsea Shirleys, the wild poppies from the top of the Chilterns a higher value than those from the bottom, &c.

As our series of poppy plants were in no case very large, it was impossible to directly test this result by taking only a single pair from each plant, even if it had been possible after the counting and marking to select a pair of capsules at random. I accordingly adopted a third or intermediate method. Miss FAWCETT had recorded her capsules under the letters *a*, *b*, *c*, &c., and not grouped them directly under the number of the stigmatic bands. Accordingly, it was possible to take *a* with *b*, *c* with *d*, *e* with *f*, &c., and thus repeat no single capsules, but get a number of pairs sensibly taken at random. Thus large and vigorous plants will give more pairs than feeble ones, and yet much fewer than if we take every possible pair. This series will be referred to as Series (ii.)^{bis}. Its general result confirms the anticipation referred to above. It will be noticed that nothing of this kind can be done in the case of leaves of trees, where we cannot gather *every* leaf. One great advantage of the 26 leaves from every tree is the means it provides of testing the accuracy of the lengthy process of tabulation and calculation. Each leaf is used 25 times, and thus the mean and standard-deviation of the total number of pairs ought to be the same as those of the total series of leaves (say, 2600 for 100 trees). This valuable method of verification is lost when we do not take the same number of like organs from each individual. It then becomes of considerable interest to ascertain how far the mean and variability of the total number of organs observed are identical with the mean and variability of the total number of pairs of organs observed. This comparison is made in Series (v.)^{bis}, the mean and variability of the 4443 capsules being compared with those of the corresponding 197,478 pairs. The whole of the tabulation and arithmetic involved in Series (ii.)^{bis} is mine, but Series (v.)^{bis} is again due to Dr. LEE.

I now proceed to sum up the results obtained from the above seven series.

---

[1] I worked out the theory of this some time ago, and may publish it on another occasion.

Resemblance of Poppy Capsules from the same Plant.

Series.	Number of			Mean number of bands.	S. D. of bands.	Correlation.
	Plants.	Capsules.	Pairs.*			
Wild poppy (i.) . (top of Chilterns)	200	1305	11,026	10·0443 ± ·0381	1·7704 ± ·0234	·5620 [± ·0128]
Wild poppy (ii.) . (southern slope of Chilterns)	236	2343	25,160	9·9055 ± ·0210	1·5087 ± ·0149	·3997 [± ·0117]
Wild poppy (ii.)ᵇⁱˢ (no capsule repeated)	236	2268	2,268	9·8444 ± ·0211	1·4878 ± ·0149	·4452 [± ·0114]
Wild poppy (iii.) . (Quantocks)	127	1324	19,790	8·7733 ± ·0270	1·4553 ± ·0191	·5333 [± ·0133]
Shirley poppy (iv.) (Chelsea)	325	1020	2,756	12·3679 ± ·0355	1·6803 ± ·0251	·6149 [± ·0131]
Shirley poppy (v.) (Hampden)	176	4443	197,478	12·6103 ± ·0191	1·8653 ± ·0135	·5238 [± ·0073]
Mean (i.), (ii.), (iii.).	563	4972	55,976	9·5744	1·5781	·4983
Mean (iv.), (v.) .	501	5463	200,234	12·4891	1·7828	·5693
Mean (i.) to (v.) .	1064	10,435	256,210	[10·6524]	[1·6600]	·5267

Now this table contains a good deal of interest. Upwards of a thousand poppy plants were dealt with, involving the counting of the bands on more than ten thousand capsules, and the entry of more than a quarter of a million of pairs in the correlation tables. We see at once that the mean and variability of the Shirley poppy differ so widely from those of the wild poppy, that it is idle to group these species for these characters together. It is interesting to note, however, that both in the number of bands and in the variability the Hampden poppies, whether wild or Shirley, stand at the top of their respective series. With regard to the wild poppy, I probably gathered from a wider area with a greater variety of environment than was the case in Series (ii.) or (iii.). That mine were gathered from much poorer soil is evidenced by my plants having on an average only 6 capsules, while the Quantocks and southern Chiltern poppies had about 10 capsules. The difference is possibly even greater than these numbers indicate, for I counted every bud I possibly could. The two Chiltern series do not differ much in the mean number of bands, but the difference of variability is very significant. I should take it to be a very fair measure of the relative range of environment in the two cases. When we turn to the Shirley series the difference is almost too great for any comparison. While the Hampden series had some twenty-five capsules, on an average, to the plant, the Chelsea series numbered

* The reader must bear in mind that this column gives the number of pairs of the symmetrical tables below, or *twice* the number of independent pairs.

only about three! In fact, besides being extremely crowded a process of selection had gone on in the transit. From examination of the uninjured plants, I think that five capsules a plant would be more than the average of these poppies before packing, and that at least one to two capsules per plant were destroyed on the journey. Accordingly, I do not consider that much weight should be given to the results of this series, but I felt bound to include it as I have included all the series dealt with, even when I hold them to be really unsuited to the present investigation. If we consider the two methods of dealing with Series (ii.), *i.e.*, that of including all possible pairs, and that of including pairs in which no capsule was repeated, we find that no very sensible change is made in the mean or variation, but a rather greater increase in the correlation than might have been anticipated. I think this is largely due to unconscious selection in the choice of pairs, *i.e.*, we took the records for (*a*) and (*b*), (*c*) and (*d*), &c., as our pairs; it is quite possible that in recording the capsules successive letters would correspond to capsules gathered from the same part of the plant, and so the members of the pair would be subject to the same or somewhat similar conditions of nutriment, light and warmth during development. Valuable investigations have yet to be made on the relative degree of resemblance of leaves or seed-vessels coming at random from the same tree or plant, and coming from the same branch or shoot. The effects of position on the individual undoubtedly produces some differentiation, and all we can hope to do at present is to get round numbers approximately describing the degree of resemblance of these *theoretically* undifferentiated like organs.

If we exclude the Chelsea series, we have for the

$$\begin{array}{ll} \text{Wild poppy} & \cdot 4983 \\ \text{Shirley poppy} & \cdot 5238 \end{array}$$

For the four series of trees we have

$$\begin{array}{ll} \text{Ash} & \cdot 3918 \\ \text{Chestnut} & \cdot 5284 \\ \text{Holly} & \cdot 5035 \\ \text{Beech} & \cdot 5699 \\ \hline \text{Mean} & \cdot 4984 \end{array}$$

So far as the seed-vessels of poppies and the leaves of trees are concerned we have a degree of resemblance in undifferentiated like organs which approximates to the value ·5.

It will be as well to put together the frequency distributions of the capsules as we have done in the case of the trees :—

Wild Poppies (*Papaver Rhœas*).   Frequency of Stigmatic Bands.

Series.	5.	6.	7.	8.	9.	10.	11.	12.	13.	14.	15.	16.	Totals.
Chilterns (i.) . .	2	17	86	182	255	272	260	130	67	25	9	—	1305
Chilterns (ii.) . .	1	13	103	304	571	634	428	198	56	26	6	3	2343
Quantocks (iii.) . .	17	61	189	358	316	227	96	39	20	1	—	—	1324
Total . . .	20	91	378	844	1142	1133	784	367	143	52	15	—	4972

Shirley Poppies.   Frequency of Stigmatic Bands.

Series.	5.	6.	7.	8.	9.	10	11.	12.	13.	14.	15.	16.	17.	18.	19.	Totals.
Chelsea (i.). .	0	1	1	4	40	95	167	245	202	145	86	26	7	1	0	1020
Hampden (ii.)	1	11	32	56	148	363	628	925	954	709	397	155	51	12	1	4443
Total . .	1	12	33	60	188	458	795	1170	1156	854	483	181	58	13	1	5463

These distributions appear to be uniform although asymmetrical, and they give no obvious sign of a mixture.   The mode is 10 for the Chilterns series and 8 for the Quantocks series, while the Shirley poppies give 12 for Chelsea and 13 for Hampden.

It remains to tabulate the several results obtained for variation.   These are given in the following table.   We see at once that the general effect of weighting the capsules with the number of pairs on the plant has the effect of increasing the mean and reducing the variability ; in other words, the plants with most capsules on and leading to most pairs, are those with the greatest number of stigmatic bands. We have some evidence accordingly that the most vigorous plants exhibit most bands on the capsules :—

Individual and Racial Variability.

Series.	Stigmatic bands.		Coefficient of variation.	S. D. of array.	Percentage variability.
	Mean.	S. D.			
Wild Poppy.					
(i.) from pairs	10·0443	1·7704	17·6259	1·4644	82·71
„ „ capsules	9·9318	1·7684	17·8052	—	—
(ii.) from pairs	9·9055	1·5087	15·2309	1·3829	91·66
„ „ capsules	9·8288	1·5007	15·2677	—	—
„ pairs without repetition of capsules	9·8444	1·4878	15·1132	1·3322	89·54
(iii.) from pairs	8·7733	1·4553	16·5878	1·2311	84·59
„ „ capsules	8·7236	1·5402	17·6556	—	—
Mean from pairs. . . .	9·5744	1·5781	16·4809	1·3661	86·32
Mean from capsules . . .	9·4947	1·6031	16·9095	—	—
Shirley Poppy.					
(iv.) from pairs	12·3679	1·6803	13·5860	1·3251	78·86
„ „ capsules	12·3873	1·7326	13·9869	—	—
(v.) from pairs	12·6103	1·8853	14·9505	1·0060	85·18
„ „ capsules	12·5091	1·8977	15·1705	—	—
Mean from pairs.	12·4891	1·7828	14·2682	1·4655	82·02
Mean from capsules. . .	12·4482	1·8151	14·5787	—	—
Total mean from pairs. .	—	—	15·5962	1·4019	84·60
Total mean from capsules.	—	—	15·9772	—	—

But we might well have expected a good deal more evidence of this. The differences of the means calculated from capsules and pairs is not very great, and in two out of three series of the wild poppies the variability as deduced from pairs is very slightly greater than (sensibly equal to) that deduced from capsules. The mean values of the coefficient of variation, as calculated from pairs and from capsules, are in close agreement. The only sensible deviation is that of the Quantocks series, and this arises from the large difference in the standard deviations of the two methods. The work has been carefully revised, but no error discovered.

Generally for the coefficient of variation we have—

Ash	17·56	Wild poppy	16·91
Spanish chestnut	15·05	Shirley poppy	14·58
Beech	10·77		
Holly	23·36		

Mean . . 16·67

So that the variability of both may be expressed in round numbers as 16. It is clearly not a constant for the whole vegetable kingdom, but 16 enables us to form some rough appreciation of the plasticity of vegetable forms, and may be compared with values obtained from other series later.

Comparing again the ratio of the racial and individual variabilities, we see that the latter reaches, on an average, over 85 per cent. of the former, the minimum value being 79 and the maximum 91. It is difficult to test this point directly, for the very simple reason that a single plant will, as a rule, have very few flowers; and upon small numbers it is impossible to test satisfactorily the full variability of the plant. The probable error becomes enormous if we have only four or five capsules to determine the standard deviation from. The method adopted has been that which arises in the treatment of parental inheritance; all the offspring of parents of a given stature are clubbed together and form an array. This array has a certain variability, and this variability is taken to be that of a single family of offspring having parents of the given stature. It would clearly be impossible to find the variability inside the family from numbers such as occur in a single human family. To some extent we can verify the assumptions thus made. Thus I took the wild poppy with the largest number of capsules I had come across, and determined their variability; there were 36 such capsules, and the standard deviation was 1·4519. I then took two of the most prolific Shirley poppies which had 126 and 85 capsules on them, and their variabilities were given by standard deviations of 1·3290 and 1·5889, i.e., a mean of 1·4589. These are actually in excess of the standard deviations of the arrays, namely, 1·3661 and 1·4019 of the two series of poppies! I think we may accordingly take the latter as a reasonable measure of the possible variation of the individual.

TABLE IX.—Wild Poppy (Chilterns' Top) Series (i.).

Number of Stigmatic Bands on First Capsule.

	5.	6.	7.	8.	9.	10.	11.	12.	13.	14.	15.	Totals.
5	0	8	4	—	—	—	—	—	--	—	---	12
6	8	38	46	23	11	5	2	—	—	—	...	133
7	4	46	184	163	146	78	31	8	1	--	---	661
8	—	23	163	390	398	279	111	75	32	4	—	1475
9	-	11	146	398	554	415	250	161	68	9	5	2017
10	---	5	78	279	415	514	520	240	112	27	10	2200
11	—	2	31	111	250	520	770	366	160	52	9	2271
12	---	-	8	75	161	240	366	252	178	35	12	1327
13	—	—	1	32	68	112	160	178	92	23	11	677
14	—	—	—	4	9	27	52	35	23	28	12	190
15	—	—	—	—	5	10	9	12	11	12	4	63
Totals	12	133	661	1475	2017	2200	2271	1327	677	190	63	11026

Number of Stigmatic Bands on Second Capsule.

TABLE X.—Wild Poppy (Chilterns' Base) Series (ii.).

Number of Stigmatic Bands on First Capsule.

	5.	6.	7.	8.	9.	10.	11.	12.	13.	14.	15.	16.	Totals.
5	0	1	2	1	2	1	4	—	—	—	—	—	11
6	1	4	40	25	19	10	9	1	1	—	—	—	110
7	2	40	184	289	228	145	48	20	4	1	—	—	961
8	1	25	289	822	964	563	283	105	20	8	—	—	3080
9	2	19	228	964	1868	1633	855	353	78	32	5	4	6041
10	1	10	145	563	1633	2106	1469	682	151	60	17	8	6845
11	4	9	48	283	855	1469	1220	585	154	63	20	13	4723
12	—	1	20	105	353	682	585	336	134	54	14	14	2298
13	—	1	4	20	78	151	154	134	52	44	17	7	662
14	—	—	1	8	32	60	63	54	44	26	8	1	297
15	—	—	—	—	5	17	20	14	17	8	2	1	84
16	—	—	·	...	4	8	13	14	7	1	1	0	48
Totals	11	110	961	3080	6041	6845	4723	2298	662	297	84	48	25160

TABLE XI.—Wild Poppies (Chilterns' Base) (ii.)[bis]

(No repetition of any individual capsule.)

Number of Stigmatic Bands on First Capsule.

Number of Stigmatic Bands on Second Capsule.

	5.	6.	7.	8.	9.	10		
5	—	—	—	—	1	—		
6	—	2	5	1	1	0		
7	—	5	24	28	19	13		
8	—	1	28	80	79	73		
9	1	1	19	79	202	140		
10	—	0	13	73	140	198		
11	—	3	1	24	74	127	0	2
12	—	—	1	8	25	50	2	0
13	—	—	—	2	5	10	1	0
14	—	—	—	..	4	7	0	0
15	—	—	—	—	—	1	0	1
16	—	—	—	—	—	—	1	0
Totals	1	12	91	295	550	619	5	3

TABLE XII.—Wild Poppy (Quantocks) Series (iii.).

Number of Stigmatic Bands on First Capsule.

	8.	9.	10.	11.	12.	13.	14.	Totals.
	22	16	7	4	0	1	0	136
	202	93	40	9	0	9	0	863
	895	375	113	22	2	7	0	2666
895	1716	1312	687	178	22	11	0	5045
375	1312	1590	1275	337	43	19	0	5060
	687	1275	1234	406	92	52	0	3906
	178	337	406	308	138	51	1	1454
	22	43	92	138	102	37	1	437
7	11	19	52	51	37	26	4	217
	0	0	0	1	1	4	0	6
	5045	5060	3906	1454	437	217	6	19790

TABLE XIII.—Shirley Poppies (Chelsea) Series (iv.).

Number of Stigmatic Bands on First Capsule.

	6.	7.	8.	9.	10	11.	12.	13.	14.	15.	16.	17.	18.	Totals.
6	—	—	1	—	—						—	—	—	1
7	—	—	0	0	1	—					—	—	—	1
8	1	0	0	0	2	0	0	3			—	—	—	6
9	—	0	0	20	37	29	15	9	—	—	—	—	—	110
10	—	1	2	37	88	64	39	19	4	1	—	—	—	255
11	—	—	0	29	64	140	138	70	19	4	1	—	—	465
12	—	—	0	15	39	138	196	154	68	25	2	—	—	637
13	—	—	3	9	19	70	154	174	105	55	8	—	—	597
14	—	—	—	—	4	19	68	105	112	63	17	2	1	391
15	—	—	—	—	1	4	25	55	63	52	16	4	0	220
16	—	—	—	—	—	1	2	8	17	16	4	4	1	53
17	—	..	..	—	—	—	—	—	2	4	4	6	1	17
18	—	—	—	—	—	—	—	—	1	0	1	1	0	3
Totals	1	1	6	110	255	465	637	597	391	220	53	17	3	2756

TABLE XIV.—Shirley Poppies (Hampden) Series (v).

Number of Stigmatic Bands on First Capsule.

	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	Totals.
5	0	10	19	10	4	3	8	4	5							63
6	10	90	190	101	51	42	80	40	50							654
7	19	190	354	208	122	140	308	194	145	7						1687
8	10	101	208	114	181	251	365	335	238	112	44	11				1970
9	4	51	122	181	762	1144	1085	721	390	119	50	16				4645
10	3	42	140	251	1144	2858	3943	3357	1948	698	249	86	15	7		14741
11	8	80	308	365	1085	3943	8044	6401	5327	2305	758	200	54	2		28880
12	4	40	194	335	721	3357	6401	10574	9096	5129	2234	715	179	49		39027
13	5	50	145	238	390	1948	5327	9095	11198	8494	4008	1321	303	57		42579
14			7	112	119	698	2305	5129	8494	9388	5569	2151	594	144	2	34712
15				44	50	249	758	2234	4008	5569	3664	1717	513	115	2	18923
16				11	16	86	200	715	1321	2151	1717	792	325	97	1	7432
17						15	54	179	303	594	513	325	160	58	4	2205
18						7	2	49	57	144	115	97	58	20	1	550
19										2	2	1	4	1	0	10
Totals ..	63	654	1687	1970	4645	14741	28880	39027	42579	34712	18923	7432	2205	550	10	197478

Number of Stigmatic Bands on Second Capsule.

(13.) C. *Nigella Hispanica.*—Besides the poppy capsules, Professor OLIVER had suggested to me that it would be possible to count the segmentation of the seed vessel in the cases of Nigella and Mallow. I accordingly procured from Messrs. VEITCH, of Slough, upwards of a hundred plants of *Nigella Hispanica.* Meanwhile, Professor OLIVER sent me a list of some he had counted himself, and pointed out that there was a substantial differentiation between the seed vessels on the main stem and those on the side shoots. It was quite clear on examination that these seed vessels were not undifferentiated like organs, the number of segments in a seed vessel on the side shoots being much more like the number on a similar vessel on a separate plant than like those of a main-shoot capsule on the same plant. Further, although I do not know the history of these special plants, they were probably from highly selected seed and in a very unstable condition. Whereas in the examination of many thousand poppy capsules I only found one remarkable abnormal case,[*] abnormalities in the case of *Nigella Hispanica* abounded and rendered the counting of the segmentation very difficult. Double and treble capsules were found in every state of union up to complete fusion, and with every variety of deformity, such as the tips of one capsule growing through the sides of a second capsule. Thus the material was by no means suited to my present purpose. However, as I had the material it seemed worth while testing the influence of differentiation in reducing correlation. Accordingly I made no attempt to separate top from side capsules,[†] and formed the following table in the manner sufficiently indicated in the preceding pages :—

---

[*] A double capsule on one stem. It contained only a few seeds. These Professor F. O. OLIVER kindly sowed for me this year; there resulted two plants with a very few poor flowers, but the abnormality was not perpetuated, and the capsules rotted off without giving seed.

[†] If the top capsules only had been taken I should often have had only one on a plant.

TABLE XV.—Segmentation of *Nigella Hispanica*.

Number of Segments in First Seed Vessel.

(Rows = Number of Segments in First Seed Vessel; Columns = Number of Segments in Second Seed Vessel.)

First \ Second	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	Totals.
2	—	—	—	5	2	3	—	—	—	—	—	—	—	—	—	—	—	—	—	10
3	—	—	—	1	1	0	4	0	0	0	0	1	—	—	—	—	—	—	—	7
4	—	—	—	2	5	4	6	1	1	0	0	0	0	0	1	—	—	—	—	90
5	5	1	2	64	58	55	101	13	1	0	2	1	0	0	0	—	—	—	—	303
6	2	1	5	58	60	76	168	25	9	2	6	0	0	0	0	—	—	—	—	412
7	3	0	4	55	76	88	240	41	7	10	9	0	0	0	1	—	—	—	—	534
8	—	4	6	101	168	240	866	107	28	15	11	4	0	0	2	—	—	—	—	1563
9	—	0	1	13	25	41	107	14	3	9	0	0	0	0	0	0	1	—	—	223
10	—	0	1	1	9	7	28	9	0	1	0	0	0	1	0	0	0	1	—	59
11	—	0	0	0	2	10	15	3	1	0	3	0	0	0	1	—	—	—	—	35
12	—	0	0	2	9	9	11	9	1	3	2	—	—	—	—	—	—	—	—	43
13	—	1	0	1	0	0	4	0	0	0	—	—	—	—	—	—	—	—	—	6
14	—	—	—	0	0	0	0	0	0	0	0	—	—	—	—	—	—	—	—	0
15	—	—	0	0	0	0	0	0	0	0	0	—	—	—	—	—	—	—	—	0
16	—	1	0	0	1	2	0	1	1	—	—	—	—	—	—	—	—	—	—	6
17	—	—	—	—	—	—	0	0	—	—	—	—	—	—	—	—	—	—	—	0
18	—	—	—	—	—	—	0	0	—	—	—	—	—	—	—	—	—	—	—	0
19	—	—	—	—	—	—	0	0	—	—	—	—	—	—	—	—	—	—	—	0
20	—	—	—	—	—	—	1	1	—	—	—	—	—	—	—	—	—	—	—	2
Totals.	10	7	90	303	412	534	1563	223	59	35	43	6	0	0	6	0	0	0	2	

Number of Segments in Second Seed Vessel.

While the modal value is conspicuously 8, the mean value is 7·4639. This is due to the second modal value of 5 on the side plants. The standard deviation is 1·4893, and the correlation ·1899. It will be seen at once how the result fully confirmed my expectation, that the degree of resemblance between capsules on the same plant would be largely reduced, owing to the existence of differentiation. The actual distribution of capsules in 210 plants was as follows :—

Number of Segments.

	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.	12.	13.	14.	15.	16.	17.	18.	19.	20.	Total.
Frequency.	1	1	5	75	108	133	420	61	19	10	8	2	0	0	1	0	0	0	1	845

Thus on an average there were four seed vessels to a plant. The mean and standard deviation found directly from the capsules are 7·5101 and 1·4559 respectively. These are in substantial agreement with the above results found for pairs. They do not confirm the conclusion based upon the data for poppies, namely that if we work with pairs we shall raise the mean and lower the variability. The fact is that vigour seems to work in opposite directions with this plant and the poppy ; with the latter it means many capsules with a high number of stigmatic bands, but with the former it means many side shoots with the differentiated capsules with few segmentations. Such capsules of course decrease the mean and raise the variability.

The coefficient of variation is 19·385, sensibly greater than that of the wild poppy, but below that of the holly leaf. The standard deviation of an array is 1·4622, or the variability of the individual 98·18 per cent. of that of the race. The plant with the greatest number of seed vessels on had only 11, and these gave a standard deviation of 1·6188, greater than that of the whole race ! But of course this is subject to a very large probable error ; it suffices to emphasise, however, how the source of variation is to be sought within the individual, and is not a mere result of sexual reproduction.

The following table sums up the results obtained for *Nigella Hispanica* :—

	Number of			Mean.	S. D.	Coefficient of variation.	S. D of array.	Percentage variation.	Correlation.
	Plants.	Capsules.	Pairs.						
Pairs. Capsules	} 210	845	3212 {	7·4639 7·5101	1·4893 1·4559	— 19·385	1·4622 —	98·18 —	·1899 —

(14.) D. *Mallow* (Malva Rotundifolia).—I had given up the idea of the ordinary wild mallow (*Malva sylvestris*) owing to the few plants to be found in the neighbourhood of Hampden, as well as on account of the immense number of seed vessels on the plants to be counted. But one day an inspection of my potato-patch revealed

the whole ground covered with low plants of *Malva Rotundifolia*.* On inquiry I found that a cottage had recently stood on the site, and that the cottagers kept in their herb-bed a small clump of this plant for the preparation of a local ointment of high repute. When the cottage was removed the mallow had spread over the whole patch, I believe very little by seed, but largely by stolons. This seemed to provide excellent material ready to hand, and from it we can draw some conclusions as to the influence of such a division on the degree of resemblance of like parts. It will be remembered that DARWIN,† in experimenting on *Origanum vulgare*, found that cross-fertilisation of flowers on plants which had separate roots, but had spread by stolons from the same individual, presented no advantage over self-fertilisation, and he concluded that this arose from the fact that the plants were really one and the same individual so far as such processes were concerned. If this were so in the case of my mallow, we ought to find a great reduction in the relationship of undiffer-entiated like organs. Two points struck me at the outset—the small variability in the segmentation of the seed vessels and the low mean value (13·7) compared with that (15) given in botanical works. I was able to collect 127 plants; of these, 76 had more than 26 capsules, some many more, and of these I took 26 only; 51 plants had fewer than 26 capsules, and although I took buds in every stage of development I could not make up my number to this, often only obtaining 10, or even fewer. As the potatoes were being removed the mallows had to be taken up before the seeds were ripe, and the seed-vessels artificially dried. The counting of the segments proved a very laborious task. To Miss M. NOTCUTT I owe a considerable portion of the counting. The buds and small seed vessels requiring a lens or small microscope were counted by Dr. LEE, Mr. RADFORD SHARPE, and myself. I am responsible for the tabulation of the data and for all the numerical calculations.

The 127 plants gave rise to 57,740 pairs distributed thus :—

TABLE XVI.—*Malva Rotundifolia.*

Number of Segments in the First Seed-vessel.

	10	11.	12.	13.	14.	15.	16.	17.	18.	19.	Totals.
10	10	24	60	78	89	36	5	—	—	—	302
11	24	64	202	270	185	57	7	—	—	—	809
12	60	202	850	1802	1905	637	95	1	—	—	5552
13	78	270	1802	5036	6195	2409	324	8	0	5	16127
14	89	185	1905	6195	9760	4293	653	13	0	10	23103
15	36	57	637	2409	4293	2428	386	5	0	8	10259
16	5	7	95	324	653	386	60	2	0	2	1534
17	—	—	1	8	13	5	2	—	—	—	29
18	—	—	—	0	0	0	0	—	—	—	0
19	—	—	—	5	10	8	2	—	—	—	25
Totals	302	809	5552	16127	23103	10259	1534	29	0	25	57740

* Misnamed "marsh mallow" throughout this district of Buckinghamshire
† 'Cross and Self-fertilisation in Plants,' 3rd edition, p. 94

The following table gives the chief numerical results :—

<div align="center">*Malva Rotundifolia.*</div>

	Number of			Mean.	S. D.	Coefficient of variation.	S. D. of array.	Per-centage variation.	Correla-tion.
	Plants.	Capsules.	Pairs.						
Pairs. . ⎫ ⎬127 Capsules ⎭		2568	57740 ⎧ ⎨ ⎩	13 7000 ± ·0139 13·6632	1·0407 ± ·0098 1 0652	— 7·7961	1·0232 —	98·32 —	·1827 ± [·0129] —

The actual distribution of frequency of the capsules was :—

No. of segments .	10.	11	12.	13.	14.	15.	16.
Frequency . .							

These results amply bear out my *a priori* conclusions.

The degree of resemblance of like parts has been very largely reduced. The variability (7·7961) is smaller than any we have yet come across, and the variability of the individual differs under 2 per cent. from that of the race. All these conclusions are compatible with the decreased individuality which may be expected to exist among the members of a group to a large extent propagated by stolons and not from seed.

Although in the majority of cases (76 out of 127) I took the same number of seed capsules (26) from the plant, still the influence of the plants with fewer capsules is quite marked; we see that weighting with the number of pairs lowers the variation and raises the mean, in other words, the large vigorous plants have more segmentation and less variability. The distribution of segmental frequency is fairly uniform, and shows a marked mode at 14 segments, not the 15 of the usual statement.

<div align="center">SECTION III.—*Members of Whorls.*</div>

(15.) *Woodruff* (Asperula odorata).—I must confess that there is divergence between the standpoints of the botanist and of the statistician. To the latter a character is good or bad according as it affords facilities for fairly easy measurement or enumeration. He has first to seek such characters as a *sine quâ non*, and then inquire how far they occur in undifferentiated like organs. On the other hand, the biologist will readily provide a list of such characters in plant or animal life, and not

one of them be available for measurement or counting. I had long fixed my eye on the members of the whorls of the woodruff. They were so fascinatingly easy to collect and count ; they exhibited such a moderate amount of variation, and one felt sure the collecting, counting, tabulating, and calculating could all be done in the inside of a week ; and the impulse to such a light task after the elaborate work on the earlier series was very great. Professor F. O. OLIVER and others warned me that the members of the whorls were differentiated in their origin* and also in their position on the stem, and that such a series was unsuited to illustrate the degree of resemblance between undifferentiated like organs. However, the task seemed, comparatively speaking, so easy that I felt I would undertake it, if only to compare with the *Nigella Hispanica*, and ascertain in another case how differentiation does weaken the degree of resemblance of like parts. I accordingly collected 201 single sprays of woodruff† well distributed along about a mile of lane on the bank of a hedgerow facing south. These were gathered at Great Hampden, Buckinghamshire. In counting the members on the whorls I soon found evidences of differentiation in position, the whorls towards the top of the spray having, as a rule, fewer members than those lower down. The following is the table of frequency of different pairs of whorls :—

TABLE XVII.—Woodruff. Great Hampden.

Number of Members on First Whorl.

		4.	5.	6.	7	8.	9.	10.	Totals.
Number of Members on Second Whorl.	4	8	8	41	7	1	—	—	65
	5	8	32	273	126	91	4	—	534
	6	41	273	2376	1363	1150	41	4	5248
	7	7	126	1363	1216	953	34	3	3702
	8	1	91	1150	953	1284	54	2	3535
	9	—	4	41	34	54	4	—	137
	10	—	—	4	3	2	—	—	9
	Totals ...	65	534	5248	3702	3535	137	9	13230

I now give the numerical statement of the constants deduced from the above table and the frequency distribution of the whorls :—

---

* Some of the members of the whorls are true leaves and others modified stipules.

† They were broken off as close to the ground as possible ; they included all parts branching off above ground, but the forked sprays were only few in number.

Woodruff.

Number of			Mean.	S. D.	Coefficient of variation.	S.D. of array.	Percentage variation.	Correlation.
Sprays.	Whorls.	Pairs						
} 201	1465	13230 {	6·7978 ± ·0164	·9318 ± ·0116	—	·9177	98·49	·1733 [± ·017]
			6·9010	·9291	13·4631	—	—	—

The frequency distribution of the 1465 whorls was :—

Number of	4.	5.	6.	7.	8.	9.	10.	Total.
Frequency . . .	6	40	537	410	455	16	1	1465

The double modes 6 and 8 are brought out by the above distribution, and this is evidence either of differentiation or of a tendency to an even number of members. Contrary to the usual rule, the mean number of members is reduced when we take pairs, indicating that when we take a spray with many whorls—such as occasionally arises from forking—the number of members in the whorls are reduced, and since the variation is also very slightly increased (as in *Nigella*), they are spread out over a greater range.

The great reduction in the correlation, the value of which is only ·1733, shows how unsuited the material was for the purposes of the present investigation. At the same time the ratio of the individual to the racial variability reaches the very high value of 98·5 ! We have selected a character to test individuality on, which exhibits differences which are largely racial, and not peculiar to the individual plant.

SECTION IV.—*Ferns.*

(16.) In choosing ferns for considering the resemblance between undifferentiated like organs I had in view not only the comparative ease of counting (as compared, for example, with the segmentation of mallow), but the fact which students of the fern strongly impressed upon me that it was peculiarly subject to its environment. One great authority went so far as to assert that the presence or absence of *sori* on an individual hartstongue depended *solely* on the environment and not at all on the individuality of the plant. While hardly prepared to accept to the full such a statement (for I had already learned in a variety of types of the existence of individuality in a marked manner apart from the influence of environment), I

anticipated in accordance with it a fairly high correlation of undifferentiated like organs, and this has, indeed, turned out to be the case.

(17.) *Somersetshire Hartstongue* (Scolopendrium vulgare).—I owe the counting of the *sori* on 8 to 12 fronds of each of 101 hartstongue ferns to Miss AGNES FRY. It was not always possible to obtain the full suggested number of 10 fronds. In a few cases Miss FRY also counted more than 10. But the ferns with fewer than 10 are not many, and the distribution of the fronds is sensibly identical with that of the pairs of fronds. The number of *sori* was often very large, running up to 180 ; the grouping of the leaves was accordingly arranged in units of 10 *sori*, and, to save fractionising in the table, these groups were taken 0—9·5, 9·5—19·5, 19·5—29·5, &c. The grouping on the basis of Miss FRY's record was carried out by Mr. LESLIE BRAMLEY-MOORE. With regard to the counting of the *sori*, all the sterile as well as the fertile markings were included. Fronds with no markings were included ; one plant, the only one found, with absolutely no markings on any of the fronds was excluded, and to this I shall recur immediately. It was difficult to procure ten perfect fronds ; many of those gathered were found to be more or less torn ; in many cases the loss of *sori* was easily ascertained ; in others an estimate had to be formed. Such estimates were made, however, in only forty-four cases out of more than 1,000 fronds, and in the majority of these cases it meant the addition of very few *sori*, and a possible error in the estimate of still fewer. Sometimes one or two *sori* may have been missed at the tip, where the small markings are indistinct, but the errors due to this source as well as to the occasional error of a unit or two in counting such a large number are well under the unit of grouping of ten *sori* finally selected. The environment of the series was not quite uniform, some thirty-three coming from a stream bed and the remainder from a park glade. The plants varied considerably in age, but the necessity of procuring eight to twelve fronds was probably effective in excluding any quite young ferns. The absence or presence of *sori* on the frond was not directly associated with its age or size. Miss FRY has recorded a number of large and medium fronds with no *sori*, and some of the fronds recorded as young have occasionally a very considerable number of *sori* (as many as ninety-five on one at least). Still, a glance at the table on p. 330 shows that the fronds with zero *sori* form in same way an anomalous group ; they give a hump in the frequency distribution in a manner somewhat similar to that due to barren wives in fertility-frequency curves.* Dr. LEE having tabulated the *sori*, and calculated the constants on the basis of including all fronds with zero number of *sori*, I went through Miss FRY's data again and excluded, not all the entries in the first row and column of Miss LEE's table, but only such of them as arose from the fronds with no *sori*. The result was a great smoothing of the table, a raising of the mean, and a lowering of the variability ; but *no appreciable effect whatever was made on the degree of resemblance of like organs*, although I

* See 'The Chances of Death and other Studies in Evolution,' vol. 1, essay on "Reproductive Selection," and diagram.

threw out upwards of 1212 pairs! This was hardly to be anticipated, but may certainly give us confidence in the substantial validity of the numerical result reached. This result is clearly independent of any extraneous source hindering the production of any sori at all on the frond. About a third of the ferns in the series (31 in 101) had one or more fronds in ten without sori at all, the total number of fronds without sori being seventy-six. The following table gives the chief numerical constants:—

Hartstongue Fern.

	Number of			Mean number of sori.	S. D.	Coefficient of variation.	S D. of array.	Percentage variation.	Correlation.
	Plants.	Fronds.	Pairs.						
Fronds without sori included .	101	1024	9384	73·7398 ± ·7419	35·1967 ± ·5246	47·7310	27·3136	77·60	·6307 [± ·0127]
Fronds without sori excluded .	101	943	8173	74·3948 ± ·5246	31·2176 ± ·4836	41·9640	24·2358	77·64	·6303 [± ·0139]

The following is the actual distribution of pairs, which in this case is very nearly proportional to the distribution of fronds:—

Number of sori.	0- 9·5.	9·5- 19·5.	19·5- 29·5.	29·5- 39·5.	39·5- 49·5.	49·5- 59·5.	59·5- 69·5.	69·5- 79·5.	79·5- 89·5.	89·5- 99·5.	99·5- 109·5.	109·5- 119·5.	119·5- 129·5.	129·5- 139·5.	139·5- 149·5.	149·5- 159·5.	159·5- 169·5.	169·5- 179·5.	179·5- 189·5.	Totals.
Fronds without sori included .	883	254	352	727	788	896	979	1040	897	908	717	446	277	191			27	18	9	9384
Fronds without sori excluded .	103	211	364	684	716	889	925	1004	866	893	709	439	276	184			27	18	9	8173

We see from the last table how the "hump" disappears from the frequency distribution when we exclude the fronds without *sori*. The double mode indicates at least considerable flatness at the top of the distribution, and may possibly mark some heterogeneity of material. I am inclined, however, to think that the great variability (about forty-two even when the fronds without *sori* are excluded) is fully accounted for by the sensitiveness of the fern in the matter of *sori* to very slight differences of environment. I look upon the high resemblance of like parts here as having been intensified by this cause ; upon an inherent individuality we have superposed an individuality due partially, perhaps, to age, but largely to small differences of immediate environment. Although the variation in the race is so large, and the degree of individuality so great, the ratio of individual to racial variability is still 78 per cent. Thus while the variability of the hartstongue (as far as *sori* is concerned) is double that of the very variable holly, and almost thrice that of the poppy, still the percentage variability of the two latter species is to that of the former only as about 85 to 78. Thus in the most variable and most individual species we have yet come across, we still find the variation within the individual is more than three-fourths of the entire variation of the race. In view of facts like this, it seems impossible to maintain the position taken up by Mr. ADAM SEDGWICK, that variation is the outcome of bi-sexual reproduction. The source of variation exists within the individual and is extensively active without the occurrence of any form of mating whatever.

I place here the table for the distribution of *sori* in pairs of fronds. The numbers in brackets are those which must replace the unbracketed numbers, if fronds without *sori* be excluded.

Table XVIII.—Hartstongue

Number of Sori on First Frond.

This is a symmetrical bivariate frequency table (band/staircase form). Rows = Number of Sori on First Frond; Columns = Number of Sori on Second Frond of Pair. Figures in parentheses are the corresponding smoothed/theoretical values. Only the occupied band is printed; blank cells are zero or not given.

First \ Second	5–19	5–29	5–39	5–49	5–59	5–69	5–79	79·5–89·5	89·5–99·5	99·5–109·5	109·5–119·5	119·5–129·5	129·5–139·5	139·5–149·5	149·5–159·5	159·5–169·5	169·5–179·5	179·5–189·5	Totals
79·5–89·5	35 (4)						109	145	124	113	29	9	3						1040 (1004)
89·5–99·5	18 (3)	7	13	17	27	49	89	124	133	190	130	69	32	9	1	0	0	0	908 (893)
99·5–109·5	10 (2)	2	5	11	41	31	64	113	130	168	79	40	11	7	0	0	0	0	717 (709)
109·5–119·5	7 (0)	0	0	7	10	12	16	29	79	90	81	18	5	1	3	2	1	0	446 (439)
119·5–129·5	1 (0)	2	0	1	5	8	3	9	13	32	40	81	38	5	0	0	0	0	277 (376)
129·5–139·5	12 (5)	0	0	3	0	7	6	3	10	9	11	34	39	4	0	3	2	1	191 (184)
139·5–149·5									9	7	1	5	4						37 (37)
149·5–159·5											3		0						18 (18)
159·5–169·5											2		3						37 (27)
169·5–179·5											1		2						18 (18)
179·5–189·5													1						9 (9)
Totals	838 (103)	254 (211)	352 (394)	727 (634)	783 (716)	886 (829)	979 (925)	1040 (1004)	908 (893)	717 (709)	446 (439)	277 (376)	191 (184)	37 (37)	18 (18)	37 (27)	18 (18)	9 (9)	9384 (9172)

Number of Sori on Second Frond of Pair.

(18.) B. *Somersetshire Ceterach* (Ceterach officinarum).—Some time after I had tried in vain to count any characters in bracken, I had a letter from Miss AGNES FRY saying that she thought it just possible that the lobes on the fronds of ceterach might be counted. The chief difficulty, of course, was the indefinite character of the lobes near the tip—a difficulty which had rendered the lobes in bracken impossible. The rather indefinite tip is here of greater consequence than in the hartstongue or the chestnut leaf, for the total number of lobes is comparatively small. In some samples Miss FRY sent me, however, I agreed fairly closely with her estimates, and although the tip must form a difficulty,* we settled that an attempt should be made to include ceterach in the present series. Miss FRY accordingly counted 9 to 11 fronds on each of 99 plants. It was necessary to take plants of very different ages, and even if ceterach be not so sensible to its environment as hartstongue, it is quite possible that part of the correlation observed is due to similarity of age. The absolute agreement of the result obtained with that for hartstongue is one of the most striking things in the whole collection of data. Considering the difficulty of the tip in ceterach and the sensibility of the *sori* of hartstongue to environment, it may be a chance agreement, but it is certainly one that gives ground for pause, and suggests further investigations of the degree of resemblance between like organs in ferns. The tabulation of the data and the calculation of the constants are again due to Dr. LEE.

Ceterach.

Number of			Mean No. of lobes.	S. D.	Coefficient of variation	S. D. of array.	Percentage variation.	Correlation
Plants.	Fronds.	Pairs.						
99	999	9098	23·1677 ± ·0902	4·2278 ± ·0638	18·2485	3·2795	77·57	·6311[± ·0128]

The distribution of frequency of the fronds is almost the same as that of the pairs, since the attempt was made to take 10 fronds; actually 80 plants had 10, 14 had 11, and 5 only 9 fronds.

Frequency from Pairs.

No. of lobes.	13.	14.	15.	16.	17.	18.	19.	20.	21.	22.	23.	24.
Frequency	55	73	146	229	375	418	584	575	811	726	800	805

No. of lobes.	25.	26.	27.	28.	29	30.	31.	32.	33.	34.	35.	36.	37.	38.	39.	Total
Frequency	863	679	668	386	288	215	184	91	37	45	27	0	9	0	9	9098

* Miss FRY writes: "It is very difficult to keep to a standard of counting for the tip, but I have tried to do so."

TABLE XIX.—Ceterach, Somersetshire.

Number of Lobes in First Frond.

Number of Lobes in Second Frond	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	Totals
Totals	55	78	146	229	375	418	584	675	811	726	900	905	968	679	698	568	866	288	215	194	91	97	45	27	9	0	9	9096

Number of Lobes in Second Frond.

It will be seen at once that while in the case of ceterach the variability has fallen to less than half that of the hartstongue—being close to that of the *Nigella Hispanica* and less than that of holly—still the measure of the variability, *i.e.*, the degree of sameness in like organs, is to the limit of accuracy defined by its probable error identical with that of hartstongue. Considering the difficulties arising from environment, difference of age, and from standards of reckoning, this agreement is remarkable. It seems to me that the basis of the identity, the bulk of the agreement, must have other sources than a balance between the resemblances produced in undifferentiated like organs by such diverse causes as environment on the one hand and age on the other acting upon the individual plants. I hope that further investigations upon ferns may be forthcoming. Table XIX. contains the classified data for ceterach.

Section V.—*Onions* (Allium cepa), *Great Hampden.*

(19.) The veins in the successive tunics of onions occurred to me as a possible character for computation, no differentiation of the number of veins with the position of the tunic being discoverable. The number of tunics on which the veins were countable was about six, but the process of counting itself was for more than one reason a most trying one. As there were so few "like organs" to each individual, it was needful to deal with at least 200. I accordingly purchased about 250 English onions; these were in three groups, and came from as many cottage gardens. Their average horizontal diameter was about 2 inches, and they appeared very uniform in character. It is quite possible, however, that there may be some heterogeneity introduced into the series by this mixture of onions from different sources, but I did not at the time see my way to obtaining the whole series from one homogeneous sowing.* The entire work of counting and calculating on 200 of these onions was most valiantly undertaken by Dr. Lee.† Each onion was cut through horizontally; the tunics then came easily apart, and these were held up to the light, and the number of veins, ranging from 18 to 54, then counted and recorded. The following table contains the chief numerical constants :—

---

* The onions were purchased at an auction following a harvest festival; the great bulk of them had formed a large chancel crucifix constructed of two different species from one cottage garden. The English onions from this crucifix formed a most homogeneous group, having been already selected for uniformity of size and appearance. I supplemented these by the purchase of two other smaller "offerings," closely resembling them.

† If any of my readers wish to appreciate the purely *physical* difficulties of the task, they should try the effect of slicing and counting four or five onions, and they will, I think, be inclined, as I soon was, to give up the task in despair.

English Onion.

Number of			Mean No. of veins	S. D.	Coefficient of variation.	S. D. of array.	Percentage variation.	Correlation.
Plants.	Tunics.	Pairs.						
200	1085	4924	29·8725 ± ·1066	5·2060 ± ·0754	17·4274	4·1222	79·18	·6108 [± ·0128]

The actual number of tunics with veins countable was made up as follows: 121 onions with 6, 44 with 5, 34 with 4, and 1 with only 3 available tunics. It will be seen that the onion has a degree of resemblance in its undifferentiated-like organs closely resembling that of the Chelsea Shirley poppies, of the Dorsetshire hollies (without No. 91), or the mixed Spanish chestnuts, and somewhat less than that found for the two series of ferns. Its variability approaches closely that of the wild poppy. On the whole it fits well into our series of results, the somewhat high value of the correlation approaching those of several sets, the homogeneity of which is not entirely above suspicion. The accompanying table gives the classified data from which the constants were deduced.

SECTION VI.—*Leguminous Plants.*

(20.) I had at an early stage considered that the counting of seeds in the pods of leguminous plants would be a very easy manner of testing the intensity of homotypic correlation. But as the number of seeds fully developed would depend on the extent to which the flower had been fertilised, it seemed to me, after further thinking about the problem, that unless I counted all the abortive as well as the fully developed seeds, I should find the factor of homotyposis weakened by the external chances upon which I thought cross-fertilisation must depend, and which I feared would have nothing in the main to do with the individuality of the plant. I made some attempts to count all abortive as well as fully developed seeds in certain species, but I found the task not only very laborious, but my estimates doubtful. Examining for another purpose DARWIN'S 'Cross- and Self-Fertilisation of Plants.' I was struck by the fact that he did not hesitate for the purpose of comparing their vigour to count the seeds of plants living in *the open and all alike subject to free visitation from insects and bees.* He distinctly states (3rd Edn., p. 115) that "the difference in the number of the contained seeds* must depend upon the constitution of the plants." This view of the matter impressed me without entirely removing my earlier doubts. Given a number of plants living under the like conditions, and with a superabundant visitation from insects, there would be ample and equal chance of fertilisation for all

* DARWIN in all cases appears to have calculated the good or fully developed seed.

TABLE XX.—English Onion, Great Hampden.

Number of Veins in First Tuber.

[P. Jan. p. 384.

of them, and excess or defect from the average number of fertile seeds would depend only on the constitution of the individual plant. It would seem accordingly that in the case of cross-fertilisation ample visitation was a *sine quâ non*, and this led me to select broom in the first place.

Broom (Cytisus Scoparius), *Danby Dale.*—We collected 10 pods from each of 120 broom bushes. These were growing within some quarter of a mile of each other, on the roadside at Botton, towards the head of Danby Dale, Yorkshire.

I roughly estimated that if abortive seeds were to be included, 16 would be about the modal number of ovules ; the actual average of fully developed seeds in 1200 pods was 9·6425, with a modal value at 9, so that some 6 to 7 seeds failed of fertilisation on the average in each pod. The following is the frequency distribution of the seeds in the pods :—

	3	4.	5	6	7	8	9.	10	11	12	13	14.	15	16	17	18.	19	20
Frequency	29	47	54	77	98	126	186	134	108	100	98	67	48	33	17	5	4	1

I came across no pods with an entire absence of seeds, although such may exist. I refer to this because the reader might think that such were discarded. The distribution is fairly regular, but it is clear that the variation is very large, the distribution being very flat topped. As in nearly all the cases that I have dealt with in this memoir, it is markedly skew. I reserve, however, for the present the full consideration of variation in the vegetable world, as my data extend far beyond the material considered in this paper, which is limited to the cases in which the homotypic correlation has also been worked out—a far more laborious enquiry. I would only remark that among plants and trees I know of nothing approximating even to the "normal law," and that in many cases we appear to have mixtures of local races hardly yet differentiated by the botanist.

The following table gives the constants for broom :—

Broom. Seeds in Pods.

Number of			Mean No. of seeds.	S. D. of seeds.	Coefficient of variation.	S. D. of array.	Percentage variation	Correlation.
Plants.	Pods.	Pairs.						
120	1200	10,800	9·6425 ± ·0691	3·54655 ± ·0488	36·7804	3·22595	90·96	·4155 [± ·0161]

The actual distribution of pairs is given in the table on the following page. It will be seen that some of the arrays are rather irregular, but in its results I look upon broom to be as satisfactory as any material I have dealt with. In the first place, the table fully bears out the conclusion drawn from the simple frequency

TABLE XXI.—Broom. Danby Dale.

Number of Seeds in First Pod.

No. of Seeds in Second Pod	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.	12.	13.	14.	15.	16.	17.	18.	19.	20.	Totals.
1	8	10	9	4	2	7	9	9	5	5	3	1									72
2	10	8	20	24	11	17	13	6	7	8	5	2	1	2	0	1					135
3	9	20	34	49	25	34	16	23	18	15	9	6	4	0	0	0	0	1	2		261
4	4	24	49	52	39	52	37	33	40	39	19	11	11	11	1	1	1	0	2		423
5	3	34	25	39	40	53	60	41	61	53	37	25	16	13	4	5	4	3	0		486
6	7	17	34	52	53	64	63	77	89	76	50	42	33	21	7	7	8	0	4		693
7	9	13	16	37	60	63	88	97	116	103	84	63	53	35	25	14	8	0	3		882
8	9	6	22	33	41	77	97	144	133	141	106	97	106	64	36	14	17	4	1		1134
9	5	6	18	40	61	89	116	133	136	136	125	97	86	67	47	27	7	3	4	1	1224
10	3	8	15	39	53	76	103	141	136	148	122	107	96	68	67	28	14	0	6	0	1206
11	1	5	9	19	37	50	84	106	125	122	88	77	91	51	38	20	16	8	7	1	997
12		2	6	11	25	42	63	97	97	107	77	88	107	72	33	31	13	4	5	0	900
13		1	4	11	13	21	53	106	86	96	91	107	82	76	47	37	17	2	2	1	882
14		2	0	1	4	7	35	64	67	68	51	72	76	39	61	35	16	4		2	603
15			0	1	5	7	25	36	47	38	32	47	61	39	39	28	10	5		2	432
16			0		0	1	14	14	7	17	20	31	37	35	40	20	4	4		1	297
17					1	0	4	8	16	3	7	16	16	13	28	16	7	2		0	153
18							3	0	4	2	0	8	4	2	17	5		5			45
19									2	1	0	4	3	1	4	6		0			36
20									1	0	1	1	0		2	2					9
Totals…	72	135	261	423	486	693	882	1134	1224	1206	997	900	882	603	432	297	153	45	36	9	10800

Number of Seeds in Second Pod.

distribution of the high degree of variability in the broom pod. We have so far reached nothing but the *sori* on the Hartstongue fern with anything like such a high coefficient of variation. We might expect, if Mr. ADAM SEDGEWICK's view were correct and great variation meant small intensity of heredity, that homotyposis, of which heredity is only a special case, would also be small if variation were large. But we find nothing of the kind. The resemblance of undifferentiated like organs reaches the value ·4155, fully equal to that of the *pinnæ* of the ash, which have less than half the variability of broom pods. This value is close also to the ·4000 given by the Ancestral Law of Heredity for the degree of resemblance between brothers. In the next place this normal behaviour as to the degree of resemblance of like organs is associated with a variability in the individual which amounts to upwards of 90 per cent. of the racial variability. It is impossible to form standard deviations for groups of 10, but if the reader will reduce any array of the above table to the number 10, rejecting all fractions of unity, he will obtain quite fair samples of what I actually found in the case of individual pods. I think we may feel fairly confident that the variability of a race is not in inverse proportion to either its heredity or its homotyposis.

It will be seen that broom, so far as it goes, supports the view held by DARWIN that the number of ripe seeds is a measure of individual constitution. At the same time the question of self-fertilisation arises. Would completely self-fertilised plants exhibit full homotyposis? May not outside influences—wind, shaking due to visits of insects or other causes of a random character—be also needful for the ripening of the seeds even in the case of self-fertilisation? Again, can we always suppose that a plentiful visitation will take place in the case of all cross-fertilised plants? I must confess that the value for the homotyposis found in the case of broom did not suffice to remove all my doubts as to any character depending on fertilisation being a suitable one for the determination of the intensity of homotyposis. The subject, however, is one of such great interest that it deserves an independent and fuller treatment than can be provided here.*

V. CROSS-HOMOTYPIC CORRELATION.

(21.) *Mushroom* (Agaricus campestris).—A more complete study was now made of two species—the mushroom and the ivy. We found some difficulty in discovering two easily measurable or countable characters in one organ, neither of which was largely influenced by the growth of the organ or the age of the organism. In selecting an organ in the mushroom, I was guided by the desire to take a simple organism, and an organ upon which fertilisation had no influence. The gill of the mushroom seemed to satisfy to some extent these conditions. Had we been able to grow our own mushrooms we might have succeeded in taking them all at the same stage of

---

* This has been to some extent provided in an Appendix to this memoir, added since its completion.

development, and our results would in this case have been more homogeneous and thus more satisfactory. But simply gathering them in the fields we could only make a very rough approximation to uniformity in age or development. We aimed at a condition in which the underskin had disappeared and the gills had lost all signs of a crease or notch. Dr. LEE and I collected more than 100 mushrooms in the Great and Little Hampdens. With the assistance of my wife we cut 10 gills out of each of these, distributed, as far as the condition of the mushroom would allow, uniformly round the axis. These gills were then placed on ruled paper, the horizontal lines of which served to mark the maximum length of the gill. This was marked by two fine needle-pricks, one at either end. Two further needle-pricks were now made at the points farthest removed from this length line, one on the upper and one on the lower contour of the gill. Thus by drawing through the four needle points lines parallel and perpendicular to the paper-ruling we obtained a rectangle approximately circumscribing the gill. The length and breadth of this rectangle are what I shall term the length and breadth of the gill. At first I endeavoured to take these measurements by the use of a micrometer microscope, but this method was very laborious, and gave an apparent exactness wholly out of keeping with the somewhat rough and ready manipulation of the gill. Even when the gill itself was placed under the microscope the gill length could only be placed parallel to the spider wires by a somewhat vague appreciation, and the upper boundary of the gill was under the lens rather less definite than without it. Accordingly we resorted to the needle-pricks and the ruling by fine lines of the above-mentioned rectangle. I determined, after various trials with engraved glass scales, &c., the length and breadth of these rectangles by the use of a pair of proportional compasses, set to multiply by ten, and an ordinary millimetre scale. I was thus able to get readings to ·01 of a millimetre, which agreed well on repetition. As the millimetre itself was afterwards adopted as the unit of grouping, I do not think the errors of manipulation will seriously affect our results. The needle-prick method enabled us to deal with the mushrooms quite fresh and reserve the measuring till another season.* Dr. LEE found what we may term the *gill-index*, *i.e.*, the ratio of length to breadth for the 1070 gills, by aid of a FULLER's slide rule. We had thus three characters to deal with—length, breadth, and index. Of these we considered that the first two would undoubtedly be largely influenced by the stage of development of the mushroom, but we hoped that the index (like, for example, the cephalic index of children over two years of age) would not be much influenced by the growth factor. On the other hand, the correlation of lengths of gills or the correlation of breadths of gills from the same mushroom would of course be influenced by growth as well as

---

* In all such systems of mass-measurement, the method adopted has to be chosen with due regard to time and eyesight. The actual measurement of the gills occupied me about ten full days, the microscopic method would have taken about four times the time, even if my eyes would allow of such continuous working and the mushrooms could have been preserved fresh.

by individuality. Thus we should find the degree of resemblance between two gills of the same mushroom, when measured by length or breadth, to be influenced not only by the individuality of the mushroom, but by the stage of its growth, and accordingly likely to exceed, and possibly considerably exceed, the value about ·5, which we have determined in several other cases.*

In order to make a more complete study of the influence of stages of growth on the degree of resemblance of undifferentiated like organs, Dr. LEE with her characteristic energy undertook the five needful tables.

They are the following :- -

(i.) Organic correlation of the length and breadth of the same gill.
(ii.) Homotypic correlation of the lengths of two gills of the same mushroom.
(iii.) Homotypic correlation of the breadths of two gills of the same mushroom.
(iv.) Cross homotypic correlation of the length of one, and the breadth of a second gill of the same mushroom.
(v.) Homotypic correlation of the indices of two gills of the same mushroom.

Of these tables the cross homotypic correlation between length and breadth is a very laborious business, offering no means of readily testing the accuracy of the entries until the whole table is completed and its columns and rows added up. Ultimately the frequency distributions (as given by row or column marked totals) for the lengths and breadths in tables (ii.), (iii.), and (iv.), ought to be $n - 1$ times the frequency distributions of lengths and breadths as given by the like row and column in the organic correlation table (i.), n being the number of organs measured in each individual ; in our case this is ten.

I begin by considering the results obtained for the degree of resemblance of gill-indices, i.e., Table (v.) referred to above. The following were the results obtained :—

Mushroom. Gill Indices.

Number of			Mean index.	S. D.	Coefficient of variation.	S. D. of array.	Percentage variation.	Correlation.
Mushrooms.	Gills.	Pairs.						
107	1070	9630	2·8695 ± ·0108	·5246 ± ·0076	18·2819	·4385	83·58	·5490 [± ·0144]

The frequency distribution of the gills was the following :—

* The absurdity of disregarding the growth-factor may be realised if we consider how much the degree of resemblance in brethren would be increased if we took pairs of brothers, differing, say, by less than two years, and these pairs were scattered at all ages from four to twenty !

2 x 2

Gill Index.

Frequency	1	1	4	7	13	18	30	39	55	78	101	90	100	93	78

Frequency	66	62	59	43	31	43	15	20	10	9	1	8	3	3	4

												Total.
Frequency	0	3	1	2	0	1	0	1	0	1	0	1	1070

This distribution is rather irregular and may suggest heterogeneity; the tail is somewhat inordinately extended. But the numerical constants given in the table above are quite comparable with the best of our other series, and are rather better than I had anticipated *a priori*, for I expected the diverse stages of growth would influence the index as well as the absolute measures. It would be of great interest to repeat the series on an artificially grown bed of mushrooms, taking each at the same stage.

It will be seen that the mushroom, with degree of resemblance of like organs about that of the Spanish chestnut, has a variability comparable with that of the ash. The variation of the individual is even in such a comparatively simple organism more than 80 p.c. of that of the race. It will thus be clear that the mushroom takes, so far as our present investigation is concerned, no special place apart in the vegetable world, but quantitatively is comparable with highly developed organisms like trees. The accompanying table, Table XXII., gives the data for the measurement of the gills.

I now turn to the relationships of the absolute measurements, which, as I have noted, we have good grounds for believing are much affected by growth.

The numerical constants are summed up in the following table, the dimensions being given in millimetres :—

TABLE XVII.—Madrasese (Hindus).

Index of First Gill.

4·5495.	4·6495.	4·7495.	4·8495.	4·9495.	5·0495.	5·1495.	5·2495.	5·3495.	5·4495.	5·5495.	5·6495.	Totals.
—	—	—	—	—	—	—	—	—	—	—	—	9
—	—	—	—	—	—	—	—	—	—	—	—	9
—	—	—	—	—	—	—	—	—	—	—	—	36
—	—	—	—	—	—	—	—	—	—	—	—	63
—	—	—	—	—	—	—	—	—	—	—	—	117
—	—	—	—	—	—	—	—	—	—	—	—	162
—	—	—	—	—	—	—	—	—	—	—	—	270
—	—	—	—	—	—	—	—	—	—	—	—	351
—	—	—	—	—	—	—	—	—	—	—	—	495
—	—	—	—	—	—	—	—	—	—	—	—	702
0	0	0	0	0	0	0	0	0	1	—	—	909
0	1	0	0	0	0	0	0	0	2	—	—	810
0	1	0	1	0	0	0	0	0	1	—	—	900
0	0	0	1	0	0	0	0	0	1	—	—	837
0	3	0	1	0	0	0	0	0	1	—	—	657
0	0	0	0	0	0	0	0	0	0	—	—	594
0	2	1	1	0	1	0	1	0	1	0	1	558
0	3	0	1	0	0	0	0	0	0	0	1	351
0	3	1	3	0	1	0	1	0	1	0	1	387
0	2	1	2	0	1	0	1	0	1	0	0	279
0	3	0	1	0	0	0	0	0	0	0	0	387
0	0	0	1	0	0	0	0	0	0	0	0	135
0	1	0	1	0	0	0	0	0	0	0	0	180
0	1	0	1	0	0	0	0	0	0	0	0	90
0	2	2	1	0	2	0	2	0	0	0	2	81
0	0	0	0	0	0	0	0	0	0	0	0	9
0	0	0	0	0	0	0	0	0	0	0	0	72
0	0	0	0	0	0	0	0	0	0	0	0	27
0	0	0	0	0	0	0	0	0	0	0	0	27
0	0	0	2	0	0	0	0	0	0	0	0	36
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	1	0	1	0	1	0	0	0	1	27
0	1	0	0	0	1	0	1	0	0	0	1	9
0	1	0	0	0	0	0	0	0	0	0	0	18
0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	1	0	0	0	0	1	0	0	0	1	9
0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	1	0	0	0	0	0	0	0	0	1	9
0	0	0	0	0	0	0	0	—	—	—	—	0
0	0	0	0	0	0	0	0	—	—	—	—	9
0	0	0	0	0	0	0	0	—	—	—	—	0
0	1	1	0	0	1	0	1	—	—	—	—	9
0	27	9	18	0	9	0	9	0	9	0	9	9630

Absolute Dimensions of Mushroom Gills.

Character.	Mean.	S D.	Coefficient of variation.	S. D. of array.	Percentage variation.	Direct homo-typic correlation.
Length. .	19·1847 ± ·0831	4·0282 ± ·0587	20·9969	2·0510	50·92	·8607 [± ·0053]
Breadth .	6·8529 ± ·0331	1·6047 ± ·0234	23·4164	1·0859	67·67	·7363 [± ·0094]

Cross-correlations.

Length and breadth, organic . .	·7000 ± ·0105
Length and breadth, homotypic .	·6275[± ·0125]

The results here given seem to be of considerable suggestiveness. While the two means and standard deviations differ very considerably, the coefficients of variation for the length and breadth of the gill are approximately equal, and in addition very high. This is precisely what we might expect when dealing with a series of organisms in various stages of growth. Further, the variability of the individual as compared with that of the race is reduced immensely below the 80 to 90 per cent. with which our previous investigations have made us familiar. This might again be anticipated as a result of heterogeneity in the stage of growth. The organic correlation of length and breadth, as well as the homotypic correlations of the same two characters, both direct and cross, are all very high, and the latter are much beyond what we might legitimately put to the credit of the pure homotyposis factor. It seems, therefore, somewhat hazardous to consider how far they satisfy the relationship suggested in the introductory part of this paper. The direct homotypic correlations for length and breadth are not approximately equal as we might expect; this suggests that growth may be a somewhat more marked factor in the length than the breadth of the gill. If we take the product of direct homotypic correlation with the organic correlation, this ought, on our hypothesis, to give us roughly the cross-homotypic correlation. We have the following results :—

Product of direct homotypic length correlation with organic length-
breadth correlation . . . . = ·6025.

Product of direct homotypic breadth correlation with organic length-
breadth correlation . . . . = ·5154.

Product of mean of direct homotypic breadth and length correlation with
organic length-breadth correlation . . . . = ·5590.

These numbers have to be compared with the cross homotypic length-breadth correlation, *i.e.*, with ·6275.

TABLE XXIII.—Mushroom. Length of Gill.

Length of First Gill.

Length of Second Gill.

Hundredths millimetre	800·5–899·5	900·5–1099·5	1000·5–1199·5	1199·5–1299·5	1299·5–1399·5	1399·5–1499·5	1499·5–1599·5	1599·5–1699·5	1699·5–1799·5	1799·5–1899·5	1899·5–2099·5	2099·5–2199·5	2199·5–2299·5	2299·5–2399·5	2399·5–2499·5	2499·5–2599·5	2599·5–2699·5	2699·5–2799·5	2799·5–2899·5	2899·5–2999·5	2999·5–3099·5	3099·5–3199·5	3199·5–3299·5	3299·5–3399·5	3399·5–3499·5	3499·5–3599·5	Totals
Totals	18	37	109	180	342	648	711	909	1044	964	673	756	618	551	353	360	543	343	111	34	31	36	33	42	0	9	8633

TABLE XXIV.—Mushroom. Breadth of Gill.

Breadth of First Gill.

Breadth of Second Gill.

Hundreds of millimetres	299·5–349·5	349·5–399·5	399·5–449·5	449·5–499·5	499·5–549·5	549·5–599·5	599·5–649·5	649·5–699·5	699·5–749·5	749·5–799·5	799·5–849·5	849·5–899·5	899·5–949·5	949·5–999·5	999·5–1049·5	1049·5–1099·5	1099·5–1149·5	1149·5–1199·5	1199·5–1249·5	1249·5–1299·5	Totals
Totals	72	162	306	468	1044	972	1215	1259	1008	888	810	504	309	361	162	117	18	9	27	9	**9690**
1249·5–1299·5									1	0	1	0		4	2						9
1199·5–1249·5								0	0	0	3	8	0	3	8	6					27
1149·5–1199·5								0	0	0	1	1	0	8	1	0	8				9
1099·5–1149·5			2	0	0	2	0	0	1	0	1	1	2	3	3	0	1	3			18
1049·5–1099·5			0	0	0	2	4	0	5	0	16	18	19	13	14	3	3	0	2		117
999·5–1049·5			4	0	0	4	0	2	12	19	30	26	22	24	13	2	0	0	4		162
949·5–999·5			3	0	0	5	11	14	15	46	57	28	22	19	2	1	8	0			361
899·5–949·5		2	0	1	6	10	18	38	74	55	62	57	26	18	1	1	3	1			360
849·5–899·5		5	2	4	11	26	49	59	116	82	55	46	30	16	1	0	0	1			504
799·5–849·5	1	1	5	10	23	68	108	157	170	116	74	43	19	9	0	0	0	0			810
749·5–799·5	1	7	24	34	56	92	165	170	157	59	33	14	12	5	1	0	0	1			888
699·5–749·5	2	14	29	55	75	129	197	164	135	108	49	18	11	2	0	0					1008
649·5–699·5	3	14	34	123	167	243	296	197	92	68	30	10	6	0	4	0					1259
599·5–649·5	8	39	60	181	192	253	243	169	56	33	11	10	5	4	2						1215
549·5–599·5	15	40	67	206	168	192	167	73	34	10	4	1	0	0	0						972
499·5–549·5	55	69	88	250	206	181	128	55	24	5	2	0	0	0							1044
449·5–499·5	32	56	70	88	67	60	24	29	7	5	5	2	4	0	2						468
399·5–449·5	22	44	56	63	10	30	14	14	1	1	1										306
349·5–399·5	32	32	33	38	25	15	8	3	2	1											162
299·5–349·5	14	29	4	15	11	3	2	1													72

TABLE XXV.—Mushroom. Length and Breadth of Gill, Organic Correlation.

Length of Gill.

Breadth of Gill.

Hundredths of millimetres.	899·5–999·5	999·5–1099·5	1099·5–1199·5	1199·5–1299·5	1299·5–1399·5	1399·5–1499·5	1499·5–1599·5	1599·5–1699·5	1699·5–1799·5	1799·5–1899·5	1899·5–1999·5	1999·5–2099·5	2099·5–2199·5	2199·5–2299·5	2299·5–2399·5	2399·5–2499·5	2499·5–2599·5	2599·5–2699·5	2699·5–2799·5	2799·5–2899·5	2899·5–2999·5	2999·5–3099·5	3099·5–3199·5	3199·5–3299·5	3299·5–3399·5	3399·5–3499·5	Totals
299·5– 349·5	1	0	4	0	1	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	8
349·5– 399·5	0	2	0	3	7	1	1	0	1	1	1	0	0	1	—	—	—	—	—	—	—	—	—	—	—	—	18
399·5– 449·5	0	1	2	3	5	5	7	3	1	3	1	0	2	1	—	—	—	—	—	—	—	—	—	—	—	—	34
449·5– 499·5	1	0	7	2	5	7	9	6	8	4	1	0	1	1	—	—	—	—	—	—	—	—	—	—	—	—	52
499·5– 549·5	—	—	6	5	2	16	16	22	17	17	10	2	2	1	—	—	—	—	—	—	—	—	—	—	—	—	116
549·5– 599·5	—	—	0	2	6	14	12	21	15	18	5	8	3	1	1	0	2	—	—	—	—	—	—	—	—	—	108
599·5– 649·5	—	—	1	2	6	11	13	20	25	18	16	4	7	2	2	2	1	3	1	0	1	—	—	—	—	—	135
649·5– 699·5	—	—	1	1	3	12	6	16	28	16	15	20	9	4	4	1	1	2	1	0	1	—	—	—	—	—	141
699·5– 749·5	—	—	—	1	3	3	6	5	10	21	17	14	10	9	3	5	3	1	1	0	0	—	—	—	—	—	112
749·5– 799·5	—	—	—	1	0	1	2	4	5	8	14	14	22	10	2	5	2	5	1	1	0	—	—	—	—	—	98
799·5– 849·5	—	—	—	—	—	1	5	2	3	2	11	13	16	11	9	8	3	2	1	2	0	0	0	1	—	—	90
849·5– 899·5	—	—	—	—	—	—	1	1	1	2	2	5	14	10	4	7	1	3	0	1	3	1	0	0	—	—	56
899·5– 949·5	—	—	—	—	—	—	—	0	1	1	2	2	3	5	8	5	5	5	1	1	1	0	0	1	—	—	41
949·5– 999·5	—	—	—	—	—	—	—	1	0	0	2	0	2	6	2	4	5	2	3	0	0	1	0	1	—	—	29
999·5–1049·5	—	—	—	—	—	—	—	—	1	0	0	0	0	4	1	2	3	2	2	0	0	0	2	0	0	1	18
1049·5–1099·5	—	—	—	—	—	—	—	—	—	—	—	2	0	3	0	1	1	1	1	1	2	0	0	1	—	—	13
1099·5–1149·5	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	0	0	0	0	0	0	1	0	0	—	—	2
1149·5–1199·5	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	0	0	0	0	0	0	—	—	1
1199·5–1249·5	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0	0	1	1	0	1	—	—	3
1249·5–1299·5	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	1
Totals	2	3	21	20	38	72	79	101	116	106	97	84	91	69	37	40	27	27	13	6	9	4	2	5	0	1	1070

TABLE XXVI.—Mushroom. Length and Breadth of Gill, Homotypic Correlation.

Length of First Gill.

Breadth of Second Gill. Hundredths of millimetres	899·5–999·5	999·5–1099·5	1099·5–1199·5	1199·5–1299·5	1299·5–1399·5	1399·5–1499·5	1499·5–1599·5	1599·5–1699·5	1699·5–1799·5	1799·5–1899·5	1899·5–1999·5	1999·5–2099·5	2099·5–2199·5	2199·5–2299·5	2299·5–2399·5	2399·5–2499·5	2499·5–2599·5	2599·5–2699·5	2699·5–2799·5	2799·5–2899·5	2899·5–2999·5	2999·5–3099·5	3099·5–3199·5	3199·5–3299·5	3299·5–3399·5	3399·5–3499·5	Totals.
299·5– 349	3	4	22	9	10	8	7	6	3	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	72
349·5– 399	4	7	24	10	31	24	17	10	7	8	4	2	6	5	3	—	—	—	—	—	—	—	—	—	—	—	162
399·5– 449	0	5	14	12	18	45	60	49	29	31	13	3	12	12	3	—	—	—	—	—	—	—	—	—	—	—	306
449·5– 499	3	5	41	20	64	55	72	55	40	38	24	8	21	12	10	—	—	—	—	—	—	—	—	—	—	—	468
499·5– 549	2	4	44	29	39	129	112	158	158	141	107	51	36	16	5	2	2	7	0	0	2	—	—	—	—	—	1044
549·5– 599	2	2	11	24	57	107	112	159	150	123	87	47	39	25	6	2	3	11	1	1	3	—	—	—	—	—	972
599·5– 649	2	0	19	34	58	107	127	179	197	142	115	78	52	25	15	15	8	35	4	4	4	—	—	—	—	—	1215
649·5– 699	1	0	10	24	41	86	81	149	195	175	136	140	104	60	17	8	12	14	4	4	8	—	—	—	—	—	1269
699·5– 749	1	0	4	17	27	53	55	64	108	124	120	104	127	82	23	41	21	16	6	10	5	—	—	—	—	—	1008
749·5– 799	—	—	—	1	2	10	27	39	71	56	88	136	162	92	32	42	23	14	16	7	3	1	2	3	0	1	828
799·5– 849	—	—	—	—	—	24	30	25	39	58	86	114	115	81	51	70	53	32	16	10	5	1	2	2	0	1	810
849·5– 899	—	—	—	—	—	—	7	11	22	30	47	33	69	70	51	60	48	22	14	5	7	1	4	6	0	2	504
899·5– 949	—	—	—	—	—	—	0	0	9	12	19	22	25	53	57	55	35	32	18	8	6	5	4	7	0	2	369
949·5– 999	—	—	—	—	—	—	2	2	8	12	13	14	26	42	35	35	19	22	12	2	7	3	2	4	0	1	261
999·5–1049	—	—	—	—	—	—	2	3	6	7	4	1	14	24	16	17	13	12	15	2	8	3	4	9	0	2	162
1049·5–1099	—	—	—	—	—	—	—	—	0	0	9	3	10	18	7	12	10	15	9	1	9	9	0	5	—	—	117
1099·5–1149	—	—	—	—	—	—	—	—	2	2	1	0	1	1	2	0	0	2	0	0	3	2	0	2	—	—	18
1149·5–1199	—	—	—	—	—	—	—	—	—	—	—	—	—	0	0	0	0	1	0	0	3	3	0	2	—	—	9
1199·5–1249	—	—	—	—	—	—	—	—	—	—	—	—	—	0	0	0	0	6	0	0	8	8	0	5	—	—	27
1249·5–1299	—	—	—	—	—	—	—	—	—	—	—	—	—	3	0	1	1	2	2	—	—	—	—	—	—	—	9
Totals. .	18	27	189	180	312	648	711	909	1044	954	873	756	819	621	333	360	243	248	117	54	81	36	18	45	0	9	9630

It will be seen at once that they are sensibly too small (say, ·56 as compared with ·63). But having recognised the influence of heterogeneity in the growth stages of our material, the divergence is, I take it, of a magnitude rather to confirm than confute the hypothesis as to the relationship of direct and cross homotypic correlations. It must be remembered that we are dealing with *average* results. Our main proposition is that heredity is not a factor of life peculiar to sexual reproduction, but merely a phase of the larger factor, which we have termed homotyposis, or the tendency of the individual to put forth undifferentiated like organs with a certain degree of resemblance. Our subsidiary proposition is : that if the direct homotypic correlation oscillates about a certain mean value, and the cross homotypic correlation be the product of organic and direct homotypic correlations, then we should expect to find the average degree of resemblance of brethren equal to the average degree of resemblance of undifferentiated like organs in the individual.

All then I think we can safely say for the gills of mushrooms is that the relation of the direct and cross homotypic correlations is by no means such that it condemns our hypothesis, or enforces us to reject our subsidiary proposition. The difficulty lies in finding *adult* organisms with undifferentiated like organs with two characters in sufficient quantity, easily counted or measured, upon which the hypothesis can be tested. The investigation of the length and breadth of ivy leaves, to which we shall soon turn, suffers to some extent from the same defects as that for the gills of mushrooms.

Tables XXIII.–XXVI. contain the length-breadth data for the mushrooms. The frequency distributions for length and breadth are given in the last row and column of Table XXV. above, and exhibit in their irregularity something of the heterogeneity of growth to which I have referred. Of the two distributions, I consider that for the breadth as the more irregular and consequently the less satisfactory. It is, of course, harder to determine a definite breadth for the gill than a definite length, and I should be well content to compare the product of the homotypic length correlation and the organic length-breadth correlation, *i.e.*, ·6025, with the cross homotypic length-breadth correlation, *i.e.*, ·6275, as the best basis for our subsidiary proposition available from these measurements on mushrooms.

(22.) *Wild Ivy* (Hedera Helix).—This series was originally undertaken by Dr. E. WARREN, his object being to measure the degree of resemblance between ivy leaves on the same plant, using as his character the index, or ratio of length to maximum breadth. It was hoped that in this manner, the growth factor might be fairly well eliminated.

The figures (page 348) indicate how the breadth and length were determined by Dr. WARREN in leaves of somewhat different shape.

It will be seen that, as in the case of the gill of the mushroom, the breadth is taken as the maximum breadth between tangents to the contour parallel to the length, and these tangents do not necessarily go through points like *a, a*. Twenty-five leaves were taken from each of 42 plants by Dr. WARREN, in the neighbourhood of Canterbury,

TABLE XXVII.—Ivy. Leaf Index.

Index of First Leaf

Index	·295-·395	·395-·495	·495-·595	·595-·695	·695-·795	·795-·895	·895-·995	·995-1·095	1·095-1·195	1·195-1·295	1·295-1·395	1·395-1 495	1·495-1·595	1·595-1·605	Totals
·295- ·395	—	3	5	6	11	10	6	4	2	0	1	—	—	—	48
·395- ·495	3	8	14	31	55	71	36	12	5	3	0	2	—	—	240
·495- ·595	5	14	36	132	294	267	98	40	23	2	1	0	—	—	912
·595- ·695	6	31	132	1048	2186	1556	633	243	92	30	14	4	1	—	5976
·695- ·795	11	55	294	2186	5512	4728	2043	878	314	89	41	16	8	1	16176
·795- ·895	10	71	267	1556	4728	5536	3193	1420	578	195	102	40	12	4	17712
·895- ·995	6	36	98	633	2043	3193	2466	1182	459	190	101	41	12	4	10464
·995- 1·095	4	12	40	243	878	1420	1182	628	303	130	68	39	15	6	4968
1·095-1·195	2	5	23	92	314	578	459	303	142	70	33	26	13	4	2064
1·195-1·295	0	3	2	30	89	195	190	130	70	36	17	2	4	0	768
1·295-1·395	1	0	1	14	41	102	101	68	33	17	4	1	1	0	384
1·395-1 495	—	2	0	4	16	40	41	39	26	2	1	12	5	4	192
1·495-1·595	—	—	—	1	8	12	12	15	13	4	1	5	0	1	72
1·595-1·605	—	—	—	—	1	4	4	6	4	0	0	4	1	0	24
Totals	48	240	912	5976	16176	17712	10464	4968	2064	768	384	192	72	24	60000

Index of Second Leaf.

and 25 leaves from each of 58 plants by me, in the neighbourhood of Great Hampden in Buckinghamshire. Care was taken to avoid the reproductive shoots with their larger uncut leaves. The Hampden plants were, I feel certain, a fairly homogeneous group, the plants were young runners up the trees, only a few feet in length. The Canterbury plants appear to have had a somewhat larger leaf, although in themselves an apparently homogeneous group.
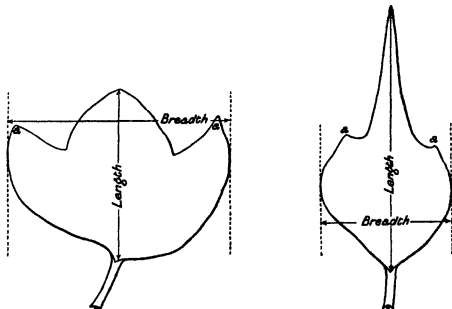
It would undoubtedly have been better to have had the leaves collected by one person from one district; but until the leaves had all been measured and their indices found, I did not know of this difference between the two groups. It was very little noticeable in the table of index distribution, but became manifest to me when finding the homotypic correlation table of the absolute lengths, which I did in two sections, one for either group. The very considerable labour involved in measuring and determining the indices of 2500 leaves precluded a repetition of the work on more homogeneous material. The whole of this labour was undertaken by Dr. WARREN, who further formed the table and determined the correlation coefficient.[*]

I give in the following scheme the frequency distribution of the indices to show its uniformity.

738   136 | 207 | 86 | 32

---

[*] After measuring the gills of only 1070 mushrooms, I very fully appreciate Dr. WARREN's task on the 2500 ivy leaves!

Of course the groupings here are large, but the distribution notwithstanding the mixture of material seems more regular than in some of our other series.

The following scheme gives the constants in the usual manner :

Leaf Index of Wild Ivy.

Number of			Mean index.	S. D.	Coefficient of variation.	S. D. of array.	Percentage variation.	Correlation.
Plants.	Leaves.	Pairs.						
100	2500	60,000	·8473 ± ·0020	·1506 ± ·0014	17·7735	·1449	96·21	·2726 [± ·0125]

It will be observed that the correlation is lower than we might have expected, although the mean of the index correlations for mushroom gills and ivy leaves, *i.e.*, $\frac{1}{2}$ (·5490 + ·2726) = ·4108, is almost identical with the value ·4 given for brothers by the law of ancestral heredity. The low value of the correlation gives of course a high value to the percentage variation, the variation of the individual plant being within 4 per cent. of the racial variation. The variability of the ivy leaf, as judged by this character, has not the intensity which is popularly associated with it ; it is sensibly less than that of the mushroom gill, and about equal to that of the tunics of the onion.

I now pass to the absolute measurements of the leaf.

Dr. WARREN being much pressed with other work, kindly placed all his measurements at my disposal, and I proceeded to draw up the same four tables as in the case of the mushroom gills. The work here was, however, much more laborious as I had to deal with 25 leaves instead of 10 gills, and this involved 300 pairs for each plant, instead of only 45 ! In the course of a fortnight's work 1 had completed the length-breadth organic correlation, the length-length homotypic correlation, and about a third of the length-breadth homotypic correlation. At this point Dr. LEE took the work off my hands and finished the last table and the breadth-breadth homotypic correlation. So that the results are again the product of co-operation.* I give below the data arranged as in the case of the mushroom gills. The dimensions are given in eighths of inches.

---

* We have now systematised the working of these long tables, involving 10,000 to 60,000 entries providing appropriate checks for accuracy up to each stage of construction. It seems unnecessary to describe these here, but we shall be glad to put our experience at the service of any one working at similar problems. At the same time the collection, measurement and formation of a table for the cross homotypic correlation of two characters in 25 undifferentiated like organs of 100 individuals will cost a *single* worker at least three weeks to a month's fairly continuous labour.

Absolute Dimensions of Wild Ivy Leaves.

Character	Mean.	S. D.	Coefficient of variation.	S. D. of array.	Percentage variation.	Correlation.
Length. .	10·9504 ± ·0457	3·3885 ± ·0612	30·9442	2·8033	82·73	·5618 [± ·0092]
Breadth .	13·2148 ± ·0323	4·5384 ± ·0433	34·3430	3·8376	84·56	·5332 [± ·0096]

Cross-Correlations.

Length and breadth, organic . .	·8718 ± ·0032
Length and breadth, homotypic .	·5157 [± ·0099]

These results differ very considerably from those for the mushroom gills. The coefficients of variation are even higher than those for the gills, and I think this is a result of a certain amount of heterogeneity, as well as of the leaves on individual runners not being all quite at the same stage of development. In the case of the deciduous trees the leaves were gathered in the fall of the year, and no further development of veins was possible ; but in the ivy leaves, taken indeed at the same time, the equal development of all the leaves taken from the runner could hardly have been reached when they were gathered. On the other hand the correlations are much lower than in the case of the gills,—much more within the range of the results obtained for other characters. I am inclined to think, therefore, that there has been some balancing of opposing factors here, heterogeneity due to locus of collection and to stages of growth being to some extent counteracted by a differentiation due to position of the leaf on the runner.* I have not included these results for lengths and breadths of ivy leaves in my homotypic series because the values, although well within the range of the other determinations, appear to me to be somewhat fictitiously so. The disturbing factors referred to above seem to me to have also given the cross homotypic correlation a higher value than it ought to have. It will be seen that the correlation of length and breadth of pairs of leaves from the same runner is almost as high as that of the breadths of pairs of leaves from the same runner. I should expect a considerably greater inequality. I regret that the great labour of these cross-correlation investigations has hindered their being carried further than these two cases in the vegetable kingdom, but in the zoological data, which we have at present in hand, I hope to get material less open to criticism than in the cases of mushroom gills and ivy leaves. These two series were originally undertaken with the view of ascertaining how far the use of indices would cancel the influence of the factor of growth. As we have

---

* The reader must of course bear in mind that the leaves were taken from runners which had not reached the tops of their walls or trees, i.e., they were not from reproductive shoots.

TABLE XXVIII.—Wild Ivy. Homotypic Correlation of Lengths of Leaves.

Length of First Leaf.

Eighths of Inches.	1 9/8.–2 9/8.	2 9/8.–3 9/8.	3 9/8.–4 9/8.	4 9/8.–5 9/8.	5 9/8.–6 9/8.	6 9/8.–7 9/8.	7 9/8.–8 9/8.	8 9/8.–9 9/8.	9 9/8.–10 9/8.	10 9/8.–11 9/8.	11 9/8.–12 9/8.	12 9/8.–13 9/8.	13 9/8.–14 9/8.	14 9/8.–15 9/8.	15 9/8.–16 9/8.	16 9/8.–17 9/8.	17 9/8.–18 9/8.	18 9/8.–19 9/8.	19 9/8.–20 9/8.	20 9/8.–21 9/8.	21 9/8.–22 9/8.	22 9/8.–23 9/8.	23 9/8.–24 9/8.	24 9/8.–25 9/8.	Totals.
Totals	96	640	2284	3744	4440	6004	6988	7968	6588	6460	4464	3490	1924	1360	984	648	828	316	216	216	144	120	120	24	60,000

Length of Second Leaf.

TABLE XXIX.—Wild Ivy.   Homotypic Correlation of Breadths of Leaves.

Breadth of First Leaf.

Breadth of Second Leaf.

TABLE XXX.—Wild Ivy. Organic Correlation of Length and Breadth of Leaves.

Length of Leaf (vertical) · *Breadth of Leaf* (horizontal)

Eighths of Inches	Totals
2·95- 3·95	2
3·95- 4·95	18
4·95- 5·95	61
5·95- 6·95	83
6·95- 7·95	117
7·95- 8·95	134
8·95- 9·95	197
9·95-10·95	238
10·95-11·95	233
11·95-12·95	217
12·95-13·95	246
13·95-14·95	181
14·95-15·95	148
15·95-16·95	112
16·95-17·95	101
17·95-18·95	74
18·95-19·95	38
19·95-20·95	35
20·95-21·95	23
21·95-22·95	14
22·95-23·95	7
23·95-24·95	18
24·95-25·95	10
25·95-26·95	5
26·95-27·95	7
27·95-28·95	5
28·95-29·95	3
29·95-30·95	2
30·95-31·95	0
31·95-32·95	1
32·95-33·95	1
33·95-34·95	0
34·95-35·95	1
35·95-36·95	
Totals	2500

Column totals (Breadth of Leaf): 4, 85, 96, 156, 185, 246, 307, 817, 387, 270, 186, 145, 76, 65, 41, 27, 22, 9, 9, 6, 6, 5, 6, 1 — Total 2500.

TABLE XXXI.—Wild Ivy.   Homotypic Cross Correlation of Length and Breadth of Leaves.

Length of First Leaf.

Breadth of Second Leaf.

seen the mean of the index-correlations for mushroom gills and ivy leaves = ·4108, a result by no means bad, or widely divergent from the mean result of all our data.

If we test the hypothesis as to the value of the cross-correlation we have the following results arranged as in the case of the mushrooms :

Product of direct homotypic length correlation with organic length-breadth correlation = ·4897

Product of direct homotypic breadth correlation and organic length-breadth correlation = ·4648

Product of mean of direct homotypic length and breadth correlations with the organic length-breadth correlations = ·4773

These numbers have to be compared with the cross length-breadth correlation, *i.e.*, with ·5157.

We see that, as in the case of the mushroom, they are somewhat too small, 48 say as compared with 52. But the difference is considerably less here, and allowing for the action of disturbing factors, I think we may say that the two quantities under investigation are at least of the same order of magnitude. There I think we must leave the hypothesis until my zoological measurements are reduced.

I give the four tables of classified data for the absolute lengths and breadths of the ivy leaves. In the first or organic correlation table the reader will be able, in the distribution at least of breadths, to find something of the irregularity to which I have already referred.

VI. SUMMARY OF RESULTS.

(23.) In summing up my results and comparing them with those obtained for fraternal correlation by my co-workers and myself, I felt some difficulty. If I made a selection of what I considered the best homotypic correlation series, and the best fraternal correlation, I might well lay myself open to the charge of selecting statistics with a view to the demonstration of a theoretical law laid down beforehand. Accordingly, I determined to include all my homotypic results, except those for the absolute dimensions of mushroom gills and ivy leaves, where it was pretty evident that we had to a greater or less degree an influence exerted by the growth factor. I thus drew up Table XXXII., containing a summary of all my results. I am quite sure that heterogeneity due to one or another cause, exerts an influence of one kind at the top, and differentiation an influence of another kind at the bottom of this table. The amount of these influences and of other disturbing causes, one cannot measure and allow for. I can only hope that having taken a fairly wide range of races and characters, the influences tending to obscure the homotypic correlation, on the one hand by raising it, and on the other by reducing it, will about balance, and

the average of the whole series be a fairly close approximation to the true mean value of homotypic correlation. The result is ·4570. Turning now to our results for fraternal correlation, I put into a second table every single result that we have

TABLE XXXII.—General Results for Homotypic Correlation.

Race.	Character.	Percentage variation.	Correlation.	Remarks.
Mushroom, Hampden	Lengths of gills .	50·92	·8607	
,, ,,	Breadths of gills .	67·67	·7363	All these results introduce
,, ,,	Lengths and breadths of gills	—	·6275	a correlation due to stages of growth and accordingly
Wild Ivy, mixed localities	Lengths of leaves .	82·73	·5618	are not included in the
,, ,, . . .	Breadths of leaves .	84·56	·5332	determination of means.
,, ,, . . .	Lengths and breadths of leaves	—	·5157	
(i.) Ceterach, Somersetshire	Lobes on fronds .	77·57	·6311	Said to be largely affected
(ii.) Hartstongue, Somersetshire	Sori on fronds .	77·64	·6303	by growth and environment.
(iii.) Shirley Poppy, Chelsea .	Stigmatic bands .	78·86	·6149	Much selected in transit.
(iv.) English Onion, Hampden	Veins in tunics .	79·18	·6108	Possibly slightly heterogeneous.
(v.) Holly, Dorsetshire	Prickles on leaves	80·11	·5985	
(vi.) Spanish Chestnut, mixed	Veins in leaves .	80·65	·5913	Heterogeneous.
(vii.) Beech, Buckinghamshire	Veins in leaves. .	82·17	·5699	
(viii.) *Papaver Rhoeas*, Hampden	Stigmatic bands .	82·71	·5620	
(ix.) Mushroom, Hampden	Gill indices . . .	83·58	·5490	Possibly influenced by individual growth.
(x.) *Papaver Rhoeas*, Quantocks	Stigmatic bands .	84·59	·5333	
(xi.) Shirley Poppy, Hampden	Stigmatic bands .	85·18	·5238	
(xii.) Spanish Chestnut, Buckinghamshire	Veins in leaves. .	88·51	·4655	
(xiii.) Broom, Yorkshire . .	Seeds in pods . .	90·96	·4155	
(xiv.) Ash, Monmouthshire .	Leaflets on leaves.	91·44	·4047	
(xv.) *Papaver Rhoeas*, Lower Chilterns	Stigmatic bands .	91·66	·3997	All from one field.
(xvi.) Ash, Dorsetshire . . .	Leaflets on leaves.	91·81	·3964	
(xvii.) Ash, Monmouthshire .	Leaflets on leaves.	92·73	·3743	
(xviii.) Holly, Somersetshire .	Prickles on leaves	93·12	·3548	
(xix.) Wild Ivy, mixed localities	Leaf indices. . . .	96·21	·2726	From two localities and possibly slightly influenced by differentiation.
(xx.) *Nigella Hispanica*, Slough	Segments of seed-capsules	98·18	·1899	Differentiation of organs due to position on stem.
(xxi.) *Malva Rotundifolia*, Hampden	Segments of seed-vessels	98·32	·1827	Principally spread from one clump by stolons.
(xxii.) Woodruff, Buckinghamshire	Members of whorls	98·49	·1733	Members really different in morphological origin.
Mean of 22 cases . .	—	87·44	·4570	—

worked out up to the date of writing this memoir. I felt this was the only safe method, although I might be justified in cutting out several values from the top and from the bottom of this table. I am sure the value for Daphnia is much too high,

TABLE XXXIII.—General Results for Fraternal Correlation.*

Race.	Sex.	Character.	Source of material.	No. of cases.	Reduced by	Correlation.	Remarks.
(i.) Daphnia	♀ & ♂	Length of spine	ERNEST WARREN	330	K. PEARSON	·6934	Probably much too high, owing to heterogeneity introduced by the selection of a few mothers only.
(ii.) Horse	♀ & ♂	Coat-colour.	WEATHERBY'S Studbooks	1000	K. PEARSON, L. BRAMLEY-MOORE, and A. LEE	·6928	Probably much too high, owing to heterogeneity introduced by the use of comparatively few sires.
(iii.) "	♂ & ♂	"	"	1000		·6233	
(iv.) "	♀ & ♀	"	"	1000		·5827	
(v.) Man.	♀ & ♀	Forearm.	PEARSON, family data	441	A. LEE	·5424	One pair only from each family.
(vi.) Hound (Basset)	mixed	Coat-colour.	GALTON, from studbook	—	K. PEARSON and A. LEE	·5257	All members of litter without regard to sex.
(vii.) Man.	♂ & ♂	Eye-colour.	GALTON, family data	1500	K. PEARSON	·5169	All possibly pairs in family taken.
(viii.) "	♀ & ♀	Cephalic index	FRANZ BOAS, N. A. Indians	—	C. FAWCETT	·4890	Paternity doubtful
(ix.) "	♂ & ♂	Eye-colour.	GALTON, family data	1500	K. PEARSON	·4615	See remark to (vii.).
(x.) "	♀ & ♀	Stature .	"	1500	"	·4463	" "
(xi.) "	♀ & ♂	"	"	595	"	·4436	" "
(xii.) "	♂ & ♂	"	"	605	"	·3913	" "
(xiii.) "	♂ & ♂	Cephalic index	FRANZ BOAS, N. A. Indians	—	C. FAWCETT	·3790	See remark to (viii.).
(xiv.) "	♂ & ♂	Stature .	GALTON, family data	1181	K. PEARSON	·3754	See remark to (vii.).
(xv.) "	♂ & ♀	Cephalic index	FRANZ BOAS, N. A. Indians	—	C. FAWCETT	·3400	See remark to (viii.).
(xvi.) "	♀ & ♀	Longevity	Quaker records	1050	M. BEETON	·3323	Reduced below true value by non-selective deaths.
(xvii.) "	mixed	Temper.	GALTON, family data	1294	K. PEARSON	·3167	Character very indefinite, and difficult to estimate.
(xviii.) "	♂ & ♂	Longevity	Peerage records	1000	M. BEETON	·2602	See remark to (xvi.).
(xix.) "	♂ & ♀	"	Quaker records	1947	"	·1973	" "
	—			—	Mean of 19 series	·4479	—

* [Since the above memoir was written I have deduced another exceedingly interesting value for fraternal correlation from the measurements of Prof C E DAVENPORT on the acetabulate of the Hydroea (Pectunella Angulina, LAMARCK). See 'The American Naturalist,' vol 34, p. 84, 1900. DAVENPORT gives for the average standard deviation of columns of acetabula ·198. We represent the former by r, the latter will be $\sigma\sqrt{1-r^2}$, whence I find for the "fraternal correlation" $r = ·302$, a result in excellent agreement with the mean values we have just found.—July, 1901.]

and the value for longevity in man much below the true fraternal correlation. In the former case, the mothers were few in number, in the latter the non-selective death-rate reduces very considerably the intensity of collateral inheritance. Both longevity and temper are included in this table for the same reason as *Nigella Hispanica* and woodruff in the first table. I would not run the risk of any apparent selection to reduce either homotypic or fraternal correlation to a closer range of values. The mean of this second table gives the value ·4479 for fraternal correlation. Now I do not propose to lay great stress on what at first sight might look like a most conclusive equality between the mean values of homotypic and fraternal correlations,— within the limits of the probable errors ·4479 and ·4570 are indeed equal. I am quite aware that a few further series added to either the homotypic or fraternal results might modify to some extent this equality. But what I would ask the reader to do is to examine the two tables side by side, to note how the first and last several results of both may fairly be held to be subject to quite definite modifying factors, and then to consider whether there is not very substantial evidence gathered from a fairly wide range of characters in nearly as wide a range of species to show that both homotypic and fraternal correlation fluctuate about a mean value between ·4 and ·5.

I will not venture to assert that either are absolutely constant, but I do realise that it is extremely difficult with the complex system of factors influencing living forms to reduce our conditions to that theoretically perfect state in which we shall measure solely the factor we are investigating. If the intensity of homotyposis were exactly ·45, I should be inclined to distrust any long series of results, one and all of which gave the answer ·45 exactly. There are so many other disturbing factors which only those who have endeavoured to collect series of this kind will fully appreciate. In the first place, the theoretical conception of undifferentiated like organs is very hard to realise practically ; position of the organ on the branch or of the branch on a plant, however careful be the collector, may really have introduced differentiation, and so weakened the apparent homotyposis. Secondly, the environmental factor comes into play. It is difficult to obtain a hundred individuals with like environment ; soil, position with regard to other growths, sunlight, insect life, &c., may differ in a manner that the collector cannot appreciate. Unlike environment may produce a fictitious likeness in the organs of the same individual when we pass from one individual to a second. The fact that half our series grew in one field, the other half in another, that part came from one side of a road, part from another, may introduce an unperceived heterogeneity which increases the apparent homotyposis. Thirdly, the difficulty of ensuring that all individuals are of the same age or in the same stage of development, is very great. The leaves of an old tree may have a fictitious likeness when compared with those of a young tree ; we may gather organs from one individual when it is in a stage of development, which would only be reached in another individual some days or weeks later. These and other factors may perhaps be to some extent eliminated—far better of course by the trained botanist than by

the mere mathematician—but I very much doubt the possibility of their complete elimination. If homotyposis had a practically constant value throughout nature, I should only expect this value to be ascertained as a result of the average of many series in which the opposing factors of differentiation, environment, age, stage of growth, &c., may more or less counteract each other. In this manner we may approach to a fair appreciation of the bathmic influence of individuality in the production of undifferentiated like organs. What I should accordingly deduce as legitimate from the above general results would be this, the intensity of pure homotyposis throughout the vegetable kingdom probably lies between ·4 and ·5 ; this is also the mean value found up to the present for fraternal correlation. We may accordingly conclude that heredity is really only a phase of the wider factor of homotyposis. At bottom it is only part of the principle that when an individual puts forth undifferentiated like organs these are not exactly the same, but with a definite intensity of variation have a definite degree of likeness. When we associate heredity with sexual reproduction, we are only considering the result of homotyposis (variation and likeness) between individual spermatozoa and between individual ova. Such homotyposis leads to a likeness of the individuals resulting from the zygotes, which has probably the same mean value as homotypic correlation itself. Thus if the continuity of the germ cells between parent and offspring be realised, we face no longer the problem of heredity, but that of homotyposis, and this again might possibly be reduced to the simplest problem of budding or cell multiplication. Why does the fundamental life-unit on self-multiplication produce homotypes with a definite degree of likeness and a definite degree of variation ? I shall hope for further light on this problem when my data for homotyposis in the animal kingdom, already being collected, are somewhat more complete ; but only the biologist, not the mathematician, can solve it.

(24.) Now let us turn to another point : homotyposis involves, as we have seen, not only a certain degree of likeness in the group of homotypes but a certain degree of variation. Our series does not include any groups of more than twenty-six homotypes, except in the case of the Hampden Shirley poppies. Hence it is not really possible to calculate directly the variability of the individual. But from theoretical considerations, as well as from the support of individual instances, we have seen that the standard deviation of the array is a reasonable measure of the variability of the individual.* Of course more elaborate direct investigations on this point would be of great interest. But I consider that the present series indicate that on an average the variation in the individual is some 87 to 88 per cent. of that of the race. If the reader will examine the column headed *Percentage Variation* in Table XXXII., he will notice that excluding the lengths and breadths of mushroom gills—cases in which the stage of growth is all important—no percentage variation falls below 77. Now

* It is quite easy to find isolated individuals with a greater degree of variability than this, and even a greater variability than that of the race, it is the *average* individual variability which is represented by the S. D. of the array.

this seems to me very instructive when we consider the statements made by some writers who theorise about variation rather than actually measure it : *The undifferentiated like organs put forth by the individual have in round numbers* 80 *to* 90 *per cent. of the variation of such organs in the race.* Does not this completely refute the views frequently expressed that variation is the result of sexual reproduction, and that it is quite insignificant in the case of budding ? Sexual reproduction may produce a type which is not that of either parent, but this does not *à fortiori* alter the variability of the race with regard to any organ. With continuous variation such type would have previously existed as far as any special organ or character is concerned, and its repetition contributes nothing to the racial variability. We have to meet the fact that the individual produces undifferentiated like organs with a remarkable degree of variety, and if the investigations of the present memoir be valid the practical result of the homotyposis factor having a value of ·4 to ·5 would denote that the individual variability is 91·65 to 86·60 per cent. of the racial.

(25.) Lastly, we may consider a third point of very great interest, which is involved in Table XXXIV. Here we have the coefficients of variation given for some twenty-two series. Now this coefficient seems to me the only satisfactory comparative measure we can find at present of variability. A variation of two in the petals of a buttercup is far more significant than one of two in the florets of an ox-eyed daisy ; the measures of the *absolute* variations as given by the standard deviations seem to me of no use when we are comparing different characters in different species. In default of the suggestion of any better standard, all we can do is to get rid of absolute size or number by using the percentage variation of the character as indicated in the coefficients of variation. Now our table gives a fairly continuous series from 7·80 up to 41·96.* The mean value of 19 is in very good accord with the results I have obtained for variation in a much wider series for the vegetable kingdom. Now I think it will be admitted :

(*a.*) that this variation is based on results for a wide number of species ;

(*b.*) that this variation covers a considerable variety of characters ;

(*c.*) that it is roughly continuous in value between 8 and 40.

But if we examine the fourth column in this table, which gives the place of the corresponding homotypic correlation, we find absolutely no relationship between the intensity of the homotyposis and of the variability. The homotypic order will not compare in any way with the variation order. The mean variability of the first eleven series is 22·95, and of the last eleven series 14·28, very sensible deviations from the mean 18·62 of the whole twenty-two series. But the mean homotypic correlation of the first eleven series is ·4559, and of the last eleven series ·4581, neither of which

---

* If we excluded the hartstongue as largely influenced by environment, we should find broom heading the list, one of my most satisfactory series, both as to similarity of individual environment and as to smoothness of frequency. We should still conclude that variation might reach to nearly 40.

exhibits any sensible difference from ·4570 the mean of all twenty-two series. We are compelled therefore to conclude that there is no relationship between the variability of a species with regard to any organ, and the homotypic correlation of such organs. It would be quite impossible to assert that as the variability of a species decreased, its homotyposis increased. But according to the results reached in this paper heredity is a result of homotyposis. Hence there seems, so far as our researches go, no ground

<div style="text-align:center">TABLE XXXIV.—General Results for Variability.</div>

Race.	Character.	Coefficient of variation.	Position in correlation table.
Hartstongue, Somersetshire	Sori on fronds . . .	41·96	(ii.)
Broom, Yorkshire	Seeds in pods . . .	36·78	(xiii.)
Holly, Dorsetshire . . .	Prickles on leaves. .	26·29	(v.)
Nigella Hispanica, Slough .	Segments in capsules.	19·39	(xx.)
Holly, Somersetshire. .	Prickles on leaves. .	18·74	(xviii)
Ash, Dorsetshire	Leaflets on leaf. . .	18·65	(xvi.)
Ash, Monmouthshire. . . .	„ „ . . .	18·57	(xiv.)
Mushroom, Buckinghamshire	Gill-indices	18·28	(ix.)
Ceterach, Somersetshire. .	Lobes on fronds . .	18·25	(i.)
Papaver Rhœas, top of Chilterns . . .	Stigmatic bands . . .	17·81	(viii.)
Wild Ivy, mixed	Leaf-indices. . . .	17·77	(xix.)
Papaver Rhœas, Lower Chilterns . .	Stigmatic bands . . .	17·66	(x.)
English Onion, Hampden . . .	Veins in tunics. . .	17·43	(iv.)
Spanish Chestnut, mixed . . .	Veins in leaves. . .	15·72	(vi)
Ash, Buckinghamshire	„ „	15·46	(xvii.)
Papaver Rhœas, Lower Chilterns . .	Stigmatic bands . . .	15·27	(xv)
Shirley Poppy, Hampden	„ „ . . .	15·17	(xi)
Spanish Chestnut, Buckinghamshire. .	Veins in leaves. . .	14·31	(xii.)
Shirley Poppy, Chelsea	Stigmatic bands . . .	13·99	(iii)
Woodruff, Buckinghamshire	Members in whorls . .	13 46	(xxii.)
Beech, Buckinghamshire	Veins in leaves . . .	10·77	(vii.)
Malva Rotundifolia, Hampden	Segments in seed-vessels	7·80	(xxi)
Mushroom, Buckinghamshire	Lengths of gills .	21·00	—
„ „	Breadth of gills .	23·42	—
Wild Ivy, mixed	Lengths of leaves .	30·94	—
„ „	Breadths of leaves .	34·34	—
Mean of first 22 cases . .		18·62	—
Mean of all 26 cases . . .		19·97	—

for asserting that increased intensity of heredity means decreased intensity of variation and *vice versâ*. The general impression formed upon my mind by our measurements is the approximate constancy of the *pure* homotypic factor, or at any rate its variation within a fairly small range, and consequently the approximate constancy within a like small range of the hereditary factor for all species and all characters.

On the other hand, great diversity in the variability of different species. I do not wish to be dogmatic on this point, however, but I would definitely assert that so far no relationship between variability and homotyposis is discoverable. Further, it does not seem to me that looking down the order of variability we can assert that our table in any way corresponds to the order of evolutionary descent. It does not seem to me that the species towards the top of it are the more primitive and those towards the bottom the more complex. Indeed two series of the same species may stand at very different points in the table. I am accordingly forced to the conclusion that my table gives no support whatever to the view that variability in the vegetable kingdom is greatest for the more primitive and simpler organisms. Nor, again examining Table XXXII., can we conclude that homotyposis is least intense for such organisms, for the second half of our table cannot be said to contain more of such organisms than the first. Shortly, I see no relation between the position of an organism in the evolutionary descent and the intensity of either its variability or its homotyposis. Hence it follows that, if homotyposis be the source of heredity, the intensity of the latter cannot be related to either variability or position in evolutionary descent.

Mr. ADAM SEDGWICK may be right in his view* that in the early stages of evolution variability is large and inheritance small, and that with development this relation is changed, variability decreasing with an increasing intensity of heredity. My statistics only cover a small portion of one kingdom. But so far as I am aware they represent the only material so far published on which a definite judgment can be founded—where by definite judgment, I mean one based on quantitative facts, and not drawn from a general estimate of nature which makes no statement whatever of the intensity of either variability or heredity for any single character in any one species of living things.

The data I have given seem entirely opposed to Mr. SEDGWICK's view, but I very fully realise that they are far from exhaustive. They indicate, however, what I hold to be the only valid method of approaching any problem in variation or heredity, i.e., laborious statistical collection of actual facts. My statistics may be insufficient, I would heartily welcome additions to them; or they may be dealt with fallaciously. Here is the material; let others add to and, if need be, modify the conclusions. Only let us follow the method so clearly indicated by DARWIN himself in his 'Cross and Self-fertilisation of Plants;' let us cease to propound hypotheses illustrating them by isolated facts or vague generalities; there are innumerable species at hand in Nature ready for us to measure and count. Sine numero nihil demonstrandum est, should now-a-days be the motto of every naturalist who desires to propound novel hypotheses with regard to variation or heredity. It is equally valid for those who merely seek either to establish more firmly or to modify the old.

(26.) I would sum up the results of this memoir in the following conclusions:

* Presidential Address to Section D of the British Association, 1899.

(i.) When an individual of any species produces a number of undifferentiated like organs (homotypes) these are not exactly alike when measured with regard to any character. They exhibit a certain degree of variation combined with a certain degree of likeness (homotypic correlation).

(ii.) The homotypic correlation as tested for twenty-two series of homotypes in a variety of species in the vegetable kingdom, has a mean value between ·4 and ·5. Its exact determination is rendered difficult by a number of conflicting factors; but it seems very possible that the actual value of pure homotypic correlation, i.e., the correlation due to the individuality of the plant as apart from the influence of environment, &c., differs but little from the mean value stated above.

(iii.) If a character occurs in a series of homotypes, it will be found to have in the individual a variability 80 to 90 per cent. of the variability of this character in the species at large. It is thus seen that variability is a primary factor of living forms, and is not in any way dependent on sexual reproduction.

(iv.) With a certain hypothesis as to cross homotypic correlation, i.e., the correlation of two different characters in homotypes, namely, that it is the product of the direct homotypic and of the organic correlations, it would follow that the mean fraternal correlation would equal the mean homotypic correlation. The mean of twenty-two homotypic series is found to be sensibly identical with the mean of nineteen fraternal series. A direct investigation of the value of the cross homotypic correlation is only made for two cases, that of the mushroom gill and that of the ivy leaf. The cross homotypic correlation is not found to differ very widely from the product of the direct homotypic and the organic correlations in these cases, but the results are not close enough to be conclusive.

(v.) It would seem, if (iv.) be true, that heredity is only a phase of the principle of homotyposis (ii.), and that the numerical value of its constants may be found from that principle.

(vi.) No relation whatever could be found between the intensity of variability and that of homotyposis in the twenty-two series dealt with. Nor was any relation to be observed between the relative simplicity of the organism and the intensity either of its variability or its homotypic correlation. Regarding heredity as a case of homotyposis, there seems no reason to suppose, as it has been suggested, that variability has decreased and heredity increased in the course of evolution. On the contrary there seems some ground for supposing that homotyposis (and therefore heredity) is a primary factor of living forms, a condition for the evolution of life by natural selection, and not a product of such selection. If the mushroom, the poppy, and the beech show approximately equal homotyposis, it seems well nigh impossible to consider it as a factor of life, increasing with advancing evolution.

On the Homotyposis of the Pods of Leguminous Plants.

I have already indicated that I had initially grave doubts as to whether a true measurement of homotyposis could be obtained by counting the ripe seeds in the pods of leguminous plants; but that a passage in a work of DARWIN'S led me to modify my opinion, and when finishing my memoir, I found time to incorporate in it the broom series from Danby Dale which seemed to confirm DARWIN'S views. Broom is cross-fertilised and much visited by insects, and accordingly the statement made on p. 334 of this memoir appears to have full validity. As the matter seemed of some interest, I suggested to Miss C. D. FAWCETT a fuller study of leguminous plants from this standpoint. Her results have reached me in time to be reduced by Dr. LEE and to be incorporated in this Appendix. I have myself worked series for the common bean and the tare vetch, so that if these be included we have now a fairly extensive range of material from which to draw conclusions.

I will first give the data and state the manner in which it has been reduced, and then consider the results which flow from its consideration. The following is a list of the species dealt with; I have remarked on some of the difficulties arising in the course of our work.

(i.) *Cytisus Scoparius.* Broom from Danby Dale (see p. 334).—It was found difficult to count the total number of ovules, and only the ripened seeds were dealt with. The plant is cross-fertilised. The pods were fairly free from insects.

(ii.) *Lotus Corniculatus.* Bird's-foot trefoil from the neighbourhood of Lyme Regis, Dorsetshire.—The aborted seeds could not be counted, and the pods were largely frequented by grubs, so that it was occasionally doubtful how far ripe seeds had been destroyed. The number of pods on each plant in this first series were too few too allow those with grubs in them to be neglected.[*]

(iii.) *Lotus Corniculatus.*—A second series from the neighbourhood of Lyme Regis. This series consisted of more vigorous plants from a slightly different environment, so that fourteen to twenty pods could be gathered from each, and thus difficulties as to the possible destruction of ripe seeds by grubs avoided. It would appear that the effect of rejecting such pods has raised the mean number of seeds, but at the same time artificially reduced the homotyposis. The plant is cross-fertilised. (DARWIN : 'Cross and Self-Fertilisation of Plants,' p. 361.)

(iv.) *Lathyrus Odoratus.* Sweet Pea.—This was obtained from a row in a nursery garden at Lyme Regis. Both the fertilised and aborted seeds were counted, so that the total number of ovules could be found. The plant is said to be cross-fertilised in

---

[*] As an illustration of this sort of difficulty I may note that I found something like 90 per cent. of the pods of gorse in the Danby district so full of insect life that nothing could be done in regard to counting their seeds.

countries where it is indigenous, but DARWIN ('Cross and Self-Fertilisation of Plants,' 3rd edition, p. 155, *et seq.*) strongly believes it to be wholly self-fertilised in England.

(v.) *Lathyrus Odoratus.*—A shorter series obtained from a second nursery garden at Lyme Regis.

(vi.) *Lathyrus Sylvestris.* Everlasting pea from the sea coast not very far from Lyme Regis.—Both the ripe and the aborted seeds were counted. The plant appears to be cross-fertilised. Professor F. O. OLIVER tells me that he has watched bees effectively visiting the cultivated everlasting pea. ,

(vii.) *Vicia Faba.* Common Bean.—I took 100 plants, each having at least ten pods, from a field in Danby Dale, in which a mixture of oats, the common pea, the common bean, and three or four vetches* (described by the owner as tares), was growing as food for cattle. The perfect beans only were counted, but the plants were a poor crop, many pods being stunted in their growth, and it was not always easy to determine whether the seeds had not been fertilised or there had been failure owing to want of nutrition. The common bean is both cross and self-fertilised. I have to thank Miss J. SHARPE and Miss E. CYRIAX for aid in the work on these beans.

(viii.) *Vicia Hirsuta.*—I found twenty-eight plants of the tare vetch growing on a strip of uncut grass along a cut cornfield at Botton, Danby Dale. I could find no more plants in the neighbourhood. There was plenty of the common vetch (*Vicia Sativa*) on the same strip, but the plants had rarely more than two to three pods on them. I was not able to get ten pods from each tare vetch, thirteen plants had fewer, and seven plants more. The series being short, the probable errors are high, but it seemed worth while to include the data. I endeavoured to count both ripe and aborted seeds, but here, as in one or two of the previous cases, I much doubt whether we have succeeded in counting all the ovules. In some of the green pods the number of ovules seemed to be considerably larger than in the fully ripe pods, and I think it possible that the non-fertilised seeds shrink till they are quite unnoticeable even by a cautious observer. Again, it is by no means certain here, as in other cases, that all the seeds reckoned as aborted are really non-fertilised. It is quite possible that in some cases fertilised seeds have dwindled for want of nutriment till they appear aborted.

Vicia Hirsuta is either cross or self-fertilised (DARWIN: 'Cross and Self-Fertilisation of Plants,' p. 367). Whether this vetch, or indeed the common bean, were in our case wholly cross-fertilised or in part self-fertilised, I do not see that we have any means of settling.

Table L. gives the frequency distributions of the various series. Now in these series we must bear in mind that we are only definitely certain of one thing, the number

---

* Neither the pea nor the vetches offered enough pods per plant for an investigation of their homotyposis to be of value.

TABLE L.—Variation of the Seeds in the Pods of Leguminous Plants per mille.

Number of Seeds in Pod.

Species.	0.	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.	12.	13.	14.	15.	16.	17.	18.	19.	20.	21.	22.	23.	24.	25.	26.	27.	Totals.
Cytisus Scoparius (ripe) .	—	7	12	24	89	45	64	108	105	113	112	86	88	88	56	40	28	14	4	3	1	—	—	—	—	—	—	—	1000*
Lotus Corniculatus (L.) (ripe)	8	6	42	50	73	85	106	101	88	72	90	56	63	39	23	25	25	15	10	6	5	8	—	—	—	1	1	—	1000
Ditto (L.) (ripe) . .	3	0	11	29	87	77	96	110	110	79	76	78	67	49	57	33	25	23	21	10	12	9	5	8	1	4	—	2	1000
Lathyrus Odoratus (ovules)	—	—	—	2	—	14	95	203	303	325	47	3	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1000
Ditto (ripe) . .	—	6	11	31	108	208	230	190	134	53	30	3	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1000
Ditto (aborted) . .	74	167	253	220	178	67	31	8	1	0	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1000*
Lathyrus Odoratus (n.) .	—	—	—	—	—	9	36	119	245	394	177	19	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1000*
Ditto (aborted) . .	6	16	45	74	113	194	194	120	120	45	18	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1000*
Ditto (aborted) . .	92	168	235	195	160	74	52	15	5	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1000*
Lathyrus Sylvestris (ovules)	—	—	—	—	—	—	37	—	—	—	—	—	—	204	476	290	39	5	—	—	—	—	—	—	—	—	—	—	1000
Ditto (aborted) . .	1	7	30	74	188	227	206	132	79	49	17	9	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1000
Vicia Faba (ripe) . .	—	—	8	10	18	9	64	187	187	203	156	96	18	12	0	—	—	—	—	—	—	—	—	—	—	—	—	—	1000*
Vicia Hirsuta (ovules) .	35	43	158	406	352	9	13	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1000*
Ditto (ripe) . .	37	169	169	438	282	91	13	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1000*
Ditto (aborted) . .	225	191	654	167	15	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1000*

* These series were reduced to per mille.

TABLE LI.—The following table sums up our results.

Species.	Fertilisation.	Number of			Nature of seeds.	Mean No. of seeds.	S.D. of seeds.	Coefficient of variation.	S.D. of array.	Percentage variation.	Homotypic correlation.
		Plants.	Pods.	Pairs.							
Cytisus Scoparius .	C.	120	1200	10,800	Ripe	9·6425	3·54655	36·78	3·23596	90·96	·4135 [± ·0161]
Lotus Corniculatus (i.) .	C.	100	1000	9000	Ripe	8·5020	4·4336	52·14	4·3090	97·19	·2354 [± ·0901]
Ditto (ii.) . . .	C.	100	1000	9000	Ripe	9·9783	4·6276	46·38	4·5447	98·21	·1884 [± ·0806]
Lathyrus Odoratus (i.) .	S.	100	1000	9000	Ovules	8·7750	1·1101	12·65	1·0834	97·59	·2188 [± ·0203]
Ditto (i.) . . .	S.	100	1000	9000	Ripe	6·1310	1·6964	27·67	1·6906	99·66	·0830 [± ·0212]
Ditto (i.) . . .	S.	100	1000	9000	Aborted	2·6400	1·5442	58·49	1·4878	96·34	·2679 [± ·0198]
Lathyrus Odoratus (ii.) .	S.	80	800	7200	Ovules	8·4925	1·1369	13·39	1·0681	93·07	·3658 [± ·0207]
Ditto (ii.) . . .	S.	80	800	7200	Ripe	5·6725	1·9831	34·96	1·9393	97·79	·2091 [± ·0226]
Ditto (ii.) . . .	S.	80	800	7200	Aborted	2·8212	1·8945	67·15	1·8850	98·44	·1759 [± ·0231]
Lathyrus Sylvestris .	C.	100	1000	9000	Ovules	14·0130	·9512	6·79	·9374	98·55	·1695 [± ·0207]
Ditto	C.	100	1000	9000	Ripe	5·6080	1·8456	32·91	1·8015	97·09	·2184 [± ·0203]
Ditto	C.	100	1000	9000	Aborted	8·4150	1·9906	23·66	1·9717	99·05	·1376 [± ·0209]
Vicia Faba . . .	C. or S.	100		9000	Ripe	3·0260	1·0136	33·50	·9806	98·22	·1877 [± ·0206]
Vicia Hirsuta . . .	C. or S.	28	255	2328	Ovules	3·3612	1·0071	30·88	·9920	98·50	·1794 [± ·0410]
Ditto	C. or S.	28	255	2328	Ripe	2·2436	·8598	38·33	·8631	99·22	·1243 [± ·0416]
Ditto	C. or S.	28	255	2328	Aborted	1·0638	·8260	77·57	·8157	98·88	·1493 [± ·0413]
Mean	Four series				Ovules	—	—	13·43	—	96·93	·2315
Mean	Eight series				Ripe	—	—	37·83	—	97·365	·2077
Mean	Four series				Aborted	—	—	56·72	—	98·18	·1827
Mean	Sixteen series				All series	—	—	37·08	—	97·45	·2073

of fully ripened seeds—and this only when the pods were not infested with grubs. We did the best we could to count the aborted seeds, but we cannot be certain that all those counted as aborted were non-fertilised, or that we succeeded in counting all those which had shrunk to microscopic proportions. The number of ovules is simply the sum of the ripe and the aborted seeds counted in each pod, and this again must be somewhat doubtful. The reader will bear in mind that we proceeded with care, but that we think it right, if anything, to rather over- than under-emphasise possible sources of error. Yet allowing for such sources of error we cannot, on examining the results given in Table LI., allow that Broom is in the least representative of the degree of homotyposis to be found in the pods of leguminous plants. Whether we consider the ovules, the ripe, or the aborted seeds, our results are sensibly below that for broom, and the mean of the whole sixteen series gives us a value about *one-half* that of the homotypic correlation based upon characters not depending on fertilisation. If we deal with averages, it would certainly seem that in the results flowing from fertilisation, we have reduced the intensity of the individuality to about half its previous value.

To this extent only does the individual constitution appear influential in the number of seeds in the pod, the remainder of the homotypic intensity seems to have disappeared under random influences having nothing to do with the individuality of the plant. This is perhaps what we might expect in the case of *ripe* seeds in cross-fertilised plants, where the fertilisation may depend on the chance or not of insect visitation and the effectiveness or not of the pollen brought on such occasions. We should have to assert that the bird's-foot trefoil and the everlasting pea, whose average for homotypic correlation in the case of ripe seeds is about ·21 to ·22, lose half the intensity of their individuality through the random nature of the chances of the cross-fertilisation. But although this might be fairly satisfactory for these cases, what are we to say for the species which are self-fertilised absolutely or self-fertilised in default of cross-fertilisation ? We might have expected a high degree of homotyposis in the field bean or the tare vetch, where failing cross-fertilisation we are told there will be self-fertilisation. We find on the contrary, however, in these results some of the lowest homotypic correlations of the whole series. The sweet pea also in its two series presents some very remarkable results. If the sweet pea be entirely self-fertilised then we should expect the homotyposis of both ovules and ripe seeds to be fully up to the average. In neither case is such a result reached, although in one series we have a value for the ovules higher than that obtained for anything except the broom. The noteworthy fact, however, is that the results for the two series of sweet peas differ so widely in character ! In the first series it is the abortion which is most individual in character ; in the second series it is the ovules. In the first series the ripe seeds have far less individuality than the aborted seeds, in fact, the smallest homotyposis I have yet observed ; in the second series the ripe seeds have a value rising to ·2, which is higher than that of the aborted seeds. Ash

trees from Monmouthshire, Buckinghamshire, and Dorsetshire, gave results in good agreement; but sweet peas from two different nursery gardens in the same district give strikingly divergent homotypic correlations!

According to DARWIN ('The Effects of Cross and Self-Fertilisation,' 3rd edition, p. 153) the sweet pea "in this country seems invariably to fertilise itself." He bases this statement on (a) the difficulty of access to bees and other insects, (b) the fact that the varieties are habitually grown by seed growers close together, and yet the colours do not blend, (c) the experience that when the varieties are artificially crossed the colours do change. Professor DELPINO, of Florence, in a letter to DARWIN, writes that it is the fixed opinion of the Italian gardeners that the varieties do cross; but in Italy other insects may of course be available. Now it is most remarkable that in Series I., whether the sweet pea be crossed or self-fertilised, the number of ripe seeds should be hardly individual at all. It would indicate that, with the particular environment of this series, the chance even of self-fertilisation depends upon extraneous causes. The comparatively high individuality in the tendency to abort may mark some peculiarity of this variety; there may be a distinct inheritance of sterile tendencies to be kept separate from an inheritance of fertility. But it is probably idle to guess at explanations of such discrepancies. I have gone carefully into the differences of environment in the two series which are interesting, although they do not obviously provide any key to the mystery. The first series were thickly planted in a long row, and the seed pods were gathered unripe. It was more difficult to separate each plant and to be quite certain that some of the seeds would not have had to be reckoned as aborted in the fully ripened pods. The second series consisted of withered plants, the pods being almost all completely ripe; the peas were planted in separate groups a few feet apart. There was no difficulty in ascertaining the individuality of the plant nor as to the number of ripe or aborted seeds. About forty hives of bees were kept in the immediate neighbourhood of this second series;* the first series were at the other end of Lyme Regis, and removed from hives. Both series had had their flowers freely cut. I might have been prepared to attribute the low value of the homotyposis in the sweet pea to this latter cause, but then the common bean and the tare vetch had not been subjected to any similar process, and they give on the whole much lower values. The ovules in either self-fertilised or cross-fertilised plants ought indeed to give a fair measure of homotyposis, but, for reasons already stated, it is not certain that our procedure in counting the perfect and the aborted seeds in the ripened pods has led to a just estimate of them. I am inclined on the whole to attempt no explanations for the anomalies observed in these seed investigations. The influences of self- and cross-

---

* I mention this, as I noticed in Yorkshire fairly frequent but apparently ineffectual visits of the hive bees to sweet peas. It is possible that their attempts may assist the self-fertilisation, just as shaking or wind has been observed to do in other species.

fertilisation on homotyposis deserve still fuller and more direct investigation.* I
feel we know little as to the influence of external causes even on the completeness or
incompleteness of self-fertilisation. So far as homotyposis in the pods of leguminous
plants is concerned, I would draw conclusions based solely on averages, and state
that :—

(i.) The homotyposis in the case of either ripe or aborted seeds in the pods or
plants seems weakened to one-half the average value it has in the case of characters
not depending on fertilisation. Thus DARWIN'S view, that differences in the number
of ripe seeds depend upon the constitution of the plants, seems to be only partially
true. Extraneous causes about which we are not very clear appear to be generally
influential.

(ii.) The extraneous causes which act in a random manner on the homotypic
correlation seem to affect both self-fertilised and cross-fertilised plants, and this both
with regard to ripe and to aborted seeds.

(iii.) The order of intensity of homotypic correlation is ovules, ripened seeds, and
aborted seeds.

The reduction of homotypic correlation in the case of the fertility of pods may be
profitably compared with the like reduction which we find in the case of the
coefficients of inheritance of fertility and fecundity in man and the horse.†

Other points worth noting, I think, in our results are those of Table LII., which
gives the correlation between ovules and ripe and aborted seeds. As we might
anticipate, the more ripe the fewer aborted seeds, and *vice versâ*. The correlation is
negative, high for the everlasting pea, remarkably low for the tare vetch. It is
interesting to see that the correlation between the ovules and either ripe or aborted
seeds is not very high. A large number of ovules not necessarily connoting either
a very large number of ripe or of aborted seeds. In fact, in the case of the ever-
lasting pea, the number of ovules has very small influence indeed on the number
of seeds which ripen. In the tare vetch only is the relationship more marked.
Professor F. O. OLIVER tells me that in certain cases evolution appears to be tending
in the direction of the pod containing one ripe seed only. It seems, therefore, that
such relations as are indicated in Table LII. (and others of a like kind, for further
observations ought certainly to be made) may be useful in indicating the degree of
fixity between the number of ovules and the number of ripe or aborted seed which
are ultimately to be found in the seed vessel.

While the ovules have the least and the aborted seeds the greatest variability, as
measured by the coefficient of variation, the ovules have the most and the aborted
seeds the least homotypic correlation. This might at first sight appear to be opposed
to the view expressed on p. 363, that there is no relationship between the intensities

---

* It seems to me that investigations of this kind ought to be carried out by those who have, what I
unfortunately have not, the needful land for experimental investigations.

† 'Phil. Trans.,' A, vol. 192, p. 277 *et seq.*

of variation and of homotyposis. But it is really not so, for if we proceed as on that page to divide up our series into two groups in order of their variability, we find that the first eight give an average coefficient of variation of 22·68, and have an average homotypic correlation of ·1941; the last eight have an average coefficient of variation of 51·47—double and more that of the first eight—while their homotypic correlation is ·2207. Thus, so far from the homotyposis being reduced by greater variability, it actually appears to be slightly increased.

TABLE LII.—Relationship between the Number of Ovules and those of Ripe and of Aborted Seeds.

Species.	Correlation.		
	Ripe and aborted.	Ripe and ovules.	Aborted and ovules.
Lathyrus Odoratus (i.) . . .	− ·7092	·4581	·2155
,, ,, (ii.) . .	− ·8290	·3628	·2203
Lathyrus Sylvestris	− ·8797	·0992	·3660
Vicia Hirsuta	− ·2859	·6195	·5751
Means . . .	− ·69095	·3849	·3442

Lastly, we may notice (Table LI.) that the variation in the individual amounts to upwards of 97 per cent. of that in the race, or we have another strong fact to aid in demolishing the theory that variability is a result of sexual reproduction. Note in particular the 97 per cent. in the case of the ovules whose number is determined before fertilisation !

Tables XXXV.–XLIX. give the data from which the constants of Tables L.–LII. have been calculated.

TABLE XXXV.—*Lotus Corniculatus.* Dorsetshire. First Series.

Number of Perfect Seeds in First Pod.

	0.	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.	12.	13.	14.	16.	17.	18.	19.	20.	21.	22.	23.	24.	25.	Totals.
0	6	1	8	8	18	9	9	8	2	8	1	6	—	2	2	0	2	—	—	—	—	—	—	—	—	81
1	1	—	5	1	4	5	5	6	5	1	6	2	2	2	1	1	—	1	1	—	—	—	—	—	—	54
2	8	5	24	35	43	42	60	37	31	31	18	10	16	8	4	4	4	3	1	1	—	1	—	—	—	378
3	18	1	35	35	45	53	59	39	37	24	11	12	5	6	7	1	2	1	1	1	—	—	—	—	—	460
4	13	4	43	46	58	73	95	64	65	43	42	38	34	30	10	5	4	2	2	2	1	—	—	—	—	667
5	9	6	42	53	73	87	97	61	63	60	60	43	34	39	23	17	13	6	8	2	4	1	1	—	—	766
6	9	9	60	53	95	97	140	136	109	75	90	54	68	31	14	24	18	8	10	2	5	4	0	—	—	973
7	8	5	39	39	67	57	136	58	82	67	69	54	68	39	10	17	15	8	7	7	4	0	4	0	1	909
8	2	6	37	38	65	61	109	82	67	76	83	48	48	25	18	21	10	8	6	5	4	8	4	0	1	798
9	8	1	31	32	61	68	82	67	76	67	77	43	45	24	16	18	6	5	6	2	4	5	0	0	1	648
10	1	6	21	45	46	87	90	48	83	48	85	48	47	31	12	12	10	2	6	1	4	2	4	0	0	730
11	6	2	18	24	38	67	48	54	54	49	31	34	33	17	12	21	4	6	2	1	4	8	0	—	—	604
12	2	2	24	23	34	58	84	89	64	38	47	36	36	12	8	11	9	5	6	3	8	0	4	—	—	667
13	6	2	11	27	18	35	89	75	67	16	16	23	29	18	2	10	6	3	2	1	6	0	—	—	—	381
14	7	2	8	24	14	36	24	68	48	67	10	13	17	5	6	21	4	2	2	1	2	1	—	—	—	307
16	3	1	9	11	5	30	14	54	34	44	32	15	5	12	3	14	0	5	6	3	4	0	4	—	—	335
17	1	1	6	12	8	14	26	32	14	16	22	10	8	8	8	2	10	4	10	6	4	0	4	—	—	188
18	2	1	6	11	5	5	24	15	15	16	10	7	8	2	6	9	4	2	6	2	4	8	4	0	1	78
19	1	—	6	12	7	4	14	9	10	18	22	5	6	2	3	11	4	5	7	4	8	2	4	0	1	90
20	2	—	7	5	6	8	7	10	8	8	10	5	3	1	3	2	2	2	6	1	4	0	4	0	5	45
21	1	—	3	4	4	8	9	7	6	7	6	4	0	1	2	4	8	2	0	0	9	6	4	0	1	72
22	—	—	1	2	2	1	1	5	0	4	5	4	0	1	0	4	1	2	0	0	0	0	4	0	4	86
23	—	—	—	1	1	1	0	0	0	4	0	0	2	0	9	0	0	0	4	1	2	1	4	0	1	36
24	—	—	—	1	1	1	0	2	0	0	0	0	0	1	1	1	2	0	—	—	—	1	—	—	—	0
25	—	—	—	—	1	1	2	2	0	0	0	—	1	1	1	3	2	—	—	—	—	—	—	—	—	9
Totals.	81	54	378	460	667	766	973	909	798	648	730	604	567	381	307	335	188	78	90	45	72	86	36	0	9	9000

Number of Perfect Seeds in Second Pod.

TABLE XXXVI.—*Lotus Corniculatus.* Dorsetshire. Second Series.

Number of Perfect Seeds in First Pod.

Number of Perfect Seeds in Second Pods.	0.	L.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.	12.	13.	14.	15.	16.	17.	18.	19.	20.	21.	23.	24.	25.	26.	27.	Totals.
Totals.	27	0	89	261	838	698	684	711	990	864	738	702	608	387	613	397	235	307	189	90	117	81	46	73	9	89	9	18	8000

TABLE XXXVII.—*Lathyrus Odoratus.*   First Series.

Number of Ovules in First Pod.

	5.	6.	7.	8.	9.	10.	11.	12.	Totals.
5	—	—	—	7	7	3	1	—	18
6	—	22	24	23	35	20	2	—	126
7	—	24	172	282	205	134	36	2	855
8	7	23	282	1168	786	371	81	9	2727
9	7	35	205	786	1114	625	132	21	2925
10	3	20	134	371	625	592	127	9	1881
11	1	2	36	81	132	127	42	2	423
12	—	—	2	9	21	9	2	2	45
Totals	18	126	855	2727	2925	1881	423	45	9000

TABLE XXXVIII.—*Lathyrus Odoratus.*   First Series.

Number of Ripe Seeds in First Pod.

Number of Ripe Seeds in Second Pod.	0.	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.	12.	Totals.
0	—	—	1	—	—	2	1	3	2	1	—	—	—	9
1	—	—	1	2	14	13	8	10	2	3	1	—	—	54
2	—	1	2	1	9	34	23	20	6	3	0	—	—	99
3	—	2	1	6	45	61	59	41	33	22	8	1	—	279
4	—	14	9	45	120	193	201	164	124	41	14	2	—	927
5	2	13	34	61	193	426	461	351	208	86	29	6	2	1872
6	1	8	23	59	201	461	564	418	251	111	48	3	3	2151
7	3	10	20	41	164	351	564	338	298	87	29	7	4	1710
8	2	2	6	33	124	206	251	238	234	76	29	3	—	1206
9	1	3	3	22	41	86	111	87	76	30	15	2	—	477
10	—	1	0	8	14	29	48	29	29	15	4	3	—	180
11	—	—	—	1	2	6	3	7	3	2	—	—	—	27
12	—	—	—	—	—	2	3	4	—	—	—	—	—	9
Totals	9	54	99	279	927	1872	2151	1710	1206	477	180	27	9	9000

TABLE XXXIX.—*Lathyrus Odoratus.* First Series.

Number of Aborted Seeds in First Pod.

Number of Aborted Seeds in Second Pod.	0.	1.	2.	3.	4.	5.	6.	7.		
0	78	151	167	129	92	35	12	2		666
1	151	286	388	296	225	104	37	14		1503
2	167	388	636	468	375	161	61	17		2277
3	129	296	468	510	355	127	72	18	3	1980
4	92	225	375	355	362	117	61	10	2	1602
5	35	104	161	127	117	32	16	10		603
6	12	37	61	72	61	16	18	1		279
7	2	14	17	18	10	10	1	—		72
8	—	1	1	2	3	1	1	—		
9	—	0	0	0	0	—	—	—		9
10	—	1	3	3	2	—	—	—		
Totals	666	1503	2277	1980	1602	603	279	72	0 9	9000

TABLE XL.—*Lathyrus Odoratus.* Second Series.

Number of Ovules in First Pod.

Number of Ovules in Second Pod.	5.	6.	7.	8.	9.	10.	11.	12.	Totals.
5	2	17	24	14	6	—	—	—	63
6	17	62	95	59	26	2	—	—	261
7	24	95	196	300	150	82	8	—	855
8	14	59	300	1066	689	321	34	1	2484
9	6	26	150	689	754	439	48	3	2115
10	—	2	82	321	439	394	37	3	1278
11	—	—	8	34	48	37	6	2	135
12	—	—	—	1	3	3	2	—	9
Totals	63	261	855	2484	2115	1278	135	9	7200

TABLE XLI.—*Lathyrus Odoratus.* Second Series.

Number of Ripe Seeds in First Pod.

	0.	1.	2.	3	4.	5.	6.	7.	8.	9	10.	11.	Totals.
0	2	3	2	10	13	11	3	6	3	1	—	—	54
1	3	2	8	13	19	31	11	12	15	1	1	1	117
2	2	8	22	50	44	68	45	38	29	16	2	0	324
3	10	13	50	62	85	105	67	70	46	17	4	2	531
4	13	19	44	85	108	178	149	129	65	20	5	4	819
5	11	31	68	105	178	300	257	251	146	42	6	0	1395
6	3	11	45	67	149	257	278	284	174	50	20	3	1341
7	6	12	38	70	129	251	284	258	183	71	18	3	1323
8	3	15	29	46	65	146	174	183	118	59	23	3	864
9	1	1	16	17	20	42	50	71	59	36	10	1	324
10	—	1	2	4	5	6	20	18	23	10	0	1	90
11	—	1	0	2	4	0	3	3	3	1	1	—	18
Totals	54	117	324	531	819	1395	1341	1323	864	324	90	18	7200

Number of Ripe Seeds in Second Pod (row label, rotated left)

TABLE XLII.—*Lathyrus Odoratus.* Second Series.

Number of Aborted Seeds in First Pod.

	0	1.	2.	3	4.	5	6.	7.	8.	9.	10.	Totals
0	112	166	138	116	57	27	25	15	6	3	1	666
1	166	266	280	230	131	51	39	26	13	4	0	1206
2	138	280	396	312	249	115	71	29	18	10	2	1620
3	116	230	312	278	237	106	75	18	23	6	3	1404
4	57	131	249	237	156	113	82	30	19	5	1	1080
5	27	51	115	106	113	48	36	18	12	4	1	531
6	25	39	71	75	82	36	30	12	4	4	0	378
7	15	26	29	18	30	18	12	8	6	0	0	162
8	6	13	18	23	19	12	4	6	6	0	1	108
9	3	4	10	6	5	4	4	0	0	—	—	36
10	1	0	2	3	1	1	0	0	1	—	—	9
Totals	666	1206	1620	1404	1080	531	378	162	108	36	9	7200

Number of Aborted Seeds in Second Pod (row label, rotated left)

TABLE XLIII.—*Lathyrus Sylvestris.* Sea Coast, Dorsetshire.

Number of Ovules in First Pod.

	8.	9.	10.	11.	12.	13.	14.	15.	16.	17.	Totals.
8	—	—	—	..	—	2	3	3	0	1	9
9	—	—	—	—	—	0	0	0	0	0	0
10	—	—	—	—	2	9	7	0	0	0	18
11	—	—	—	—	5	10	28	10	1	0	54
12	—	—	2	5	28	90	167	38	2	1	333
13	2	0	9	10	90	486	837	348	51	3	1836
14	3	0	7	28	167	837	2162	920	143	17	4284
15	3	0	0	10	38	348	920	610	121	20	2070
16	0	0	0	1	2	51	143	121	30	3	351
17	1	0	0	0	1	3	17	20	3	0	45
Totals	9	0	18	54	333	1836	4284	2070	351	45	9000

Number of Ovules in Second Pod. (row label, left side)

TABLE XLIV.—*Lathyrus Sylvestris.* Sea Coast, Dorsetshire.

Number of Ripe Seeds in First Pod.

	0.	1.	2.	3.	4.	5.	6.	7.	8.	9.	10	11.	12.	Totals.
0	—	—	—	1	1	1	0	1	1	4	—	—	—	9
1	—	—	4	4	16	12	11	7	7	1	1	—	—	63
2	—	4	16	10	40	33	37	29	5	6	0	—	—	180
3	1	4	10	42	146	170	135	65	53	20	12	6	2	666
4	1	16	40	146	412	378	299	197	86	42	16	13	1	1647
5	1	12	33	170	378	476	426	267	145	77	33	16	9	2043
6	0	11	37	135	299	426	402	251	163	84	31	13	2	1854
7	1	7	29	65	197	267	251	168	107	59	19	15	3	1188
8	1	7	5	53	86	145	163	107	70	39	24	10	1	711
9	4	1	6	20	42	77	84	59	39	38	12	5	—	387
10	—	1	0	12	16	33	31	19	24	12	2	3	—	153
11	—	-.-	—	6	13	16	13	15	10	5	3	-.-	—	81
12	—	—	—	2	1	9	2	3	1	—	—	-.-	—	18
Totals	9	63	180	666	1647	2043	1854	1188	711	387	153	81	18	9000

Number of Ripe Seeds in Second Pod. (row label, left side)

TABLE XLV.—*Lathyrus Sylvestris.* Sea Coast, Dorsetshire.

Number of Aborted Seeds in First Pod.

		2.	3.	4.	5.	6.	7.	8.	9.	10.	11.	12.	13.	14.	15.	16.	Totals.
Number of Aborted Seeds in Second Pod.	2	—	—	1	6	4	2	1	7	5	1	—	—	—	—	—	27
	3	—	—	3	4	11	21	18	13	10	8	2	—	—	—	—	90
	4	1	3	2	12	18	23	24	27	29	14	5	3	0	0	1	162
	5	6	4	12	28	46	61	73	66	45	25	7	5	0	0	0	378
	6	4	11	18	46	84	155	141	123	107	42	15	6	0	4	0	756
	7	2	21	23	61	155	250	286	262	164	123	41	14	0	2	0	1404
	8	1	18	24	73	141	286	344	354	241	143	41	16	0	0	1	1683
	9	7	13	27	66	123	262	354	396	305	195	55	22	0	0	2	1827
	10	5	10	29	45	107	164	241	305	254	148	65	19	0	1	2	1395
	11	1	8	14	25	42	123	143	195	148	100	39	15	0	1	1	855
	12	—	2	5	7	15	41	41	55	65	39	18	7	0	1	1	297
	13	—	—	3	5	6	14	16	22	19	15	7	0	0	0	1	108
	14	—	—	0	0	0	0	0	0	0	0	0	0	—	—	—	0
	15	—	—	0	0	4	2	0	0	1	1	1	0	—	—	—	9
	16	—	—	1	0	0	0	1	2	2	1	1	1	—	—	—	9
	Totals	27	90	162	378	756	1404	1683	1827	1395	855	297	108	0	9	9	9000

TABLE XLVI.—*Vicia Faba.* Danby Dale, Yorkshire.

Number of Ripe Seeds in First Pod.

		0.	1.	2.	3.	4.	5.	Totals.
Number of Ripe Seeds in Second Pod.	0	46	38	35	120	76	—	315
	1	38	46	70	142	89	2	387
	2	35	70	274	608	386	4	1377
	3	120	142	608	1542	1237	23	3672
	4	76	89	386	1237	1332	48	3168
	5	—	2	4	23	48	4	81
	Totals	315	387	1377	3672	3168	81	9000

TABLE XLVII.—*Vicia Hirsuta.* Danby Dale, Yorkshire.

Number of Ovules in First Pod.

		1.	2	3.	4.	5.	6.	Totals.
Number of Ovules in Second Pod.	1	6	22	33	21	4	—	86
	2	22	98	169	81	21	2	393
	3	33	169	360	292	86	10	950
	4	21	81	292	194	57	11	656
	5	4	21	86	57	38	6	212
	6	—	2	10	11	6	2	31
	Totals	86	393	950	656	212	31	2328

TABLE XLVIII.—*Vicia Hirsuta.* Danby Dale, Yorkshire.

Number of Ripe Seeds in First Pod.

		0.	1.	2.	3.	4.	5.	Totals.
Number of Ripe Seeds in Second Pod.	0	—	—	3	1	1	—	5
	1	—	160	165	89	31	—	445
	2	3	165	492	304	45	11	1020
	3	1	89	304	278	38	6	716
	4	1	31	45	38	10	—	120
	5	—	—	11	6	—	—	22
	Totals	5	445	1020	716	120	22	2328

TABLE XLIX.—*Vicia Hirsuta.* Danby Dale, Yorkshire.

Number of Aborted Seeds in First Pod.

		0.	1.	2.	3.	4	Totals.
Number of Aborted Seeds in Second Pod.	0	160	263	88	9	6	526
	1	263	788	176	52	10	1289
	2	88	176	92	16	16	388
	3	9	52	16	10	2	89
	4	6	10	16	2	2	36
	Totals	526	1289	388	89	36	2328

IX. *Wave-length Determinations and General Results obtained from a Detailed Examination of Spectra Photographed at the Solar Eclipse of January 22, 1898.*

By John Evershed, *F.R.A.S.*

Communicated by Dr. A. A. Rambaut, *F.R.S.*

[Plates 10–12.]

The results are here given of a detailed study and measurement of a series of ten spectra, photographed by me with a small prismatic camera, at the eclipse camp of the British Astronomical Association stationed at Talni, India.

The instrument referred to had an aperture of 50 millims., and a focal length of about 890 millims. It was fitted with two crown-glass prisms, each of 60° angle, placed in front of an ordinary visual objective, the component lenses of which were slightly separated in order to shorten the focus of the ultra-violet rays relatively to those of the visible spectrum.

[The prisms, which were set at minimum deviation for K, were originally intended for use in a spectroheliograph designed by Professor Hale, and were used by him in an attempt to photograph the corona in sunlight. They were made by the Zeiss Optical Company, of carefully selected glass, and are 59 millims. in height and 65 millims. measured on the faces, giving an effective aperture of about 42 millims. The total thickness of glass, or mean length of path traversed by the rays in glass, is therefore 65 millims.—*March* 9, 1901.]

The camera box was fitted with a long sliding plate-holder, moved by rack and pinion across the field of the lens and at right angles to the length of the spectrum. By this means it was possible to change the plates and make the successive exposures at very short intervals of time.*

In all the eclipse negatives obtained with this instrument, the extension of the spectra in the ultra-violet is greater than had been anticipated, and the density of the silver deposit is surprisingly uniform throughout the spectrum. In this connection it is of interest to compare the eclipse plates with a series of trial plates made

* For a full description of the methods of working see 'Report on the Indian Eclipse of 1898,' published by the British Astronomical Association.

16.11.1901.

in England for the purpose of securing a good focus over the greatest possible range of spectrum.   In these preliminary trials the prismatic camera was mounted in front of a large reflecting telescope, the eyepiece of which was replaced by a slit receiving light from a clear sky.

All the spectrum photographs obtained in this way are very dense in the region near G compared with other parts of the spectrum, and those which are correctly exposed for G give only a very feeble impression of the spectrum in the ultra-violet above K.   A very long exposure gave an extension nearly as great as in the eclipse negatives, but the image is extremely feeble at O (λ 3441), whilst the whole of the lower spectrum from λ 3700 down to D is very greatly over-exposed.

In the eclipse negatives, on the other hand, there is only a slight indication of over-exposure at G, and yet the lines are strongly shown up to the limits of the plates, one of which extends to λ 3340.   (Compare fig. 2 with figs. 1 and 3 in Plate 10.)   The same kinds of plates were used in both cases, and the very satisfactory results obtained at the eclipse are probably to be attributed to the extreme dryness of the air in Central India at the time of the eclipse.

The following table gives the approximate exposure times, the plates used, and the limits to which the spectra can be traced in the ten eclipse negatives.   As I was working without any assistance, the times of exposure could only be very roughly estimated.   In the case of the four cusp spectra, the times have been calculated from measures of the width of the strip of continuous spectrum.

Number.	Exposure times.		Plate used.	Spectrum photographed.	Limits of wave-length.	
		seconds			λ	λ
1	Second contact	– 20 (inst.)	EDWARDS' isochromatic medium	Cusp	336	to 590
2	,, ,,	– 10 (inst.)	Ditto	Cusp	340	,, 590
3	,, ,,	– 2 to + 2	Ditto, instantaneous	Flash	334	,, 600
4	,, ,,	+ 8 (inst.)	Ditto	Chromosphere	343	,, 580
5	,, ,,	+ 40 to + 75	Ditto	Corona	335	,, 590
6	,, ,,	+ 76 to + 85	Ditto, snapshot	Corona and chromosphere	350	,, 590
7	,, ,,	+ 86 to + 110	Ditto		344	,, 656
8	Third contact	+ 1 (inst.)	Sandell triple coated	Flash	345	,, 530
9	,, ,,	+ 9 (inst.)	Ditto	Cusp	345	,, 530
10	,, ,,	+ 19 (inst.)	Ditto	Cusp	345	,, 530

All the exposures made yielded good negatives, and Nos. 1, 2, 3, 4, and 7 were selected for measurement.   The first five spectra of the series are nearly perfect in focus between the limits λ 390 and λ 340, but the remaining five, exposed after mid-eclipse, are not quite so satisfactory, the focus being good only in the extreme ultra-violet, from 370 to 340.

Owing to the imperfect chromatic correction of the lens used, it was found necessary to incline the plate-holder about 7 degrees from the normal to the axis of the instru-

ment in order to obtain a good focus over a long range of spectrum, and the slight
shift of the whole spectrum in the direction of its length, which occurred when the
chromosphere arcs changed from the east side of the sun to the west, was sufficient to
spoil the focus for most of the spectrum, and shift the region of good focus further
towards the more refrangible end than had been allowed for.

It was found possible, however, to make fairly good measures of all the lines shown
in No. 7 spectrum.

Photographs of the Cusp before Totality.

Spectra Nos. 1 and 2 (Plate 10, fig. 1).—The first two photographs of the series are
images of the cusp spectrum, the exposures being made 20 and 10 seconds before
totality respectively.   At the moment when the first exposure was made, the strip of
photosphere still uncovered had a width of 8″ at the centre of the cusp, and this
had diminished to 4″ only when No. 2 was exposed.   The cusp therefore acted the
part of an exceedingly fine slit, and gave beautiful images of the Fraunhofer lines.

The two images obtained may be considered together as they are alike in every
respect, excepting in the width of the continuous spectrum.   This is 5·20 millims. in
No. 1, and 4·78 millims. in No. 2, the moon's diameter on the plate measuring 8·45
millims.   The continuous spectrum shows most of the features of the ordinary Fraun-
hofer spectrum, the dark lines being represented by very sharply defined curved arcs.
These are, however, very much less dark than the lines in the spectrum of ordinary
sunlight, and the hydrogen lines β, γ, and δ are not present as dark lines at all.
H and K are broad and dark, with an irregular reversal at one point showing motion
in the line of sight.

With the exception of the absent hydrogen lines the relative intensities of the
dark lines in these spectra appear to be identically the same as in the Fraunhofer
spectrum throughout the whole region photographed, but the cusp spectra have a
pale "washed-out" appearance when placed beside photographs of the spectrum of
ordinary daylight, and this peculiarity does not seem to be due to any accident of
exposure or development.*   (Compare the dark lines in figs. 1 and 3, Plate 10, with
the corresponding lines in fig. 2.)

* [After returning to England a set of photographs of the Fraunhofer spectrum was obtained with
the same prismatic camera, for comparison with the cusp spectra. These were obtained in the same
manner as the preliminary trial plates mentioned above, using a reflecting collimator and a slit.

In order to reproduce as closely as possible the form of the solar cusp a circular slit was used, this was
formed by a round hole cut in a plate of brass and nearly covered by a circular disc of slightly larger
diameter.   The hole was made to subtend an angle of 32′ at the prisms, and the ratio of the diameters of
the hole and disc was made the same as that of the sun and moon at the eclipse.

In making the exposures the slit was simply directed to a clear sky at a considerable altitude. Fig. 2,
Plate 10, is a reproduction of one of the spectra so obtained, in which the breadth of the band is the same
as that of the cusp spectrum in fig. 1.   The angular width of the slit was therefore the same as that of
the solar cusp at the moment when the cusp spectrum was photographed (about 8″ at the middle point).—
March 9, 1901.]

The bright radiations of the chromosphere are in no cases strong enough to show as bright lines upon the continuous spectrum, although they are strongly impressed along the borders, where they form in many cases a continuation of the dark arcs of the Fraunhofer spectrum. The very strong bright arcs, H and K, can be traced round the limb of the moon for a greater distance than any others.

No. 2 spectrum appears to have been under-exposed and the bright arcs are rather difficult to detect upon it, although the exposure was made within 10 seconds of second contact.

Measures were not made of the bright lines in either spectrum as they are not very well adapted for accurate determinations. All of them can be identified with the stronger lines in No. 3 spectrum.

Methods of Measurement and Reduction of the Spectra.

The Fraunhofer spectrum was carefully measured in both cusp spectra and identified line by line with HIGGS's photographic map of the normal solar spectrum.

As these measures form the basis for the determination of wave-lengths in spectra No. 3 (flash) and No. 4, it is desirable to give a brief account of the methods adopted.

I am indebted to Dr. RAMBAUT for the loan of an excellent millimetre micrometer belonging to the Radcliffe Observatory.

This instrument consists of a microscope mounted on a sliding frame, and moved by a long screw with a pitch of 1 millim. The head of the screw is divided into 100 parts, and by estimation can be read to ·001 millim. All the measures were made upon this micrometer.

Each spectrum was measured completely three times over, using different sections of the screw and different readings of the head each time. By this means any systematic differences due to errors in the screw were eliminated in the mean result, and accidental mistakes in reading the divisions were readily found and corrected. No evidence of any systematic error in the screw was, however, detected, and the three sets of measures of spectra Nos. 1 and 2 showed a very satisfactory accordance throughout the whole length measured.

In measuring a line, the mean of three settings of the spider lines was taken as the scale reading of the line, and in the final mean result, in which the two cusp spectra are combined, every line represents eighteen settings of the spider lines.

The measures of the two cusp spectra when compared were found to agree as closely as two separate measures of either spectrum, thus proving that no change had occurred in the dispersion from any cause. The two spectra were therefore considered to be identical, and the measures were combined.

From the final mean of the six sets of measures about thirty well-defined and isolated lines were selected as standards; these are all distributed through the ultra-violet portion of the spectrum more refrangible than H. In the region between

H and F about twenty-six lines were used, which are, however, not so well defined or so satisfactory as standards as the others.

In constructing a scale of wave-lengths for the reduction of the bright-line spectra (Nos. 3 and 4), the best defined known lines between D and H in these spectra were also selected as standards, and used together with the twenty-six cusp spectrum lines. But in the ultra-violet the scale depends entirely upon the standards of the cusp spectrum, the whole of the lines in this region in Nos. 3 and 4 (including the hydrogen lines) being treated as unknown.   A careful comparison made between measures of known lines in Nos. 3 and 4 spectra and the mean measures of the two cusp spectra shows that it is safe to assume that all the spectra obtained before mid-eclipse may be considered to be identical as regards dispersion.

A sufficient number of standards well distributed throughout the entire spectrum being thus obtained, the approximate relation between scale reading and wave-length at all points in the spectrum was next determined by graphical methods, and from the interpolation curve obtained it was easy to compute the scale reading corresponding to a number of definite points in the spectrum separated by equal intervals of wave-length.

The scale reading corresponding to each 50 tenth-metres of wave-length was in this way computed, taking in each case the mean result given by three or four standard lines situated near ; and the values thus obtained were finally slightly corrected by smoothing differences until fifth differences were made to increase in a regular progression.

The smoothed values in no case differ from the values derived from the measures by quantities greater than the uncertainties of the measures themselves.   The mean difference in the best defined portion of the spectrum λ 4100 to λ 3400 is ·0016 millim., and the greatest difference ·005 millim., the latter corresponding to ·15 tenth-metre at λ 4100 and ·07 tenth-metre at λ 3400.

In the visible spectrum the definition under the microscope is very poor in all the negatives except No. 4, and the measures in consequence are not very consistent, but for the sake of completeness every line has been measured, and an accurate scale of wave-lengths constructed for the entire range of spectrum photographed.

From the accordance between the different sets of measures of the bright lines in No. 3 spectrum the probable errors of the wave-length determinations given in Table I. (p. 404 et seq.) would appear to be about ·15 tenth-metre at λ 4000, decreasing to ·05 tenth-metre at λ 3400.

The Bright-line Spectra.

Spectrum No. 3 (Plate 11, fig. 1).—This negative was exposed at the moment when the dark lines of the Fraunhofer spectrum were seen to disappear and innumerable bright lines flashed out across the strip of continuous spectrum which still remained visible.

I was observing with a slitless spectroscope attached to a 3-inch equatorial, and made the exposure as nearly as I could judge 2 seconds before the last trace of continuous spectrum vanished. The plate was given an exposure of at least 4 seconds in order to impress the fainter details.

In the photograph the strip of continuous spectrum is divided into four parts by longitudinal dark bands due to irregularities in the moon's limb. Most of the bright lines of the chromosphere and flash spectra are visible as bright lines upon the continuous spectrum, but the helium lines at λ 4026·5 and λ 4713·2 are an exception, and they cannot be traced even in the dark spaces between the strips of continuous spectrum, although strongly impressed on either side. The helium line at λ 4472 is visible, but weaker in the intermediate dark spaces. No trace of any absorption lines can be detected in any part of the continuous spectrum.

The total depth of the chromosphere as indicated by the calcium lines H and K is 11″·6, these lines extending over an arc of 124 degrees. The hydrogen lines β, γ, δ, ε, and the titanium lines at λ 3761·5, λ 3759·3, and λ 3685·3, as well as the helium line D‴, extend over an arc of about 102 degrees, implying a depth of 8″·2.

The bright arcs of the flash spectrum proper are approximately of the same length. On the north side of the continuous spectrum they are cut off abruptly by a dark band, due to a projecting lunar mountain, and two similar bands interrupt the lines on the south side.

Most of the fainter flash lines can be traced over an arc of 40 degrees of the sun's limb, indicating a depth of 1″·3, or nearly 600 miles.

A great many of the stronger lines, however (chiefly titanium, iron, and magnesium lines), extend faintly far beyond this limit, and can be traced over an arc of nearly 70 degrees, the depth implied in this case exceeding 4″, or 1800 miles.*

A few very faint lines due to the very lowest layers of the chromosphere are also indicated in the best defined portion of the negative. These are visible only upon the strips of continuous spectrum, and do not extend beyond.

Probably only a small proportion of the finer lines which were actually present at second contact are shown in this photograph. In the green region near b this is certainly the case. The most striking feature seen when observing the spectrum under high dispersion and without any slit was the immense number of excessively fine bright arcs which appeared in a short section of the spectrum (including the b lines) at the moment when the exposure was made. Yet very few of these lines appear on the photograph, partly perhaps on account of the poor focus in this region.

---

* [In estimating depths from the lengths of the arcs the moon's apparent semi-diameter is taken at 997″·0 and that of the sun 971″·9   A uniform distribution of the gases round the limb is of course assumed, and the results from No 3 spectrum indicate the depth measured from the photosphere to the upper limit to which each particular radiation can be traced. The depth of photosphere left uncovered by the moon in this photograph being quite inappreciable, the limb of the moon may be considered as being coincident with that of the sun at the point of second contact.—*March* 9, 1901.]

The continuous spectrum of the corona is strongly marked on this photograph, but only one true corona line is visible, the green line at λ 5303. This line can be traced only on the eastern portion of the moon's limb, and, unlike the chromosphere arcs, it is very variable in intensity, being strongly marked about the position angles 60° to 70°, and 100°, where the continuous spectrum is also strong, but extremely weak or entirely absent in other parts of the limb.

The results of measures of the corona line on this plate and No. 7 spectra are given on p. 401.

[In the reproductions of the bright-line spectra shown in Plate 11, the position angles may be inferred from the fact that the direction of dispersion is also the direction of the moon's path across the sun, the point of second contact (P.A. 56°) being at the centre of the arcs on the right-hand side, and third contact (P.A. 236°) at the centre on the left. The angles are reckoned from the north point through east, south, and west.

The upper edge of each spectrum, therefore, represents the south-east limb, and the lower edge the north-west limb. The diagram below, showing the prominences in their correct positions, will make this clear. The broken lines indicate the positions where the green corona line is strongest on the negatives. In the reproductions it can be faintly traced on the east limb, where it persists throughout totality, thus proving it to be a true coronal radiation, since the chromosphere on this limb was entirely hidden by the moon after mid-eclipse.
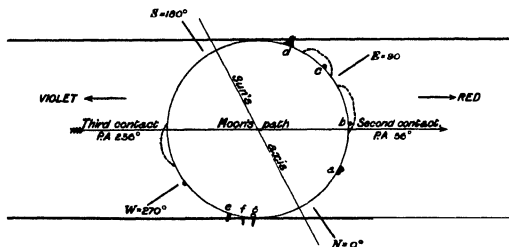
In fig. 5 the interrupted coronal arc is perhaps best seen; it almost meets the oppositely curved chromosphere arc D'''.

The wave-length scale at the top of Plate 11 must be understood as applying only to fig. 1, for which it was constructed. The designation of the principal lines in the spectra are given underneath fig. 5, at the bottom of the plate.

For further facilitating references to the bright lines a plate is given of the ultra-violet region of the spectrum of the lower chromosphere. In this a wave-length scale is given above and the designation of some of the principal lines below. The plate has been prepared by purely automatic methods from the original negative taken at second contact. By photographing a longitudinal section of this negative with a cylindrical lens interposed, the short sections of the chromosphere arcs are spread out into straight lines.

Great care was taken to exclude spurious lines due to defects in the film or to dust on the slit. By using a rather wide slit these imperfections have been reduced to a minimum, although this involved a slight sacrifice of definition and loss of detail.—March 9, 1901.]

Spectrum No. 4 (Plate 11, fig. 2).—The exposure in this photograph was probably less than half a second duration, and was made several seconds after closing No. 3 exposure.

The continuous spectrum of the corona is faintly impressed* and the green line is just traceable at the points where it is strongest in No. 3 spectrum. The chromosphere arcs H and K and the hydrogen series are narrow and well defined, the latter both in the visible spectrum and the ultra-violet. The three titanium lines at λλ 3761·5, 3759·3, and 3685·3 are shown as exceedingly fine threads, extending over an arc of about 60 degrees; they are distinctly stronger than the line Hη but fainter than Hζ. The hydrogen lines β, γ, δ and the calcium lines H and K can be traced over nearly the same extent of the limb as in No. 3 spectrum.

The prominences are well defined in this negative and their spectra exhibit some interesting features. There are seven prominences clearly shown, which I have designated by the letters of the alphabet in the order of their position angles, as follows :—

Prominence.	Position angle.	Solar latitude.
A	28°	+54°
B	59	+23
C	99	−17
D	126	−44
E	306	+44
F	315	+53
G	322	+60

All these prominences are about equally intense in the calcium lines H and K ; D is the largest and B and C are the smallest images.

* In preparing a plate suitable for reproduction it was found necessary to intensify the image with uranium. This has made the continuous spectrum of the corona appear very much stronger than it is in the original negative.

In the hydrogen images A, and the group E, F, G are much fainter than the others, and in the ultra-violet the images of these four are very difficult to trace beyond the line Hθ.   A is however very faintly indicated in the three titanium lines above mentioned.

The prominence C is much brighter in hydrogen than the foregoing and can be traced to the line Hμ.   The three titanium lines are also clearly shown.

The prominences B and D are the strongest of all in hydrogen, and although very different in size and general character (as observed in Hα about an hour before totality) they give practically the same spectra.

The prominence B being situated very near the point of second contact (P.A. 56°) its spectrum falls almost exactly along the centre of the photograph.   It is, therefore, in the best position for accurate measures ; and as the images in the ultra-violet are well defined circular black spots, much smaller and more definite than the corresponding images of the large prominence D, the wave-lengths given in Table II. are deduced from measures of this prominence only.   (See p. 411 *et seq.*)

Nearly all the lines measured can however be traced also in D ; there are fifty-two altogether and all of these are accounted for by the elements H, He, Mg, Al, Ca, Sc, Ti, Fe, and Sr.

A remarkable feature in the spectra of the prominences B and D is well shown in this negative.   The prominence D shows traces of a continuous spectrum in the region less refrangible than H, which is absent in B.   Both prominences, however, give a strong continuous spectrum in the extreme ultra-violet, beginning abruptly at λ 3668 near the end of the hydrogen series, and extending as far towards the smaller wave-lengths as the impression of the corona can be traced.

The actual limit to which the extremely narrow streak due to the smaller prominence B can be traced is at λ 3435 $\pm$.

This feature is shown more or less distinctly on all the negatives taken during totality.   In No. 3 spectrum as well as No. 7 a considerable arc of the chromosphere itself shows the ultra-violet continuous spectrum, all the flash spectrum lines more refrangible than λ 3668 being immersed in a uniformly shaded band which is absent in the less refrangible region.

(In the reproductions this delicate feature is lost in all the images except fig. 2. It is shown more satisfactorily in the enlarged image in Plate 12.)

Spectrum No. 5 (Plate 11, fig. 3).—This was exposed during about 40 seconds near the time of mid-eclipse.   The exposure began about 10 seconds after the disappearance of the chromosphere on the east limb and ended immediately after its partial reappearance on the west limb.

The continuous spectrum of the corona is strongly impressed, and can be traced as far as λ 3300 in the ultra-violet.   The green corona line is well shown on the east limb ; it is distinguished from radiations due to the chromosphere by its diffuse character and distinct maxima of intensity corresponding with the brighter regions of the

inner corona ; it is most intense at position angles 60° to 78° and 95° to 105°. On the west side it can be traced at about position angle 250°, but is extremely faint here.

No trace of any true corona line can be made out in the blue or violet part of the spectrum, but there is a fairly distinct impression of a line near the end of the plate at λ 3388, where faint maxima of intensity are indicated on the east limb at the same position angles as the maxima of the green line.

The chromosphere in this negative is represented by the lines H and K only, which form arcs extending over the western limb (P.A. 182°–289°). The prominence C gives the principal hydrogen, helium, and titanium, lines in addition to H and K.

Spectrum No. 6 (Plate 11, fig. 4).—The exposure in this case was of 10 seconds' duration, beginning about 2 seconds after closing No. 5.

The chromosphere arcs have reappeared in the hydrogen and helium lines on the west limb, and the former can be traced to Hπ. The three strongest titanium lines are also visible, extending over an arc of 60°. The prominence C shows the streak of continuous spectrum starting at λ 3668 and extending as a fine thread far into the ultra-violet.

The green corona line is faintly impressed both on the east and west limbs, but no other corona line can be made out with any certainty. There is, however, a doubtful impression of a line extending pretty uniformly over the west limb at a point in the spectrum a little less refrangible than H (λ 398/399). It cannot be traced at all on the east limb, at the points where the green line is most intense, and if it is a real corona line, the substance producing it is evidently not the same as that which gives the green line.

Spectrum No. 7 (Plate 11, fig. 5).—The exposure of 20 seconds' duration was started immediately after the last and ended only 3 or 4 seconds before the photo-sphere appeared.

In the chromosphere spectrum Hα is well shown, and in the ultra-violet the hydrogen lines can be clearly traced to Hω. Many of the stronger flash-spectrum lines are visible along the central portion of the photograph extending over arcs of from 40° to 60°. The wave-lengths deduced from measures of these lines are given in Table I., and in the region between λ 340 and λ 360, where the focus is good, the results are as accurate as those obtained from No. 3 spectrum.

A conspicuous feature in this photograph is the shaded band of continuous spectrum in which the more refrangible lines of the flash spectrum are immersed. It is absent in the region less refrangible than λ 3668, and corresponds to the fine streak or tail which the prominence C shows in all the photographs obtained during totality.

The green corona line is better defined in this photograph than in any of the others. The positions of maximum intensity correspond precisely with the bands of continuous spectrum due to the brightest regions of the inner corona. The position angles measured on this plate are as follows :—

	Position angle of maximum intensity.
East limb	{ 57° to 78° 94 „ 100
West limb	235 „ 254

As in No. 6 spectrum there is an exceedingly faint impression of a corona line near H and extending over the west limb, but excepting this no trace of any other lines are visible. The image, unfortunately, is too feeble in the extreme ultra-violet to show the line at λ 3388 found on No. 5 spectrum.

Spectrum No. 8.—This negative was exposed for about half a second almost at the moment of third contact. The reappearing photosphere has been impressed as four very narrow streaks of continuous spectrum. The flash spectrum is well developed, and the whole length of the spectrum is crowded with bright lines extending between and across the strips of continuous spectrum. The majority of the lines due to the lower chromosphere extend over an arc of 55°, the depth implied in this case being 2″·5.

The focus in this negative is unfortunately very poor throughout the spectrum, and no measurements were made for determining wave-lengths. The spectrum was, however, carefully compared with No. 3 by means of enlargements of each spectrum, which were made to correspond approximately in scale. Superposing one spectrum upon the other it was found that the two appeared to be identical in all respects. If any differences exist at all between the spectra of the east and west limbs of the sun they can only be found in the finer details, which in No. 8 are lost by reason of the imperfect focus.*

The Cusp Spectra after Totality.

Spectra Nos. 9 and 10 (Plate 10, fig. 3).—These were given an exposure of less than half a second at about 10 and 18 seconds after third contact respectively. They are both very similar to the first two spectra of the series, but apparently the exposures were appreciably of longer duration than those made before totality, and the bright lines bordering the continuous spectrum of the photosphere are in consequence much more strongly marked than in Nos. 1 and 2.

As in all the spectra of the west limb the focus is imperfect except in the ultra-violet region more refrangible than λ 360. The bright lines are, however, fairly well defined along the south edge of the cusp.

The Fraunhofer spectrum is not nearly so well developed in these spectra as it is in Nos. 1 and 2. This may be due to over-exposure as well as poor focus, but at the ultra-violet end where the spectrum is less dense and the focus good, the dark lines,

---

* No figure is given of this photograph as it was found impossible to produce a satisfactory plate suitable for reproduction.

although clearly visible, are not nearly so strongly marked as in the corresponding negatives of the east limb before totality.

All the more conspicuous lines which are characteristic of the flash spectrum can be distinguished along the south edge of the cusp in both No. 9 and No. 10 spectra.

GENERAL RESULTS AND CONCLUSIONS.

The Flash Spectrum.

In comparing the wave-length values of No. 3 and No. 7 spectra given in Table I. with ROWLAND's wave-lengths of the absorption lines in the solar spectrum, it is at once apparent that practically every strong line in the latter is present in the lower chromosphere as a bright line. In the region between λλ 3840 and 4410 there are 58 strong dark lines with an intensity exceeding 8 on ROWLAND's scale of intensities. In the flash spectrum (No. 3 photograph) 44 of these lines are certainly present as bright lines; 6 are probably present, 6 are too near to strong hydrogen lines to be separately distinguishable, and 1 is obscured by H. One line only of the 58 is unaccountably absent, that at 3788·046 (Fe, intensity 9).

It is further to be observed that of the bright lines of the lower chromosphere (hydrogen and helium lines being excluded) the great majority appear to be coincident with dark lines having an intensity on ROWLAND's scale not less than 3, and two lines only λ 3584·37 and 3812·79 occur in a blank space in the solar spectrum where the lines are weaker than intensity 0.

It may be said generally therefore, with regard to the ultra-violet region, that the bright lines of the flash spectrum are reversals of Fraunhofer lines, including *all* the very strong lines (intensity 10 and upwards).

Whilst the positions of almost all the bright lines appear to coincide with dark lines in the solar spectrum, the relative intensities of the lines in the latter are widely departed from in the flash spectrum. A negative of No. 3 spectrum is therefore very unlike a positive of the solar spectrum.

When, however, the flash spectrum is analysed into its separate constituents it is found that the relative intensities of the lines of any one element correspond very closely with the intensities of the dark lines of that element. In the iron and titanium spectra, which claim more lines of the flash spectrum than any other elements, this correspondence is most clearly shown. There are, as might be expected, many cases where the intensities are abnormal, but by taking the average intensity on ROWLAND's scale of *all* the dark lines corresponding to each unit of intensity in the flash spectrum the effects of these are eliminated. These abnormal intensities are probably due to the superposition of closely adjacent lines.

The following tables prove the general close correspondence of intensities in the flash and solar spectrum for iron and titanium. The first column gives the number of flash lines having the estimated intensity given in column 2, whilst the average

intensity of the corresponding dark lines is given in column 3.   Although the
intensities of the bright lines, estimated on a scale of 1 to 10, are not comparable
with those of Rowland ranging from 1 to 1000, the relation between the bright and
dark line spectra is, nevertheless, clearly indicated by the regular progression of the
figures in the last column of both tables corresponding with the progression between
0 and 5 in the middle column :—

Intensities of Iron and Titanium Lines in the Chromosphere and in the Solar
Spectrum.

Iron Lines.

Number of lines in flash spectrum.	Photographic intensity.	Average intensity of the corresponding dark lines in ⊙ (Rowland).
17	0	5 5
33	1	
25	2	
10	3	
Total 85	Weighted mean 1·3	Weighted mean 10·2

Titanium Lines.

Number of lines in flash spectrum.	Photographic intensity.	Average intensity of the corresponding dark lines in ⊙ (Rowland).
5	0	1·6
9	1	3·4
13	2	4·2
9	3	5·4
4	4	7·7
1	5	10·0
Total 41	Weighted mean 2·0	Weighted mean 4·5

The mean intensities given at the bottom of each column show the greater
intensity of the Ti lines in the chromosphere compared with Fe, and the much
greater average intensity of the Fe dark lines in the solar spectrum compared with
the Ti lines.

The striking dissimilarity in the relative intensities of the lines of different
elements in the bright-line and dark-line spectra is probably due to the *unequal
heights* to which the gases of the various elements ascend in the chromosphere.

The intensities as they appear in photographs of the flash spectrum are evidently
largely determined by the relative depths of the various gaseous strata, the more
extensively diffused gases giving the strongest bright arcs simply by reason of the
greater radiating area.   The low-lying gases, on the other hand, although they may be

intrinsically more luminous and denser, give fainter arcs because of the excessively small angular width of the radiating area.

It is to be borne in mind that the flash-spectrum arcs or "lines" obtained with a prismatic camera are not true images of the strata producing them, but diffraction images more or less enlarged by photographic irradiation. The deepest layers subtend an angle of only 2″, representing on the sun about 900 miles, and mono-chromatic radiations from such strata will produce lines which are as narrow as instruments of ordinary resolving power are capable of defining. The diffraction image of a shallower stratum will not be narrower in proportion but simply less bright. A layer of gas 9 miles in depth would, if sufficiently dense, be capable of producing strong absorption lines in the solar spectrum, yet it would have to be 100 times more luminous than a stratum extending for 2″ above the photosphere to give equally strong bright lines in the photograph, even were the moon to remain stationary at the contact and allow of equal exposures.

But it is obvious that, for the shallow layers, the time during which the plate is exposed to any radiation is proportional to the depth of the luminous layer, the advancing limb of the moon cutting off the shallow strata during the exposure; consequently, for an exposure timed from the moment of second contact and lasting until the whole depth of 2″ had been occulted, the plate would be exposed to the 9-mile stratum for only $\frac{1}{100}$th of the whole exposure. This layer would therefore need to be 10,000 times more luminous to give lines of equal intensity to those of the deepest layers.

[In the *absorption* spectrum the relative intensities between the different elements must depend on the total number of absorbing molecules of each element encountered by a ray of light in its passage outward from the photosphere, and not on the relative state of diffusion or depth of the different gases.

To illustrate these points, which appear to me to be of great importance in elucidating the relationship between the emission and absorption spectra, we may consider the typical cases of iron and of titanium in the chromosphere.

The total number of iron molecules encountered by a ray of photospheric light probably vastly exceeds the number of titanium molecules, since the absorption lines of the former are among the strongest and broadest in the spectrum, whilst those of the latter are narrow and mostly weak lines. But the iron vapour is concen-trated in a stratum close down upon the photosphere, whilst titanium is diffused through the entire depth of the chromosphere. Hence the *apparent* intensities of the emission lines of iron fall considerably below those of titanium.

If the relative distribution of these two elements were reversed without changing the total quantity of each, the absorption lines would probably not be materially altered in intensity, but the bright lines of iron would then rival those of calcium in H and K, whilst the titanium lines would be difficult to detect in the flash spectrum.—March 10, 1901.]

It thus appears that the intensities found in these spectra by no means represent the actual intrinsic intensities of the bright lines of the different elements. Could we obtain a sample of the incandescent gas from near the base of the flash-spectrum layer and examine it close at hand with a slit spectroscope, it is certain that the relative intensities would differ widely from those in the flash spectrum as observed at an eclipse, and it is possible that they would be found to correspond much more closely with the relative intensities in the Fraunhofer spectrum.

From the foregoing considerations it is clear that the emission lines from the lowest levels of the flash layer must be very difficult to observe, however bright they may be intrinsically. The wide divergence between the flash and Fraunhofer spectra, with respect to intensities, would appear, therefore, to afford no ground for abandoning the original interpretation of the flash proposed by Professor YOUNG from his observation in 1870, and the evidence of these photographs certainly indicates that the flash does, in fact, represent the upper, more diffused portion of a true reversing stratum.

In the flash-spectrum photograph (No. 3) fifteen elements can be identified with certainty in the lower chromosphere, in addition to hydrogen and helium, and there are three elements doubtfully represented.

Arranged according to their relative intensities, the following four groups occur :—

Group I.

(Lines strong in flash and in solar spectrum.)

Na	.	.	.	atomic weight	23·0
Mg	.	.	.	.,	,,	24·3
Al	.	.	.	,,	,.	27·1
Ca	.	.	.	.,	,,	40·0

Group II.

(Lines strong in flash but comparatively weak in solar spectrum.)

Sc	.	.	.	atomic weight	44·1
Ti	.	.	.	,,	,,	48·1
V	.	.	.	,,	.,	51·2
Cr	.	.	.	,,	,,	52·1
Mn	.	.	.	,,	,,	55·0
Sr	.	.		,,	,,	87·6
Y	.	.	.	,,	,,	88·7

Group III.

(Lines relatively weak in flash, very strong in solar spectrum.)

Fe	.	.	.	atomic weight	56·0
Ni	.	.	.	,,	,,	58·7

intrinsically more luminous and denser, give fainter arcs because of the excessively small angular width of the radiating area.

It is to be borne in mind that the flash-spectrum arcs or "lines" obtained with a prismatic camera are not true images of the strata producing them, but diffraction images more or less enlarged by photographic irradiation. The deepest layers subtend an angle of only 2″, representing on the sun about 900 miles, and mono-chromatic radiations from such strata will produce lines which are as narrow as instruments of ordinary resolving power are capable of defining. The diffraction image of a shallower stratum will not be narrower in proportion but simply less bright. A layer of gas 9 miles in depth would, if sufficiently dense, be capable of producing strong absorption lines in the solar spectrum, yet it would have to be 100 times more luminous than a stratum extending for 2″ above the photosphere to give equally strong bright lines in the photograph, even were the moon to remain stationary at the contact and allow of equal exposures.

But it is obvious that, for the shallow layers, the time during which the plate is exposed to any radiation is proportional to the depth of the luminous layer, the advancing limb of the moon cutting off the shallow strata during the exposure; consequently, for an exposure timed from the moment of second contact and lasting until the whole depth of 2″ had been occulted, the plate would be exposed to the 9-mile stratum for only $\frac{1}{100}$th of the whole exposure. This layer would therefore need to be 10,000 times more luminous to give lines of equal intensity to those of the deepest layers.

[In the *absorption* spectrum the relative intensities between the different elements must depend on the total number of absorbing molecules of each element encountered by a ray of light in its passage outward from the photosphere, and not on the relative state of diffusion or depth of the different gases.

To illustrate these points, which appear to me to be of great importance in elucidating the relationship between the emission and absorption spectra, we may consider the typical cases of iron and of titanium in the chromosphere.

The total number of iron molecules encountered by a ray of photospheric light probably vastly exceeds the number of titanium molecules, since the absorption lines of the former are among the strongest and broadest in the spectrum, whilst those of the latter are narrow and mostly weak lines. But the iron vapour is concen-trated in a stratum close down upon the photosphere, whilst titanium is diffused through the entire depth of the chromosphere. Hence the *apparent* intensities of the emission lines of iron fall considerably below those of titanium.

If the relative distribution of these two elements were reversed without changing the total quantity of each, the absorption lines would probably not be materially altered in intensity, but the bright lines of iron would then rival those of calcium in H and K, whilst the titanium lines would be difficult to detect in the flash spectrum.—
March 10, 1901.]

It thus appears that the intensities found in these spectra by no means represent the actual intrinsic intensities of the bright lines of the different elements. Could we obtain a sample of the incandescent gas from near the base of the flash-spectrum layer and examine it close at hand with a slit spectroscope, it is certain that the relative intensities would differ widely from those in the flash spectrum as observed at an eclipse, and it is possible that they would be found to correspond much more closely with the relative intensities in the Fraunhofer spectrum.

From the foregoing considerations it is clear that the emission lines from the lowest levels of the flash layer must be very difficult to observe, however bright they may be intrinsically. The wide divergence between the flash and Fraunhofer spectra, with respect to intensities, would appear, therefore, to afford no ground for abandoning the original interpretation of the flash proposed by Professor YOUNG from his observation in 1870, and the evidence of these photographs certainly indicates that the flash does, in fact, represent the upper, more diffused portion of a true reversing stratum.

In the flash-spectrum photograph (No. 3) fifteen elements can be identified with certainty in the lower chromosphere, in addition to hydrogen and helium, and there are three elements doubtfully represented.

Arranged according to their relative intensities, the following four groups occur :—

Group I.

(Lines strong in flash and in solar spectrum.)

Na	.	.	.	atomic weight	23·0
Mg	.	.	.	,,	,,	24·3
Al	.	.	.	,,	,,	27·1
Ca	.	.	.	,,	,,	40·0

Group II.

(Lines strong in flash but comparatively weak in solar spectrum.)

Sc	.	.	.	atomic weight	44·1
Ti	.	.	.	,,	,,	48·1
V	.	.	.	,,	,,	51·2
Cr	.	.	.	,,	,,	52·1
Mn	.	.	.	,,	,,	55·0
Sr	.	.	.	,,	,,	87·6
Y	.	.	.	,,	,,	88·7

Group III.

(Lines relatively weak in flash, very strong in solar spectrum.)

Fe	.	.	.	atomic weight	56·0
Ni	.	.	.	,,	,,	58·7

<div align="center">

Group IV.

(Lines weak in flash and in solar spectrum.)

Zr . . . atomic weight 90·6
C* . . .   ,,   ,,   12·0

</div>

The elements of which the identifications are somewhat doubtful are—

Si . . . atomic weight 28·4 One line only λ 3905·660
Co . . .   ,,   ,,   59·0 Many Co lines seem unaccountably absent
                                       in the flash
La . . .   ,,   ,,   138·5 Two lines only λ 4123·384 and λ 3649·654

It is probable that the elements of Group I. are diffused throughout the entire depth of the flash-spectrum layer, but become denser near the photosphere. Those of Group II. are possibly absent from the very lowest strata, but are widely diffused in the higher regions; whilst the elements Fe and Ni (Group III.) would appear to be mostly concentrated in the lowest strata where the density is great enough to produce the winged absorption lines.

Group IV. probably represents low-lying elements of small density.

The bright lines of the flash spectrum corresponding to Group II. may be considered to be true reversals of the dark solar lines of these elements; that is to say, the whole of the matter concerned in the absorption contributes to the emission spectra, and the bright and dark lines are practically of equal width.

But in the emission lines of Groups I. and III. the radiation from the very lowest region, where the density of each element is considerable, contributes very little indeed to the total light, most of which comes from the higher more extensive regions of low density; consequently the lines appear narrow and without appreciable shading. These lines, therefore, can only be considered as partial reversals of the corresponding absorption lines.

The metallic elements found in the lower chromosphere (including those doubtfully identified) include all the known metals having atomic weights between 20 and 60, with the single exception of potassium. Arranged according to their atomic weights, they are :—

<div align="center">

⎧ Na . . . approximate atomic weight 23
⎨ Mg . . .   ,,   ,,   ,,   24
⎩ Al . . .   ,,   ,,   ,,   27
   Ca . . .   ,,   ,,   ,,   40
   Sc . . .   ,,   ,,   ,,   44
   Ti . . .   ,,   ,,   ,,   48
| V . . .   ,,   ,,   ,,   51

</div>

* Probably the compound cyanogen.

Cr   .   .   .   approximate atomic weight   52
Mn   .   .   .   ,,   ,,   ,,   55
Fe   .   .   .   ,,   ,,   ,,   56
Ni   .   .   .   ,,   ,,   ,,   59
{ Co ? .   .   .   ,,   ,,   ,,   59
{ Sr   .   .   .   ,,   ,,   ,,   88
{ Y   .   .   .   ,,   ,   ,,   89
{ Zr   .   .   .,   ,,   ,,   91

Taking the whole series of known elements having atomic weights from 1 to 100, all the metals are here represented excepting the following :—

Li   .   .   .   approximate atomic weight   7
Be   .   .   .   ,,   ,,   .,   9
K   .   .   .   ,,   ,,   .,   39
Cu   .   .   .   ,,   ,,   ,.   64
Zn   .   .,   ,   ,,   65
Ga   .   .   ,   ,.   ..   70
Ge   .   .   .,   ,,   .,   72
Rb   .   ,.   ..   .,   85
Nb   .   .   .   ,,   ,,   ,.   94
Mo   .   .   .   .,   ,,   ,,   96

Of the nineteen known metalloids, the only ones indicated in these spectra are—

H   .   .   .   approximate atomic weight =   1
He   .   .   ,,   ,,   ,,   4
C } .   .   .   ,,   .,   ,,   12
N } .   .   .   ,,   ..   ,,   14
Si ? .   .   .   .,   ,,   ,,   28

None of the elements with atomic weights exceeding 91 appear to be represented in the flash spectrum, unless we include lanthanum (at. wgt. 138). It is probable, however, that with instruments of much greater aperture and focal length than the one employed in this research photographs would be obtained which would bring to light many of the heavier metals which doubtless exist in the lowest strata of the flash-spectrum layer.

Although it would seem from the above statement that there is a general, inverse, relation between the atomic weight of an element and the extent of diffusion of its vapour above the photosphere, it is evident that among the elements actually found in the chromosphere this relation does not always hold ; thus calcium and titanium ascend to far greater elevations than the three elements of smaller atomic weight, sodium, magnesium and aluminium.

Unknown Lines.—Only a small proportion of the lines in Table I. remain to be

identified. In the region between λλ 3340 and 4000 there are 29 such lines in a total of 225, and all of them are weak lines. The strongest (intensity 2) correspond in position with dark lines in ROWLAND's table which have not been identified with any element, and these are given in the following list :—

Wave-length in flash.	Intensity.	Wave-length in ⊙.	Intensity.
3567·77	2	3567·835	4
3589·72	2	3589·773	5
3645·35	2	⎰ 3645·429 ⎱ 3645·475	3 3
3655·78	1	3655·801	3

These are the only unknown lines in the flash spectrum with an intensity greater than 1, and it is just possible that they belong to the same element.

The Hydrogen Spectrum.

The wave-lengths of the hydrogen lines taken from Tables I. and II. are entered separately in Table III., together with the values computed from BALMER's formula. The latter were computed for a vacuum and corrected for air in accordance with a table of RUNGE ('Astronomy and Astro-physics,' vol. 12, No. 5).

In the formula

$$\lambda = \frac{S^2}{a(S^2 - 4)},$$

S is the series number of the line, and a is a constant derived from ROWLAND's measures of the lines Hα, Hβ, and Hγ in the solar spectrum reduced to a vacuum, the mean value adopted being $a = 27418·75$.

In obtaining the mean values given in column 4, equal weights are given to No. 3 and No. 4 spectra, both of which are in good focus beyond K, and give equally consistent measures in this region. No. 4 spectrum is, however, much the better of the two in the visible region, where the images are comparatively small, well-defined spots. This spectrum, as already mentioned, consists of a series of images of a very small bright prominence, and very consistent measures were obtained throughout. The good agreement of the wave-lengths of the metallic lines with ROWLAND's values will be seen on referring to Table II.

The very close agreement of the hydrogen lines with the computed values is shown in column 6 of Table III. (observation − calculation). It will be noticed, however, that the wave-lengths of the more refrangible members of the series are appreciably smaller than the theoretical values, the differences increasing towards the limit of the series. This may be due to the greater uncertainty of measurement of the fainter lines, which are apt to be confused with other lines closely contiguous, or it may possibly be due to a slight progressive error in the scale value for this region.

The results obtained for No. 7 spectrum given in column 7 appear to confirm the

values of the more refrangible lines in No. 3, although they were obtained by quite independent methods of reduction. The measures of this plate, however, cannot be given the same weight as those of the other two, the definition being poor owing to imperfect focus.

The lines in the visible spectrum Hα, β, γ, and δ are entered for the sake of completeness, but they are not to be compared in accuracy with the ultra-violet lines. This is largely due to the difficulty in measuring the broad over-exposed lines β, γ, and δ on No. 3 spectrum, and partly to the small dispersion in this region.

The line δ appears in all the measures to be largely displaced towards the red, but this is probably accidental. It is remarkable, however, that in ROWLAND's table of the solar lines Hδ is given at λ 4102·000, which is also less refrangible than the theoretical position, the difference being ·10 tenth-metre. The almost perfect agreement of the other absorption lines in his table, and the very close agreement of the best defined ultra-violet lines in Nos. 3 and 4 spectra with the positions assigned by the formula, suggests that the wave-length of this line has been erroneously estimated. Mr. JEWELL, however, who made the measures of ROWLAND's plates, while admitting some uncertainty in estimating the centre of the line in the solar spectrum, finds on re-examining his measures, no justification for altering his original estimate ('Astro-Physical Journal,' vol. 9, p. 211).

It is very desirable that accurate measures be made of Hδ in the upper chromosphere where the line is narrow and free from interfering lines.

The limit of the hydrogen series defined by the formula when $s = \infty$ is at λ 3646. But in the chromosphere spectra the lines fade away to invisibility long before reaching this point. I would call particular attention, however, to the very remarkable band of continuous spectrum shown by the prominences and lower chromosphere, beginning near the end of the tabulated series and extending indefinitely towards the more refrangible end of the spectrum. The interspaces between the hydrogen lines are quite clear of this continuous spectrum, which begins abruptly at about 3668. (See Plate 12.)

It seems probable that this faint spectrum may be itself due to hydrogen. In the *absorption* spectrum of white stars of Type I., Sir WILLIAM HUGGINS has observed an analogous feature. This consists in the "rather sudden fall of intensity of the continuous spectrum at about the place of the end of the series of dark hydrogen lines. The enfeebled spectrum continues to run on without further enfeeblement until it is stopped by the absorption of our atmosphere."[*]

This seems clearly to be the absorption effect of the matter giving the faint continuous spectrum in my plates, and the fact that the feature is characteristic of white stars in which the hydrogen absorption is very strong points to its being due to hydrogen itself.

---

[*] 'An Atlas of Representative Stellar Spectra' (Sir WILLIAM and Lady HUGGINS), p. 85.

Helium and Hydrogen in the Lower Chromosphere.

The part played by helium and hydrogen in the lower chromosphere may next be considered. Helium is probably absent from the very lowest strata, if indeed it is present at all in the flash-spectrum layer. The lines at λλ 3634·4, 4026·3, 4471·6, and 4713·2 are very weak, or quite invisible upon the continuous spectrum at the centre of the photograph (No. 3) although strong on either side, whilst the majority of the lines of the flash spectrum proper are strongly impressed right across the continuous spectrum. This would point to the absence of helium from the region within 2″ of the photosphere.

[But the lines at λλ 4922·1 and 5015·7 appear on the other hand to be strong on the continuous spectrum and weak at the sides. It is to be remarked, however, that these lines belong to two series of lines in the spectrum of cléveite gas which have been ascribed to parhelium by Runge and Paschen.* It would seem, therefore, that these "parhelium" lines, whatever their origin, are produced under conditions existing only in the lower levels.—March 10, 1901.]

The absence of appreciable absorption due to helium in the solar spectrum seems also to favour the view that this element exists only in a rarefied condition in the upper chromosphere. This peculiarity in the spectrum certainly does not necessarily imply equality of temperature between the radiating gas and the photospheric background; for the hydrogen spectrum affords a demonstration that chromospheric gases at a lower temperature than the photosphere, can emit strong bright lines without corresponding absorption lines. Thus, in the visible spectrum the dark lines corresponding to Hα, β, γ, and δ are sufficient evidence that hydrogen in the chromosphere is cooler than the photosphere. But in the ultra-violet the absorption becomes inappreciable at Hζ, and beyond this point there are no dark hydrogen lines corresponding with the strong bright lines in this region.

In the emission spectrum of the chromosphere there is a progressive diminution of intensity of the hydrogen lines towards the smaller wave-lengths. Nevertheless, the lines Hη, θ, ι, κ are still among the strongest lines in the spectrum beyond K. In the ultra-violet, therefore, hydrogen behaves exactly like helium in the visible spectrum.

The disappearance of the hydrogen absorption lines in the ultra-violet have been ascribed by Rowland to excessive diffuseness or widening of these lines, and if this view is correct it follows, as Sir William Huggins has pointed out, "that the hydrogen absorption in the sun must be restricted to a narrow region low down and close upon the photosphere itself,"† the hydrogen in the higher regions contributing little or nothing to the absorption.

This view, however, which seems to imply the complete independence between

---

* Runge and Paschen, "On the Spectrum of Cléveite Gas," 'Astro.-Physical Journal,' vol. 3, No. 1 (1896).

† 'An Atlas of Representative Stellar Spectra,' p. 150.

the emission and absorption lines in the sun, is extremely difficult to reconcile with the ordinary appearance of the lines C and F at the sun's limb. With a radial slit the emission lines are seen to correspond exactly both in width and in absolute intensity with the dark lines, and it is difficult to avoid the conclusion that the *whole depth* of the chromosphere is effective in producing the absorption.*

The progressive diminution of intensity of the emission lines in passing from the visible spectrum towards the ultra-violet appears to be in itself sufficient to account for the disappearance of absorption beyond H. The total quantity of hydrogen above the photosphere is probably too small to produce appreciable absorption when the lines have a certain limiting intensity, although the same lines when viewed at the sun's limb may be strikingly conspicuous, chiefly by reason of the wide diffusion of the gas as already explained in the case of the metallic lines in the flash spectrum, and also on account of the enormous depth of radiating gas through which the line of sight passes.

Unlike helium, hydrogen is very conspicuous in the lower chromosphere, and the fainter more refrangible lines are much more strongly impressed in the photographs obtained near the times of the two internal contacts than in those taken near mid-eclipse with much longer exposures. The intensity of the lines evidently increases rapidly towards the photosphere, but they still remain narrow and well defined even within 1″ of the photosphere.

The Corona Spectrum.

Although the continuous spectrum of the corona is strongly shown on most of the plates exposed during totality, the green line is the only one of which it was possible to obtain measures. The faint line near H in Spectrum No. 6 becomes quite invisible under the microscope, and the new line in the extreme ultra-violet, shown in No. 5, was not discovered until some time after making the measures ; the wave-length of this line was estimated by superposing No. 5 spectrum upon No. 3, the hydrogen lines and H and K being made coincident ; the position of the new line was then obtained with reference to the fine lines of the flash spectrum.†

The following are the wave-lengths obtained for the green line which was measured on negatives No. 3 and No. 7 :—

$$
\begin{aligned}
\text{No. 3 spectrum } \lambda &= 5307 \cdot 0 \\
\text{No. 7 } \quad\quad\;\; ,, &= 5299 \cdot 7 \\[4pt]
\text{Mean } \lambda &= 5303 \cdot 3
\end{aligned}
$$

* The same may be said of the calcium lines H and K. In the upper chromosphere these lines are narrow and well defined, and correspond exactly with the narrow dark lines at the centres of the broad absorption lines.

† It is impossible in the reproductions to show the corona lines here mentioned, except the green line.

This result is in satisfactory agreement with the value found by Mr. Fowler on the plates obtained by Sir Norman Lockyer. The difference between the two measures above is due to the opposite displacement of the line in the photographs taken at second and third contacts, the point measured being situated about 2′ above the moon's limb. This apparent displacement affects all lines of a diffuse character where the moon's limb is not well defined.

The three corona lines shown on these plates are therefore—

1. 3388.
2. Near H (on less refrangible side).
3. 5303·3.

The ultra-violet line is similar to the green line in distribution round the limb, and is probably due to the same substance, whilst the line near H differs from the others in showing no maxima of intensity, the substance producing it being evenly distributed on the west limb.

There is no trace whatever of any other coronal line on any of the plates.

SUMMARY OF RESULTS.

The principal results obtained from the study of these photographs are stated briefly in the following paragraphs :—

1. Practically every strong dark line in the solar spectrum is present as a bright line in the flash spectrum.

2. Almost all the flash-spectrum lines (excepting those due to hydrogen and helium) coincide with dark lines in the solar spectrum.

3. The relative intensities of the lines of any one element in the flash spectrum are practically the same as those of the same element in the solar spectrum.

4. The relative intensities between groups of lines belonging to different elements are widely different in the flash and in the solar spectrum.

5. The apparent intensity of the radiation from any element in the lower chromosphere is determined by the extent to which that element is diffused above the photosphere ; and the real relative intensities between the different elements cannot be judged in photographs of the flash spectrum.

6. The flash spectrum may be considered to represent the upper more extensively diffused portion of a stratum of gas which, by its absorption, gives the Fraunhofer dark-line spectrum.

7. Nearly all of the known metals having atomic weights less than 60 are represented in these spectra, and no element having a higher atomic weight than 92 is certainly represented.

8. All the strong lines in the flash spectrum can be identified with known elements, and the small proportion of unidentified lines are weak lines.

9. The wave-lengths of the hydrogen lines in the ultra-violet agree closely with those derived from BALMER's formula.

10. The prominences and lower chromosphere emit a strong continuous spectrum in the ultra-violet, beginning near the limit of the hydrogen lines and extending indefinitely in the more refrangible region.

11. The ultra-violet hydrogen lines increase in intensity towards the photosphere, but remain narrow lines in the flash-spectrum layer.

12. Some of the helium lines decrease in intensity towards the photosphere, others

In conclusion, I have to acknowledge my indebtedness to Dr. RAMBAUT for much valuable help and advice in the preparation of this paper, and for the kind interest he has taken in the work throughout.

I have also to acknowledge the great assistance rendered to me by Mr. L. E. JEWELL, of the Johns Hopkins University, who made a careful and exhaustive study of some positive copies of my plates.

In identifying the lines given in the tables, I have been guided almost entirely by his results; and I am also indebted to him for supplying me with revised values of many of the solar lines given in the last column of each table.

TABLE I.—*Flash Spectrum Wave-lengths compared with the Dark Lines of the Solar Spectrum.*

In this table the wave-lengths deduced from the measures of No. 3 and No. 7 spectra are entered in the first two columns. The third column gives the photographic intensities for No. 3 spectrum. These were estimated on a scale ranging from 1 to 10 ; 1 representing very weak lines, and 10 representing the strongest lines in the spectrum. Very faint traces of lines difficult to measure are designated 0 in this column.

In column 5 the wave-lengths of the solar lines are given with which the bright lines have been identified with a greater or less degree of probability. The intensities as given by ROWLAND are entered in column 6, and the element corresponding to the solar line in the last column. The wave-lengths of the helium lines are placed within brackets ; they are taken from the tables of RUNGE and PASCHEN.

The columns 5, 6, and 7 are taken from ROWLAND's tables of solar lines published in the 'Astro-Physical Journal' for 1895, 1896, and 1897, and supplemented by more recent determinations supplied to me by Mr. JEWELL. In a few instances the identifications and intensities will be found to differ from those given in the tables.

In column 4 an (S) means a sharply defined line, well identified, and suitable to be taken as a standard line in wave-length determinations. A (?) in this column indicates that the identification with the solar line is extremely doubtful.

In the last column a ? means that the element in question has not been identified with certainty with the solar line.

TABLE I.

| Wave-lengths in flash spectra. | | Photographic intensity. | Character. | Wave-length, solar spectrum. Rowland. | Intensity. | Element. |
No. 3.	No. 7.	No. 3.	No. 3.			
3342·3	—	2	. . . .	{3342·062 / 42·358}	7 / 3	Ti / Fe Ti
47·0	—	0	(One measure only)	{46·882 / 47·066}	5 / 2	Ti / Co
49·4	—	4	Long . . . (S)	49·508	12	Ti
54·0	..	2 }	Faintly extended	53 875	4	Sc
58·5	—	2 }		58·649	4	Cr
61·4	—	3½	Long . . (S)	61·327	12	Cr ? Ti
68 3	—	3	Long	{68·193 / 68·319}	5 / 1	Cr / Mn
73 0	—	4	Long . (S)	72·948	12	Ti
80·4	—	3	Short	{80·424 / 80·722}	6 / 6	Ti / Ni
84·0	—	4	Long . (S)	83·892	10	Ti
88·1	—	3	Long . (S)	87·988	5	Ti
92·1	—	1½	(One measure only)	92 109	2	Zr
94·7	—	3	Long . . (S)	94·716	6	Ti
99·3	–	1	Short . .	{99·375 / 99·489}	2 / 3	Fe Zr / Fe
3403·45	..	3	Faintly extended (S)	3403·404	3	Cr
05 17	—	1	Short . .	05·217	2	Co
07·32	—	1	Short . . . (?)	07·597	4	Fe
08·97	—	3	Long . . (S)	08·911	3	Cr
10·24	—	1	Visible on south side only	—	—	—
13·01	...	0	Not measured; w-l estimated	—	—	—
15·01	—	1	Visible on south side only	14·911	15	Ni
21·42	—	3½ }	Equal pair; long (S)(S)	21·353	4	Cr
22·94	—	3½ }		22 892	4	Cr
25·46	—	0 {		—	—	—
26·97	—	0	One measure only; very short	—	—	—
28·73	—	0 }		—	—	—
30·61	—	1	Short . .	30 671	1	Zr
33 54	...	4	Long . .	{33·453 / 33·715}	3 / 8	Cr / Ni
38·40	...	2	Short .	38·376	2	Zr
40·93	—	3	Faintly extended	{40·762 / 41·155}	20 / 15	Fe / Fe
42·24	3442·1	4	Long . . . (S)	42 112	6	Mn
44·39	—	3	Faintly extended (S)	44·467	4	Ti
46·34	—	2 }	Ill defined .	46·406	15	Ni
52·86	53·3	2 }		53·039	6	Ni
56·55	—	2 }	Ill defined; short	56·528	3	Ti
58 58	—	1 }		58·601	8	Ni
60·53	60·4	3	Long . . (S)	60·460	4	Mn
61·68	61·7	2	Faintly extended	{61·633 / 61·801}	5 / 8	Ti / Ni
63·05	—	1	Interrupted; very short	62·950	6	Co
64·32	...	1	ditto ditto	{64·275 / 64·608}	1 / 1	Fe / Sr ?
65 87	...	2	Very faintly extended	{65·900 / 66·015}	4 / 6	Co / Fe

TABLE I.—*continued.*

Wave-lengths in flash spectra.		Photo-graphic intensity.	Character.	Wave-length, solar spectrum. Rowland.	Intensity.	Element.
No. 3.	No. 7.	No. 3.	No. 3.			
3467·46	—	0	Stronger south than north of centre . (?)	—	—	—
68·72	—	0	Ditto	3468·821	2	Fe
71·36	—	1	} Equal pair, short and interrupted . . .	{ 71·404	3	Fe, Zr
72·58	—	1		72·680	5	Ni
74·28	3474·26	3	Long . . . . (S)	74·287	2	Mn
75·67	—	2	Short	75·594	10	Fe
77·26	77·31	3	Long	77·323	5	Ti
79·48	—	1½	} Ill defined ; short	79·531	2	Zr
81·20	—	1½		81·302	2	Zr
83·08	83·02	3		83·047	5	Mn
88·85	88·84	3	} Long . . . . (S)	88·817	4	Mn
91·16	91·10	3	A very faint line is visible about λ 3490 ±	91·195	5	Ti
93·22	93·21	2	Ill defined ; faintly extended	93·114	10	Ni
94·60	—		Short . . . .(?)	94·815	2	Fe
96·17	96·11	3	Long . . . .(?)	{ 95·853	2	Ti
				95·974	2	Mn
97·82	97·71	2	Short	{ 97·668	3	Mn
				97·982	8	Fe
99·14	—	0	Interrupted . . . .(?)	99·248	0	Ti
3500·45	—	1	Short	3500·471	3	Ti
02·30	—	0	Short (one measure only)	02·394	4	Co
05·06	3505·01	3½	} Long equal pair . . . . . . (S)	05·036	5	Ti
10·96	11·10	3½		10·985	5	Ti
13·95	—	2	Short	13·965	7	Fe
15·15	15·40	2	} Faintly extended . . . . . (S)	15·206	12	Ni
17·37	—	2		17·446	3	V
20·26	—	2½	} Short	20·397	2	Ti
21·44	—	1		21·410	8	Fe
21·84	—	1	On continuous spectrum north of centre only	21·748	4	Co
24·67	24·72	3½	Long . . . .(S)	24·677	20	Ni
26·22	—	1	} Close pair very short	26·183	6	Fe
26·59	—	1		—	—	—
30·80	30·93	1	Faintly extended .	30·919	3	V
31·83	—	1	Short . . . .(?)	{ 31·982	3	Mn
				32·143	4	Mn
33·35	—	0	Short	33 345	6	Fe
35·64	35·68	3	Faintly extended . . . . . .(S)	35·554	4	Ti
41·17	—	1	} Short, fine lines	41·237	7	Fe
42·24	—	1		42·232	6	Fe
45·25	45·39	2	Faintly extended	45·336	2	V
48·08	—	1	Interrupted	{ 47·941	5	Mn
				48·175	3	Mn
				48·332	5	Mn, Ni
49·07	—	2	} Short	49·151	2	Y ?
51·93	—	2		52·098	1	Zr
53·15	53·87	0	. . . .	—	—	—
55·02	—	1	On continuous spectrum	55·079	9	Fe
56·82	56·94	2½	} Faintly extended	56·738	2	Zr
58·61	58·63	2		58·672	8	Fe
60·88	—	0	Invisible to north side . . . . . .(?)	61·037	4	Co
61·82	—	1	Short	61·898	3	Ni

TABLE I.—*continued.*

Wave-lengths in flash spectra.		Photographic intensity.	Character.	Wave-length, solar spectrum. ROWLAND.	Intensity.	Element.
No. 3.	No. 7.	No. 3.	No. 3.			
3565·52	3565·64	2	}	3565·535	20	Fe
66·25	—	1		66·522	10	Ni
67·77	68·08	2	} Long lines	67·835	4	?
70·21	70·24	2½	(S)	70·273	20	Fe
72·61	72·63	2½	}	73·676	6	Sc
73·75	73·98	1	Short	73·874	3	Ti
76·51	76·52	3	}	76·527	3	Sc
78·76	78·88	2	} Long . . . .(S)	78·832	10	Cr
81·16	81 37	3	}	81·349	30	Fe
84·37	—	1	More distinct in clear spaces than on continuous spectra.	—	—	—
85·45	85·46	2½	}	85·479	7	Fe
87·08	—	1		87·130	8	Fe
87·71	87·51	0	Difficult to measure ; visible only on continuous spectrum on north side. (?)	87 370	7	Co
89·72	—	2	} Very faintly extended · ·	89·773	5	?
90·43	—	2	}	90·509	2	C
92·07	—	2	Faintly extended . . . . . .(?)	92·169	2	V ?
93·55	93·84	2	Long . . . .(S)	93·636	9	Cr
94·81	—	1	Short	94·784	6	Fe
96·05	96·39	2	Long	96·195	4	Ti
97·72	—	1	Short	97·854	8	Ni
3600·71	99 59	2½	Faintly extended	3600·880	3	Y
01·90	—	2	}	02·060	1	Y
03·67	3603 97	1	} Short . . . .(?)	03·922	3	Cr ?
05·35	05·64	2	Long . . . .(S)	05·479	7	Cr
06·48	—	0	Short . . . .(?)	06·838	6	Fe
08·98	09 00	2	Long . . . .(S)	09·008	20	Fe
11 05	—	2	Short	11·189	2	Y
13 91	14·00	3½	Narrow strong line with shading.	13·947	4	Sc
18·98	19·01	3	Probable double (width ·92). . .Long {	18·919	20	Fe
				19·539	8	Ni
21·26	—	1	Short	21·344	3	?
21·62	—	1	Short	21·612	6	Fe
23·45	—	0	Very short	23·362	5	Fe
24·86	25·02	2½	Faintly extended	24·979	5	Ti
28·71	—	1	Fine and sharp on continuous spectrum ; { does not extend beyond	28·847	2	Y
				28·967	2	La
30·85	—	2½	}	30·876	4	Ca ?
31·50	31 37	2	} Long	31·605	15	Fe
33 11	—	2	Very fine and distinct ; ends abruptly at sides of continuous spectrum	33·277	2	Y
34·19	34·54	1	Long ; almost invisible upon continuous spectrum	(3634·393)	—	He
35·11	—	0	} Very short on continuous spectrum only	35·091	3	Ni
36·50	—	0	}	36·608	4	Cr ?
41·32	41·65	2½	Long, sharply defined . . . . .(S)	41·473	4	Ti
42·70	42·81	3	„ „ „ „ .(S) {	42·820	7	Ti
				42·912	3	Sc
45·35	—	2	Very faintly extended	45·475	3	Sc ?
47·86	47·77	2	Long . . . .(S)	47·988	12	Fe
49·86	—	1	Short ; stronger on continuous spectrum (?)	49·654	5	La

TABLE I.—*continued.*

Wave-lengths in flash spectra.		Photographic intensity.	Character.	Wave-length, solar spectrum. ROWLAND.	Intensity.	Element.
No. 3.	No. 7.	No. 3.	No. 3.			
3651·65	3652·46	2	} Very faintly extended, a line at 3653·4 ± just visible . . . . (1) {	3651·614	7	Fe
—	—	—		51·940	4	Sc
55·78	—	1	Very short; stronger on continuous spectrum	55·801	3	?
59·58	—	2	Faintly extended . . . . (?)	59·663	5	Fe
61·16	—	1	In clear spaces on south side only . .	—	—	H₃₁
62·14	—	2	Probably a line superposed on hydrogen line	62·378	5	Ti, H₃₀
63·40	63·42	1				H₂₉
64·71	64·67	2				H₂₈
66·15	66·07	2				H₂₇
67·70	67·65	2½				Hω
69·52	69·54	2½	Long lines, not very sharply defined . .	—		Hψ
71·45	71·58	3				Hχ
73·77	73·80	3				Hφ
76·32	76·50	3				Hυ
77·64	78·6	2	Short; narrower than hydrogen lines (?) {	77·764	5	Fe
				77·831	3	Cr
				77·991	3	Cr
79·41	79·43	3½	Probably a Fe line on less refrangible side	—		Hτ
82·80	82·82	3	Long	—		Hσ
85·28	85·17	5	Very long; well defined . . . . (8)	85·339	10	Ti
86·98	86·81	3½	Long	—		Hρ
89·48	—	0	Short; interrupted . . . . . . (?)	89·614	6	Fe
91·67	91·67	3½	Long	—		Hπ
93·83	—	1	Short	—		
97·26	97·20	4	Long. A very faint line is visible half-	—		Ho
3703·87	3703·85	4	way between Ho and Hξ λ3701 ±	—		Hξ
05·92	—	3	Faintly extended . . . . . . {	3705·708	9	Fe
				06·175	6	Ca ?
07·66	—	0	} Very short; interrupted; not visible (?)	07·702	2	Ti
09·09	—	1	} on continuous spectrum (?)	09·389	8	Fe
10·14	—	2	Short . . . . (?)	10·431	3	Y
12·10	12 00	4	Long	—		Hν
15·32	—	2	Very faintly extended . . . . . {	15·319	2	Cr ?
				15·615	4	V ?
18·10	—	0	On continuous spectrum only.	—		
19·82	19·96	3	Long, very sharply defined . . . (8)	20·084	40	Fe
22·05	21·89	4½	Long	—		Hμ
24·48	—	1	Very short, continuous spectrum only .	24·526	6	Fe
27·40	27·64	2½	Short . . . . (?)	27·778	4	Fe
34·55	34·54	4½	Narrower on continuous spectrum . .	—		Hλ
36·94	37·18	3½	Long	37·281	30	Fe
39·11	—	0	Short	—		
41·51	41·83	3	Long . . . . {	41·205	4	Ti
				41·791	4	Ti
43·26	43·31	1	Short, very narrow line	43·508	6	Co
				45·401	2	Co
45·58	45·69	3	Long . . . . {	45·717	8	Fe
				46·058	6	Fe
48·01	—	1	Faintly extended, narrow	48·406	10	Fe
50·29	50·18	4½	Narrower on continuous spectrum . .	—		Hκ
57·57	—	2	Ill-defined, short. . . . . . . (?)	57·824	4	Or, Ti

TABLE I.—*continued.*

Wave-lengths in flash spectra.		Photographic intensity	Character.	Wave-length, solar spectrum. Rowland.	Intensity	Element
No. 3.	No. 7.	No 3.	No. 3.			
3759·32	3759 34	4	⎱ Very long, well defined . . . . (8)	3759·447	12	Ti
61 46	61·25	4½	⎰ Longer than H₆, and narrower . . (8)	61·464	7	Ti
63·66	—	1½	Long, faint, visible on continuous .	63 945	10	Fe
66 84	—	1	Long, faint, not visible on continuous.	67·341	8	Fe
70 77	70 77	6	Narrower on continuous spectrum . .	—	—	H₆
74 03	—	2½	Short	74 473	3	Y
75 79	—	0	Very short	75 717	7	Ni
83 28	—	1	Short	83 674	6	Ni
86·6 ±	—	0	Short (wave-length estimated)	84 046	8	Fe
88·53	89·57	2	Short . . . . (?)	⎰ 88·839	2	Y ?
90·0 ±	—	0	Short (wave-length estimated).			
94·72	—	0	Short	95·147	8	Fe
97 98	98 07	7	Diffuse outside, narrower on continuous spectrum.	—	—	H₆θ
3805 19	—	0	⎱ Trace of lines or band, poorly defined .	3805·486	6	Fe
07 46	—		⎰	07·681	6	Fe
12 79	—	2	Faintly extended .	—	—	—
14 22	—	1	Short . . .	14·698	8	Fe ?
15·70	3815 92	2	Faintly extended . . .	15 987	15	Fe
20·23	20·17	3	Long, nearly equal to Hη. . . (8)	20·586	25	Fe
24 16	24·91	2	⎱ Ill-defined pair, faintly extended .	24·591	6	Fe
25·51	25·73	2	⎰	26·027	20	Fe
27 4±	—	0	(Wave-length estimated) . . .	—	—	—
29 32	29 28	3	⎱ . . . . (8)	29·501	10	Mg
32 13	32·06	3	⎰ Long, nearly equal to Hη . .	32 450	15	Mg
35 58	35·14	7		—	—	Hη
38 36	38·32	3	Long, nearly equal to Hη . . (8)	38·435	25	Mg
40·61	—	1	Short	40·580	8	Fe
49·94	—	1	Short, hazy	50·118	10	Fe
56 21	56 36	2	Long	56·524	8	Fe
59 96	60·00	3	Long . . . . . . (8)	60 055	20	Fe
72·09	73 1 ?	1	Ill-defined, wide, faintly extended . . (?)	72·639	6	Fe
78·29	78·7	2	Long	⎰ 78·152	8	Fe
82 75	—	1	Ill-defined short	⎱ 78·720	1	Fe
89·34	89 1	8	Narrower on continuous spectrum . .	—	—	Hζ
95 29	—	2	Faintly extended . . .	95·803	7	Fe
3900·25	3900 7	3	Long, faintly the patch remaining on the continuous spectrum	3900·681	5	(Fe?)Ti
06·11	—	1	Wide, short, probably two lines . .	05·660	12	Si
13·55	13·6	3	Long, the predominating element ? .	13·609	5	Ti(Fe)
19 90	—	1	. . . . (?)	20 410	10	Fe
22·23	22·8	1	⎱ Short	—	—	—
27·52	—	1	⎰			
34·4	33·1	10	(K) width of line 2·95 . . .	33 825	1000	Ca
44·0	44 4	2½	Faintly extended . . . . (8)	44·160	15	Al
50·1 ±	—	1	Centre of group	50·102	5	Fe
56·9	—	1	Ditto	⎰ 56·476	4	Co, Ti
				⎱ 56·819	6	Fe
61·6	—	3	Long . . . . . . (8)	61·674	20	Al
69·8	68·3	10	(H) width of line 3·92	⎰ 68·625		Ca
		5		⎱ 70·177	700	H₆
82·6	83·6	1	Short . . . . (?)	82·742	3	Y
90·6	—	1	Short	—	—	—
98·8	—	1	Group of lines	98·790	4	Ti

TABLE I.—continued.

Wave-lengths in flash spectra		Photographic intensity.	Character.	Wave-length, solar spectrum.	Intensity.	Element.
No. 3.	No. 7.	No. 3	No 3.	ROWLAND.		
4005·5	—	1	Short	4005 408	7	Fe
12·7	—	1	Ditto	12·541	4	Ti
26·56	4026·8	3	Long, not visible upon continuous .	(4026·312)	—	He
30·7	—	2	} Groups of short lines	30 918	10	Mn
34·4	—	2		33·224	8	Mn
46·3	46·5	2¼	Long, visible across continuous . . (S)	43·975	30	Fe
54·7	54 1	1		—	—	—
63·9	63 7	2	} Faintly extended . . . . (S)	63·750	20	Fe
72·1	—	1	(S)	71 908	15	Fe
78·3	78·1	4	Long . . . . (S)	77·885	8	Sr
4102·5	4102·4	7	Very diffuse on violet side . . .	4102·000	40	Hδ
09·9	—	1				
18·9	21·4 {	1	} Group of faint lines . . . . (?)	23 384	12	La
23·6		1				
28·0	—	1	} Very faint group.			
33 9	—	1				
43·4	—	2	} Probable double	(4143·919)	—	He
			Visible on continuous . . .	44 038	15	Fe
63 6	—	0	. . . . . .	63 818	4	Ti
72·9	—	0	} Ill-defined . . . . (?)	72 923	4	Fe
77·7	—	0				
4216 3	4215·8	4	Long . . . . . . (S)	4215 703	5	Si
26·9	27 0	3	. . . . . .	26 901	20	Ca
33 3	—	2	} Long	33 328	4	Fe
				33 772	6	Fe
47 5	46·7	3	. . . . . (?)	46 996	5	Sr ?
54·8	—	2	. . . . . .	54·505	8	Ca
60·4	—	1	. . . . .	60 610	8	Fe
73·6	74·5	2	Probably a group of lines.			
90·2	91·7	2	. . . . . (?)	91 114	3	Ti
97·1	—	1	Probably faintly extended			
4300·5	4300·9	2	. . . .	1300·211	3	Ti
				00 732	2	Ti
08·2	—	2	(G)	07·907	3	Ca
				08 081	6	Fe
13·7	15 1	1	Faintly extended lines . . .	15 138	3	Ti
21·3	—	1	. . . .	20 167	3	Sc
26·1	—	2	. . . .	27 939	8	Fe
41 4	41 3	7	Best defined on red side . .	40 631	20	Hγ
51·7	—	2	Short	51·216	3	Cr
				51 930	5	Cr
58·7	—	1				
75·2	—	2	} Faintly extended . . .	75 103	2	V, Mn
83·6	—	2		83 720	15	Fe
95·5	95 2	2	Long	95·201	3	Ti
				95·413	2	V, Zn
4400·6	—	0	} Very difficult . . .	—		
05·4	—	0		—		
16·8	4417·0	1		—		
26·4	—	1		—		
35·7	—	0	. . . . .	4435 851	4	Ca
44·1	43·8	2	Long	43·976	5	Ti
50·4	—	0		—		

TABLE I.—*continued.*

Wave-lengths in flash spectra. No. 3.	No. 7.	Photographic intensity. No. 3.	Character. No. 3.	Wave-length, solar spectrum ROWLAND.	Intensity.	Element.
4455·3	—	0		—	—	—
72·0	4472·1	3	Long, well-defined on R side, very diffuse on B side, weak over continuous spectrum . . .	(4471·646)	—	He
82·0	—	0				
90 8	—	2		—	—	—
4501 6	4501·6	2	. . . .	4501·448	5	Ti
09 3	—	2		—	—	—
15 9	—	0		—	—	—
23·2	—	0		—	—	—
34·9	34·5	2	. . . .	34·953	4	Ti
51·6	46·0	2		—	—	
55·2	55·0	2	. . . .	55·662	3	Ti
62 0	—					
66·0	65 0	2	Group of lines, faintly extended.	—	—	—
69·9	—					
74 0	72 9	2	Ditto.	—	—	—
87 7	88·4	1		—	—	—
90·7	—	0		—	—	—
4629·4	—	1	. . . .	4629·521	6	Ti, Co
46 6	—	0	. . . .	46·347	5	Cr
68·0	—	0		—	—	—
85 6	86·3	0		—	—	—
4713 2	4713·4	2	Long, not visible on continuous spectrum	(4713·252)	—	He
4805·3	—	0		—	—	—
25·0	—	0		—	—	—
60·3	4860·9	5		4861·527	30	Hβ
4922 0	—	2	Visible across continuous spectrum. . .	(4922·096)	—	He
33·5	4930·2	2	Very difficult.	—	—	—
57·4	—	0		—	—	
5017 4	5016 1	2	Visible across continuous spectrum. . .	(5015·732)	—	He
5164·7	5167·3	1	(b''' b'''' b'''') Long	5167·497	15	Mg
73·0	74 7	3		72·850	20	Mg
83·5	83·9	3	(b') Long	83·791	30	Mg
			. . . .	5204·680	5	Cr
5205·3	—	1	. . . .	06 215	5	Cr
			. . . .	08·596	5	Cr
27·7	—	0		—	—	—
35·1	—	0		—	—	—
70·0	—	0		—	—	—
5307·0*	5299 7*	1	Very diffuse (corona line).			Fe
15·4	—	1	(1474 K)	5316·790	4	
26·5	—	0		—	—	—
35·5	—	0		—	—	—
5876·0	5875·9	5	(D''')	(5875·870)	—	He
93·7	—	2	(D', D'')	5890·186 / 96·357	30 / 20	Na / Na
—	6563·0		(C) strong on No. 7 plate	6563·045	40	Hα

* Apparent centre of line displaced to red at second contact, and to blue at third contact. Mean position = 5303 3.

TABLE II.—*Spectrum of Prominence.*

The wave-lengths given in the first column of this table are deduced from measures of the images of a very small bright prominence in spectrum No. 4. This prominence was situated almost at the point of second contact at position angle 59° (solar latitude + 23°).

The photograph was exposed at the beginning of totality immediately after the disappearance of the flash spectrum.

The prominence images are well defined circular spots in all radiations except the calcium and brighter hydrogen rays, which show wings due to motion in the line of sight.

The columns of this table are arranged as in Table I., and the intensities in column 2 were estimated in the same way as those of No. 3 spectrum given in Table I.

The helium wave-lengths given within brackets in column 4 are from RUNGE and PASCHEN.

TABLE II.

Wave-length No. 4 spectrum.	Intensity.	Remarks.	Wave-length, solar spectrum. ROWLAND	Intensity	Element.
3435	—	Limit to which continuous spectrum of prominence can be traced			
3534 { 3537 {	—	Continuous spectrum is much brighter at this point	3535·554	4	Ti
3614·0	0	. . . .	3613·947	4	Sc
3618·9	0	. . . .	3618·919	20	Fe
3625·3	0	One measure only	3624·979	5	Ti
3631·6	1	. . . .	3631·605	15	Fe
3634·6	1	. . . .	(3634·393)	—	He
3668±	—	Beginning of continuous spectrum in prominences	—	—	—
3669·52	0	. . . .	—	—	Hψ
3671·61	0	Rather difficult to measure	.	—	Hχ
3673·92	0	Silver deposit very weak .	.	—	Hφ
3676·55	0	.	—	—	Hυ
3679·55	1	. . . .	—	—	Hτ
3683·06	1	. . .	—	—	Hσ
3685·34	3	. . . . . .	3685·339	10	Ti
3687·12	1	. . . .	—	—	Hρ
3691·75	1	. . .	—	—	Hπ
3697·18	1	. . .	—	—	Ho
3704·11	1	. . . .	—	—	Hξ
3705·86	1	Well-defined and very small circular dots . .	3705·708	9	Fe
3712·19	2	. . . .	—	—	Hν
3720·02	1	. . . .	3720·084	40	Fe
3722·07	2	. . . .	—	—	Hμ
3734·52	2	. . . .	—	—	Hλ
3737·27	1	. . . .	3737·281	30	Fe
3741·79	1	. . . .	{ 3741·205 { 3741·791	4 4	Ti Ti

TABLE II.—*continued.*

Wave-length No. 1 spectrum.	Intensity.	Remarks.	Wave-length, solar spectrum. Rowland.	Intensity.	Element.
3745·98	1	. . . .	{ 3745·717	8	Fe
			{ 3746·058	6	Fe
3718·47	0	. . . .	3748·408	10	Fe
3750·21	2		—	—	Hκ
3759·53	2	Well-defined and very small dots	3759·447	12	Ti
3761·40	2	. . . .	3761·464	7	Ti
3770·80	2	. . . .	—	—	Hι
3798·12	3	. . . .	—	—	Hθ
3820·09	1	Not well defined	{ 3820·586	25	Fe
3825·77	0		{ 3826·027	20	Fe
3829·39	0	. . . .	3829·501	10	Mg
3832·56	1	. . . .	3832·450	15	Mg
3835·63	3	. . . .	—	—	Hη
3838·56	2	Well-defined nearly circular dots	3838·435	25	Mg
3860·14	1	Slightly winged	3860·055	20	Fe
3889·09	4	. . . .	—	—	Hζ
3900·5	0	. . . .	3900·681	5	Ti(?)
3913·2	0	. . . .	3913·609	5	Ti(Fe)
3933·8	10	K, triangular black spot	3933·825	1000	Ca
(3944·0)	0	Not measured wave-length in No. 3 spectrum	3944·160	15	Al
3968·6	10	H, triangular black spot	3968·625	700	Ca
4026·3	2	} Fairly good definition . . {	(4026·342)	—	He
4077·8	2	}	4077·885	8	Sr
4102·2	5	Circular black spot with wings due to motion in line of sight	4102·000	40	Hδ
4215·7	2	. . . .	4215·703	5	Sr
4340·6	5	Same as Hδ	4340·634	20	Hγ
4471·8	3	Well-defined spot	(4471·646)	—	He
4713·1	1	. . . .	(4713·252)	—	He
4861·5	5	Circular spot slightly winged	4861·527	30	Hβ

TABLE III.—Hydrogen Lines.

Designation.	Spectrum No. 3.	Spectrum No 4.	Mean of 3 and 4.	Computed.	O – C	Spectrum No. 7.
α	—	—	—	6563·07	—	(6563·07)
β	4860·3	4861·5	4860·9	1861·52	– 6	4860 9
γ	4341·4	4340·6	4341·0	4340·63	+ ·4	4341·3
δ	4102·5	4102·2	4102·3	4101 90	+1 4	1102 4
ε	Obscured	by H	—	3970·22	—	—
ζ	3889·34	3889·09	3889·21	3889·20	+ ·01	3888 87
η	3835·58	3835 63	3835·60	3835·53	+ ·07	3835·05
θ	3797·98	3798·12	3798·05	3798·04	+ ·01	3797·97
ι	3770·77	3770·80	3770·78	3770·77	+ ·01	3770 70
κ	3750·29	3750·21	3750·25	3750·30	– ·05	3750·15
λ	3734·55	3734·52	3734·54	3734·51	+ ·03	3734·50
μ	3722·05	3722·07	3722·04	3722·08	– ·02	3721 85
ν	3712·10	3712·19	3712·14	3712·11	+ ·03	3712 05
ξ	3703·87	3704 11	3703·99	3704 00	– ·01	3703·86
ο	3697 26	3697·18	3697·22	3697·29	– ·07	3697 24
π	3691·67	3691·75	3691·71	3691 70	+ ·01	3691·71
ρ	3686·98	3687·12	3687·05	3686·97	+ ·08	3686·75
σ	3682 80	3683·06	3682 93	3682·95	– ·02	3682 87
τ	3679 41	3679·55	3679 48	3679 49	– ·01	3679 47
υ	3676·32	3676·55	3676·43	3676·50	– 07	3676 53
φ	3673·77	3673·92	3673·84	3673·90	– ·06	3673 81
χ	3671 45	3671·61	3671·53	3671·48	+ ·05	3671 59
ψ	3669·52	3669·52	3669·52	3669·60	– ·08	3669·51
ω	3667·70	—	3667·70	3667 82	– ·12	3667·67
27*	3666·15	—	3666·15	3666·24	– ·09	3666·08
28	3664 71	-	3664·82	3664·82	– ·11	3664 68
29	3663·40	—	3663 40	3663 54	– ·14	3663 27
30	3662·14	—	3662 14	3662·40	·26	—
31	3661·16	—	3661·16	3661·35	– ·19	—
∞	Limit	of series	(theoretical)	3646·13		—

* Series number, α = 3.
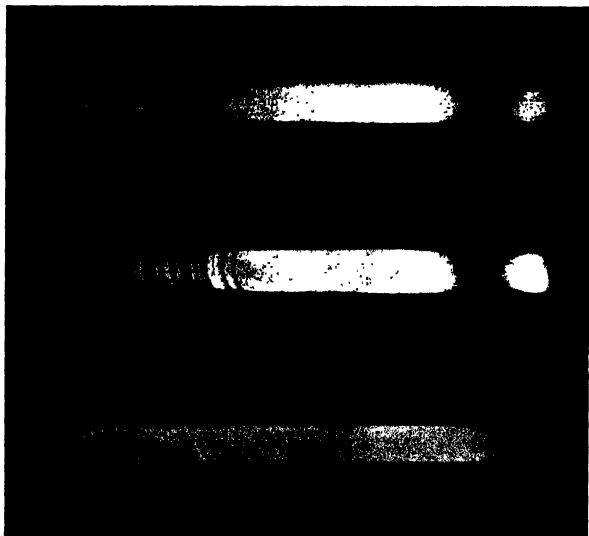
Comparison between cusp spectra photographed during the solar eclipse, and artificial cusp photographed in ordinary daylight– to show the relative faintness of the Fraunhofer lines in the eclipse spectra, and the great extension of the latter in the ultra violet   (The absence of lines between F and H in each spectrum is due to over-exposure)
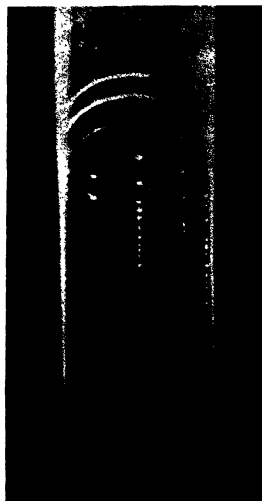
Spectrum No. 4.—Ultra-violet region, enlarged five times. Showing continuous spectrum of prominences beginning at the end of the hydrogen series of lines.

Spectrum of Lower Chromosphere—ultra-violet region. From photograph taken with a 2-inch prismatic camera at the Solar Eclipse of January 22nd, 1898.

X. On the Electrical Conductivity of Air and Salt Vapours.

By HAROLD A. WILSON, D.Sc., M.Sc., B.A., Allen Scholar, Cavendish Laboratory, Cambridge.

Communicated by Professor J. J. THOMSON, F.R.S.

THE experiments described in this paper were undertaken with the object of obtaining information on the variation of the conductivity of air and of salt vapours with change of temperature, and on the maximum current which a definite amount of salt in the form of vapour can carry. They are a continuation of the two researches* on the same subject published in 1899.

In the paper "On the Electrical Conductivity and Luminosity of Flames" some observations on the variation of the conductivity with the temperature at different heights in the flame are given. They indicate a rapid increase in the conductivity with rise of temperature.

The method employed in the experiments now to be described was the following :—
A current of air containing a small amount of a salt solution in suspension in the form of spray was passed through a platinum tube heated in a gas furnace ; this tube served as one electrode, and the other was fixed along its axis. The temperature of the tube was measured by means of a platinum platinum-rhodium thermo-couple, and the amount of salt passing through the tube was estimated by collecting the spray in a glass-wool plug. From the temperature variation of the conductivity the energy required to produce the ionization can be calculated, and this compared with the energy required to ionize bodies in solutions.

Since the publication of the researches just referred to several papers† on the conductivity of salt vapours in flames by Dr. E. MARX have appeared. The first of these is concerned mainly with the fall of potential between the electrodes, and the

---

* HAROLD A. WILSON, "On the Electrical Conductivity of Flames containing Salt Vapours," 'Phil. Trans.,' A, vol. 192, 1899. A. SMITHELLS, H. M. DAWSON, and H. A. WILSON, "The Electrical Conductivity and Luminosity of Flames containing Vaporised Salts," 'Phil. Trans.,' A, vol. 193, 1899.

† VON ERICH MARX, "Ueber den Potentialfall und die Dissociation in Flammengasen," 'Gesellschaft der Wissenschaften zu Göttingen,' 1900, Heft 1 ; 'Annalen der Physik,' 1900, No. 8. "Ueber das Hall'sche Phänomen in Flammengasen,' 'Annalen der Physik,' 1900, No. 8.

22.11.1901.

results obtained are in good agreement with those described in my paper, and show with greater detail the changes in the potential fall due to changes in the temperature of the electrodes. Cooling the positive electrode diminishes the fall of potential close to the negative electrode. This effect is easily explained, as Dr. MARX points out, on the assumption that the ionization occurs mainly on the surface of the hot platinum, and that cooling the platinum diminishes the amount of this ionization. The negative drop is due to the presence of positive ions in the gas near the negative electrode, and cooling the positive electrode diminishes the supply of positive ions, and consequently also the negative drop. In my paper, referred to above, I described this effect and explained it in this way. Dr. MARX has also used his results on the fall of potential to make an estimate of the velocities of the ions in the flame. The calculation is based on the investigation given by Professor J. J. THOMSON[*] of the distribution of potential between the electrodes when all the ionization is uniformly distributed throughout the volume of the gas, as in the case of the conductivity of air due to Röntgen rays when aluminium electrodes are used.

In the theory given by Professor THOMSON the space between the electrodes, when the current is far from its saturation value, is considered to be divided into three parts by planes parallel to the electrodes. In the middle region the intensity (X_0) is constant, so that the numbers of positive and negative ions present are equal, and the ionization is equal to the recombination. The value of the current density is given by $\iota = X_0 n e \, (k_1 + k_2)$, where n is the number of positive or negative ions present in unit volume, e the charge carried by an ion, k_1 and k_2 the velocities of the positive and negative ions respectively due to unit intensity.

In the regions near the electrodes, whose thicknesses will be denoted by λ_1 and λ_2 no recombination is supposed to take place, so that if the middle region is taken away the distance between the electrodes becomes $\lambda_1 + \lambda_2$, and then the current density has the saturation value for this distance, viz., $qe \, (\lambda_1 + \lambda_2)$, where q is the number of ions produced in unit time in unit volume of the gas.

Dr. MARX's determination of the velocities of the ions in the flame depends therefore on the assumption that the ionization occurs uniformly throughout the space between the electrodes which, according to my experiments (loc. cit.), is not even approximately the case in flames containing salt vapours. In his first paper Dr. MARX agreed with the conclusion that nearly all the ionization occurs on the electrodes, and brought forward fresh evidence in favour of its truth, and at the same time applied Professor THOMSON's theory to calculate the velocities of the ions.

In his paper in WIEDEMANN's ' Annalen,' Dr. MARX again describes the same experiments, but comes to the conclusion that at any rate when the E.M.F. is small, the surface ionization is not large compared with the volume ionization, so that when

---

[*] " On the Theory of the Conduction of Electricity through Gases by Charged Ions," 'Phil. Mag.,' March, 1899.

the E.M.F. is small it is allowable to apply Professor THOMSON's theory to flames. It is consequently necessary to consider the arguments brought forward by Dr MARX in favour of his view.

His main reason for concluding that the ionization is nearly uniformly distributed is the existence of a point of inflexion in some of the curves showing the fall of potential between the electrodes, which, as he points out, can only occur theoretically when there is some volume ionization.

It is easy to calculate the variation of the electric intensity between two electrodes when only surface ionization occurs and a saturating P.D. is applied.   We have then

$$dX/dx = 4\pi (n_1 - n_2) e \quad . \quad . \quad . \quad . \quad (1),$$

$$(d/dx)(k_1 n_1 X) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

$$(d/dx)(k_2 n_2 X) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

$$(k_1 n_1 + k_2 n_2) X e = \iota \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

If q_1 is the number of positive ions coming in unit time from unit area of the positive, electrode (2) gives

$$k_1 n_1 X = q_1,$$

and in the same way

$$k_2 n_2 X = q_2.$$

Hence substituting in (1)

$$\frac{dX}{dx} = 4\pi \left(\frac{q_1}{k_1 X} - \frac{q_2}{k_2 X} \right)$$

or

$$2X \frac{dX}{dx} = 8\pi \left(\frac{q_1}{k_1} - \frac{q_2}{k_2} \right).$$

Integrating we get

$$X^2 = 8\pi \left(\frac{q_1}{k_1} - \frac{q_2}{k_2} \right) x + C.$$

If we measure x from a point where the intensity is X_0, then $X_0^2 = C$, so that

$$X^2 - X_0^2 = 8\pi \left(\frac{q_1}{k_1} - \frac{q_2}{k_2} \right) x.$$

Therefore in this case the intensity should increase regularly from one electrode to the other unless $q_1/k_1 = q_2/k_2$, in which case $X = X_0$ everywhere.

The fall of potential in the flame when the upper electrode is negative is like that indicated by the above calculation ('Phil. Trans.,' A, vol. 192, 1899, p. 508), and so agrees with the view that the ionization nearly all occurs on the electrodes.

In Dr. MARX's experiments the upper electrode appears to have always been positively charged, in which case, as I have shown, the curves show a rapid fall of potential near both the electrodes with a nearly uniform intensity in the intervening space, and this form of curve is obtained even when very large voltages are applied.

Thus even when a very large P.D. is applied the observed curves show that the electric intensity between the plates is very small, almost zero in some cases, so that there is no reason to suppose that there is no recombination occurring between the electrodes when this form of curve is obtained; in fact, a saturating P.D. has not really been applied, so that MARX's argument breaks down. The presence of surface ionization has much the same effect as an increase in the distance between the electrodes, and consequently protects the intervening region from the applied P.D., so that it does not get saturated.

When the current is reversed so that the upper electrode is negative the supply of negative ions is greatly diminished owing to the comparatively low temperature of the upper electrode. Consequently, since the current is mainly carried by the negative ions it becomes easier to saturate the intervening space and the potential fall curve changes accordingly.

I think, therefore, that the observed potential fall curves are not inconsistent with the view that most of the ionization occurs close to the surfaces of the glowing electrodes.

The remainder of the present paper is divided into the following sections :—

(1) Description of the apparatus used.
(2) Variation of the current with the E.M.F.
(3) Variation of the current through air with the temperature.
(4) Variation of the current through salt vapours with the temperature.
(5) Summary of results.

1. *Description of the Apparatus used.*

The apparatus used is shown in fig. 1.

It consisted of a platinum tube TT, 25 centims. long and 0·75 centim. in diameter. having a narrow tube T' joined on at one end, and a flange FF, 6 centims. in diameter, joined on at the other. This tube was supported horizontally in a FLETCHER's tube furnace, the fire-clay blocks of which are shown by the dotted lines.

The flange served to keep the furnace gases from the open end of the tube. An electrode, EE, consisting of a platinum tube, 12 centims. long and 0·3 centim. in diameter, was supported on an adjustable insulated stand, along the axis of the tube TT. The end of this electrode was closed by a conical platinum cap which was about 9 centims. down the tube TT.

At T' the platinum tube was sealed on to a glass tube through which the air charged with spray entered. The spray was produced by a Gouy sprayer, S, which projected the spray into a glass bulb, G, about 8 centims. in diameter, from which the air and spray were led through an inverted U-tube in which the coarser spray settled.

The solution was contained in a reservoir R. The level of the surface of the solution being 30 centims. above the nozzle of the sprayer. The greater part of the

spray settled in the bulb and first half of the U-tube, and was returned to R through a tube, DD, up which the liquid was forced by air introduced by the tube K.
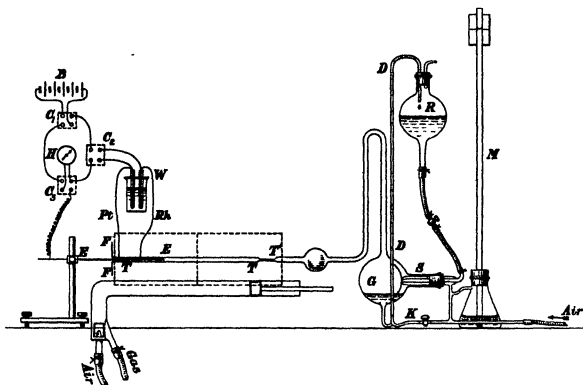

Fig. 1.—TT Platinum tube. EE Electrode. S Sprayer. R Reservoir of salt solution. M Air manometer. W Water bath. Pt, Rh Thermo-couple wires. H Galvanometer. B Battery. C_1 C_2 C_3 Commutators.

The supply of compressed air used was obtained by means of two water injector pumps similar to the one used in the previous work on the conductivity of flames. Some of the air was allowed to escape by bubbling through mercury which served to keep its pressure nearly constant, and the rest was passed through a large carboy to smooth out small oscillations in the pressure. The air pressure at the sprayer was measured by means of the water manometer, M, and was kept constant at 50 centims. This arrangement gave enough air to work the sprayer and also the furnace except at temperatures above about 1100° C., when the air supply to the furnace was supplemented by oxygen from a cylinder by means of which a temperature of 1400° C. could be obtained.

The temperature of the tube was measured by means of a platinum platinum-rhodium thermo-couple, which was simply connected directly to an Ayrton-Mather dead-beat galvanometer of about 500 ohms resistance. The platinum wire served to support the tube, and the Pt/Rh wire was fused on to the tube at a point on its upper surface, so that the tube itself formed one of the elements of the couple. The

couple was standardised by determining the galvanometer deflection corresponding to the melting point of K_2SO_4, which melts, according to Hevcock and Neville, at 1066° C. The K_2SO_4 was introduced into the tube on a small platinum spatula, and the temperature gradually increased until it was seen to melt, and the corresponding deflection noted. Then by maintaining the tube at a series of constant temperatures near the melting point, and finding when the K_2SO_4 melted, it was possible to obtain two temperatures very near together, at one of which the K_2SO_4 melted and at the other remained solid. The mean of the two galvanometer deflections was taken as corresponding to 1066°, and the temperature corresponding to any other deflection was first calculated on the assumption that the deflection was proportional to the difference between the temperatures of the two junctions. The " platinum temperatures" thus obtained have been corrected to the centigrade scale by means of the table of corrections given by Callendar (' Phil. Mag.,' Dec., 1899, p. 534).

This method of getting the temperatures was quite sufficiently accurate for the purposes of the present investigation, for which it was useless to aim at a greater accuracy than 5° or 10°, and, according to Callendar, the corrections are much more accurate than this near 1000°, while even at 300° the error is not more than 10°.

The Pt and Pt/Rh wires dipped into mercury cups kept in a water bath at a known temperature, from which copper wires led to the galvanometer.

The gas supplied to the furnace was kept at a constant pressure by means of a gasometer, and the air and oxygen supply tube was provided with a water manometer by means of which the pressure of the supply could be maintained constant if necessary, for any length of time. In this way the tube could be maintained constantly at any desired temperature within 5° without difficulty.

The current through the air and salt between the electrode EE and the tube T, due to various potential differences between them, was measured by the galvanometer used to measure the temperature. By means of commutators the connections with the galvanometer could be immediately changed from the thermo-couple to the conductivity apparatus. The difference of potential was supplied by a battery of small secondary cells.

2. *Variation of the Current with the E.M.F.*

Before making observations on the variation of the current with the temperature at constant E.M.F., some observations were made on the variation of the current with the E.M.F. at constant temperature.

The results obtained at a temperature of 1080° C. for air alone, without salt and no air current passing along the tube, are shown graphically in diagram No. 1.

When the outside tube was negatively charged the current soon attained an almost constant value as the E.M.F. was increased, but when the outside tube was

the positive electrode the current continued to increase rapidly with the E.M.F., and was much greater than the current in the reverse direction, except with very small E.M.F.'s.
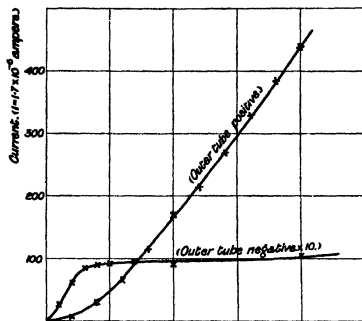

Diagram No. 1.

When no air current was passed through the tube the inner electrode was much colder than the outside tube, especially near the flange, owing to its losing heat by conduction along the tube supporting it. This difference of temperature between the two electrodes enables the great difference between the two currents to be explained in the same way as the similar differences observed in flames can be explained.

It was shown by GUTHRIE ('Phil. Mag.,' vol. 46, p. 257, 1873), that a red hot metal can retain a negative but not a positive charge in air. This is the reverse of what occurs in a flame where a negative charge leaks away much more rapidly than a positive one. Consequently when the hotter tube is positive the current is greater than when it is negative.

The electric intensity is of course greater at the inner tube than it would be if both electrodes had the same area, and it is possible that this circumstance helps to exaggerate the difference between the two currents.

An experiment was therefore tried, using a wire 1 millim. in diameter instead of the inner tube. The results are shown graphically in diagram No. 2.

The general character of the curves is the same as before, but the two currents at any E.M.F. are much smaller.
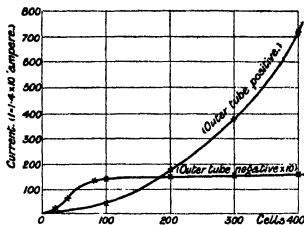
Diagram No. 2.

The effect of cooling the inner tube by blowing a current of air through it was also tried. A small hole was bored at the point of the conical cap to allow the air to escape. The results obtained are shown in diagram No. 3.

Diagram No. 3.

Cooling the inner tube greatly diminished the current in both directions, and also caused the current, with the outer tube positive, to begin to show signs of attaining a saturation value.

A very peculiar phenomenon was noticed in all the above experiments. On first applying the E.M.F. the current was much greater than the steady value to which it soon settled down. This effect was always much more marked when no current had been passed for some time than immediately after breaking the circuit. Thus on first heating the tube in the morning to about 1100°, and putting on an E.M.F. of 400 volts, the current would be as much as ten times its final value, to which it

would drop in about one minute. If after passing this current for 10 minutes, the circuit was broken for 10 seconds, then, on remaking it, the initial current would be perhaps 20 per cent. greater than the final steady value to which it would fall in a few seconds.

Long-continued heating of the tube to 1500° was found to diminish this effect, which, however, slowly reappeared, though not entirely, when the tube was cooled. It was also found that the steady currents observed through air fell off considerably as the platinum tubes became more and more aged through use.

This effect is presumably due to long-continued heating of the platinum. A similar effect was observed by ELSTER and GEITEL ('Wied. Ann.,' vol. 37, p. 315, 1889) in their experiments on the charge communicated to a neighbouring plate by an incandescent platinum wire in air. At first the plate gets a positive charge, but this diminishes with long-continued heating of the wire, and finally at low pressures actually changes sign and the plate then gets a negative charge. NAHRWOLD ('Wied. Ann.,' vol. 35, p. 107, 1888) ascribes the positive electrification first observed to the action of dust in the air, and regards the final communication of a negative charge as the normal effect.

It is clear that the falling off in the amount of conductivity as the tube gets aged is due to a change of some kind in the state of the platinum. Perhaps occluded gases are gradually expelled from it, though it seems very unlikely that the loss of a small quantity of gas should produce such a marked change in the amount of ionization as actually occurs. I think a change in the state of molecular aggregation of the platinum, due to long-continued heating, is the most probable cause of the falling off in the amount of ionization produced in the air close to it, though it is not easy to see why any such change should have this effect. Perhaps the porosity of the metal diminishes with long-continued heating, so that the effective surface of metal exposed to the gas is diminished.

The large current obtained for a short time on first applying the E.M.F. suggests the idea that the passage of the current produces some kind of polarisation of the electrodes, but on connecting the electrodes directly to the galvanometer immediately after applying a big E.M.F. for some time, no indication of a polarisation current could be detected. If the applied E.M.F. was suddenly reversed in direction, the current after reversing was for a short time greater than the steady value to which it soon settled down, but this effect was not greater than what would have been observed if the E.M.F. had merely been suddenly applied in the second direction without first applying it in the opposite direction.

The object of the investigation being to obtain information on the variation of the conductivity with temperature, the investigation of these peculiar effects was not proceeded with, but I hope before long to make experiments with a view to elucidate their real nature.

The values given for the observed currents through air throughout this paper are

the steady values attained after the E.M.F. had been applied for a short time. In most cases it was only necessary to wait a few seconds before taking the reading. The current through salt vapours did not show any such peculiar changes,* and in the later experiments, after the tube was aged, was usually very large compared with that carried by the air at the same time. When the tube was nearly new the current, carried by the air at high temperatures, was sometimes of the same order of magnitude as that due to the salt. The actual numbers obtained in many cases are given below.

When a salt solution was being sprayed the variation of the current with the E.M.F. was of a much simpler character than with air alone.

The current due to the salt was usually greater when the outside tube was negative, but the character of the relation between the current and E.M.F. was about the same in either case.

Diagram No. 4 shows the current E.M.F. curves got while spraying a 1 per cent. KCl solution at 900°.
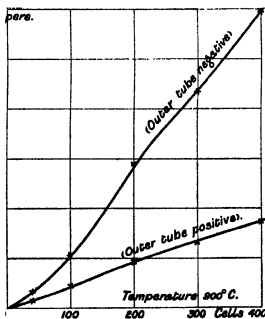
Diagram No. 4.

A comparison of the curves given below, showing the variation of the current at constant E.M.F. with the temperature for different E.M.F.'s, shows that as the temperature falls the E.M.F. at which approximate saturation is attained falls also, so that below 500° saturation is attained at about 100 volts.

The following table shows the way in which the relation of the current to the E.M.F. varies with the temperature. The numbers are for a 1 per cent. KI solution, and were taken from the curves given in Section 5 :—

* *Note, added August* 8, 1901.—This applies to KI and KCl. Some salts do show the effect as well as air. See Appendix.

Current (1 = 10⁻⁸ ampere).

	500°.	700°.	900°.	1100°.	1300°.
40 volts . . .	0·2	0·3	0·3	0·9	2·5
100 ,, . . .	5·1	5·5	5·5	6·8	9·2
800 ,, . . .	7·0	30·0	45·0	35·0	71·0

We see that for small E.M.F.'s the current increases very slowly at first, and then more rapidly. At higher E.M.F.'s it becomes nearly constant at 500°, whereas at 1300° it is nearly proportional to the E.M.F.

Diagram No. 5 shows the variation of the current with the E.M.F. for KI at 1200°.

Diagram No. 5.
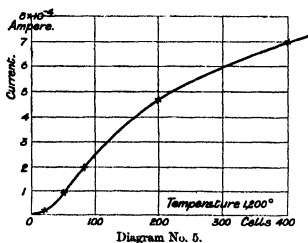
3. *Variation of the Current through Air with the Temperature.*

To determine the variation of the conductivity with the temperature at constant E.M.F., a series of measurements of the current due to a fixed number of cells at different temperatures was made. The value of the current was not read until it had become constant. A temperature reading was always taken both before and after the measurement of the current, and unless the two readings agreed the measurement was repeated. Usually no difficulty was experienced in keeping the temperature sufficiently constant, owing to the excellent working of the regulating arrangements on the gas and air supplies. The highest temperature was done first and the temperature then diminished by steps until the current became inappreciable, when the temperature was raised again and some of the observations repeated.

The usual current of air coming from the sprayer was always passed through the tube, since it served to keep the inner electrode at nearly the same temperature as the outer one. Measurements of this kind, using air alone, were made from time to time during the course of the investigation, and, as already mentioned, it was found that a marked falling off in the conductivity at any particular temperature occurred as the tube got aged.

The character of the relation between the current and temperature, however, was not sensibly affected by this ageing of the tube. The results of the first series of observations of this kind done are shown in diagram No. 6. The tube had then been used at intervals for about a fortnight.
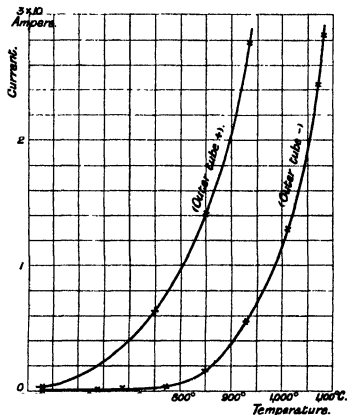

Diagram No. 6.

The gradual falling off in the conductivity is shown by the numbers in the following table, which give the values of the currents at several temperatures obtained on different dates :—

Current through Air in Amperes.

E.M.F. 800 volts.

Outer tube negative.

	Temperature 800°.	900°.	1000°.	1100°.
July 6 . . .	8 × 10⁻⁶	40 × 10⁻⁶	120 × 10⁻⁶	400 × 10⁻⁶
,, 10 . . .	2 ,,	11 ,,	50 ,,	140 ,,
,, 30 . . .	0·1 ,,	0·7 ,,	2·5 ,,	8 ,,

Diagram No. 7 shows the results obtained with an E.M.F. of 240 volts, the outer tube being the negative electrode.   Date, July 29.

700°   800°   900°   1,000°   1,100°   1,200°   1,300°C.
Temperature.
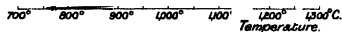
Diagram No 7.

Diagram No. 8 shows the results obtained, using 40 volts, on July 28.

10×10⁻⁶
Amperes

Current.

5

800°.   900°   1,000°   1,100°   1,200°   1,300°C
Temperature.

Diagram No 8.

3 I 2
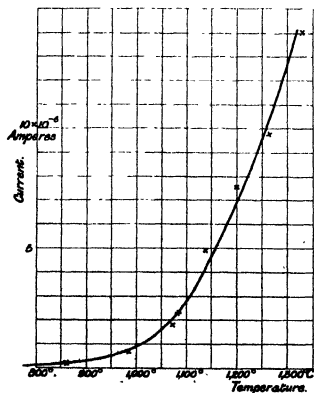
Diagram No. 9 shows the relation between the logarithms of the current and the absolute temperature, using values taken from diagram No. 7. The relation between the logarithms is roughly log C = 17·17 log θ — 59·3, where C is the current and θ the absolute temperature.
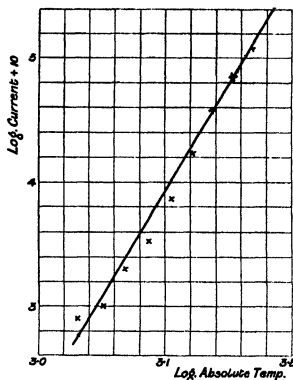
Diagram No. 9.
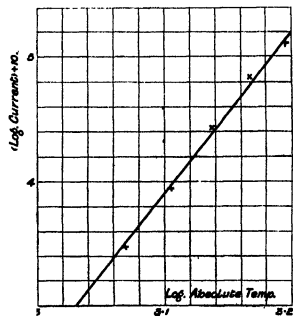
Diagram No. 10.

Diagram No. 10 shows the relation between the logs got from diagram No. 8. This gives log C = 12·8 log θ − 45·8.

According to these formulæ some current should be obtained at any temperature, however low.

The above results show that the conductivity does not begin suddenly at a definite temperature, but always increases regularly with rise of temperature, so that the lowest temperature at which conductivity can be detected depends entirely on the sensitiveness of the galvanometer.

The calculation of the energy required to ionize the conducting gas from the variation of the conductivity with the temperature will now be considered.

The ionization practically all takes place in a thin layer close to the surface of the platinum, and the number of ions in this layer will depend on the rate at which they are formed and on their rate of recombination.

For the purpose of this calculation the rate of production of ions will be regarded as a function of the temperature and concentration of the gas in this layer. At any constant temperature, if C is the concentration of the gas and C′ that of the positive or negative ions supposed equal, we have

$$C = aC'^2,$$

where a is a constant and it is assumed that one molecule of the gas dissociates into two ions. The case is evidently on these suppositions exactly analogous to, say, the dissociation of N_2O_4 into $2NO_2$, and the equation commonly employed to represent the variation of the dissociation with temperature in such cases can be employed. It is

$$\frac{q}{2}\left(\frac{1}{T_2} - \frac{1}{T_1}\right) = \log\left\{\frac{x_1^2}{x_2^2} \cdot \frac{1 - x_2^2}{1 - x_1^2} \cdot \frac{T_2}{T_1}\right\},*$$

where q is the heat developed when 1 gramme molecule of the gas is formed from the ions by recombination, and x is the fraction of the gas dissociated at the absolute temperature T. q is supposed expressed in calories and is taken as being constant between T_1 and T_2.

If x is small then approximately

$$\frac{q}{2}\left(\frac{1}{T_2} - \frac{1}{T_1}\right) = \log\frac{x_1^2}{x_2^2} \cdot \frac{T_2}{T_1}$$

The current when the potential gradient is small is equal to A n ($k_1 + k_2$), where n is the number of positive or negative ions in the gas per unit volume of the layer, k_1 and k_2 the velocities of the positive and negative ions respectively, and A a constant. Now k_1 and k_2 vary with the temperature, probably being proportional

* See 'Lectures on Physical Chemistry,' VAN'T HOFF, vol 1.

to a small power of the absolute temperature such as the square root over small ranges of temperature.

Hence at temperatures near 1000° C., k_1 and k_2 may be taken as constant over small ranges of temperature without serious error, so that the current with a small E.M.F. is proportional to n, and n is proportional to x when x is small. Hence in the formula

$$\frac{q}{2}\left(\frac{1}{T_2} - \frac{1}{T_1}\right) = \log \frac{x_1^2}{x_2^2} \cdot \frac{T_2}{T_1},$$

provided we take T_1 and T_2 very near together, we may replace $(x_1/x_2)^2$ by the square of the ratio of the two currents obtained at T_1 and T_2.

The following table gives values of q obtained from diagram No. 8 :—

$(T_1 - 273.)$	$(T_2 - 273.)$	C_1.	C_2.	q.
1000°	950°	0·9	0·5	71,000
1200°	1150°	7·0	4·7	64,000
1300°	1250°	12·6	9·6	49,000

Thus q for 1 gramme molecular weight of air is about 60,000 small calories between 1000° and 1300° C.

If we take the charges on the ions to be the same as in the electrolysis of solutions then we can obtain an estimate of the potential difference, through which the charge on an ion must fall to do enough work to ionize a molecule of air. To electrolyse 1 gramme molecular weight of any substance requires 9.6×10^4 coulombs, consequently the required P.D. is

$$\frac{60,000 \times 42 \times 10^6}{9.6 \times 10^4 \times 10^7} = 2.6 \text{ volts.}$$

The energy set free in a solution when 1 gramme molecular weight of water is formed from its ions H and OH, is well known to be about 17,000 calories, and when ozone O_3 decomposes into oxygen 30,000 calories are set free. It is interesting to note that the energy in these two cases is of the same order of magnitude as the value just calculated for the energy necessary to ionize air.

RUTHERFORD has determined the energy of the Röntgen rays absorbed by a given volume of air, and the number of ions produced in the same volume by the rays. From this he deduces the energy required to ionize one molecule of air at the ordinary temperature. His result expressed in volts multiplied by the charge on one ion is 170, which is 70 times greater than the value I have obtained at temperatures above 1000°.

It is probable that in Professor RUTHERFORD'S experiment the greater part of the Röntgen rays absorbed by the air is not used up in producing ions.

If we suppose that the energy required to ionize a molecule is due to the electric

attraction between the two charges, then it is easy to calculate the distance between them in the molecule before it is ionized. We have, if e is the charge on one ion and r the distance required,

$$\frac{2 \cdot 6\, e}{300} = \frac{e^2}{r^2},$$

and according to J. J. THOMSON* $e = 6 \times 10^{-10}$ electrostatic unit, hence

$$r = 7 \times 10^{-8} \text{ centim.}$$

This is of the same order of magnitude as the radius of molecular action in liquids at the ordinary temperature.

Professor TOWNSEND† has shown that the potential difference through which the ionic charge must fall to have enough energy to ionize a molecule of air at the ordinary temperature is less than 5 volts, and probably greater than 1 volt. My result, 2·6 volts for air at temperatures above 1000° C., is therefore nearly the same as TOWNSEND's, in spite of the great difference between the temperatures at which his and my experiments were carried out.

4. *Variation of the Current through Salt Vapours with the Temperature.*

In the earlier measurements made on the conductivity of salt vapours the conductivity of the air alone without salt at temperatures above 900° was not small
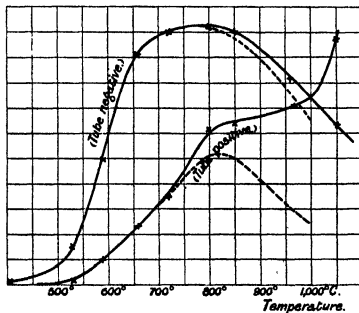

Diagram No. 11.

* 'Phil. Mag.,' December, 1898.

† "The Conductivity produced in Gases by the Motion of Negatively Charged Ions," 'Phil. Mag.,' February, 1901.

compared with that of the salt, but as the tube got aged the conductivity of the air
fell off to such an extent that it became quite small compared with that of the salt
when solutions containing 10 grammes in a litre were sprayed.

Diagram No. 11 shows the variation of the current with the temperature when
spraying a solution of KCl, done when the tube was nearly new.

The conductivity due to the KCl has a maximum value at about 800°. The
current due to the salt is greater when the outer tube is negative, while the reverse
holds good for air, so it is best to have the outer tube negative when measuring the
conductivity of salt vapours. The dotted curves drawn were got by subtracting the
current due to the air alone from the current due to both salt and air.

In all the measurements given below the outer tube was negative, and the con-
ductivity of the air alone was negligible compared with that of the salt.

Diagrams Nos. 12, 13, and 14 show the variation of the current with the
temperature when a solution of KI containing 10 grammes in a litre, was sprayed
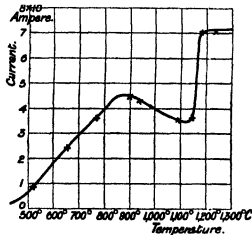with E.M.F.'s of 800, 100, and 40 volts respectively.

Diagram No. 12.

It is evident from these curves that the relation between the conductivity of salt
vapours and the temperature is of a much less simple character than is the case
for air alone.

As the temperature rises the conductivity at first increases rapidly and then
attains a maximum value at about 900° C., above which it falls again until about
1150° C. is reached, when it suddenly begins to rise very rapidly. Above 1200° C.
the current again becomes nearly independent of the temperature.

With an E.M.F. of 800 volts, the sudden rise at 1150° just doubles the current,
which suggests that at this temperature some kind of dissociation occurs and doubles
the number of ions available.

The amount of salt passing through the tube was determined by substituting for

the platinum tube a large glass tube about 4 centims. in diameter and 50 centims. long, tightly packed with glass wool. The resulting increase in the air pressure
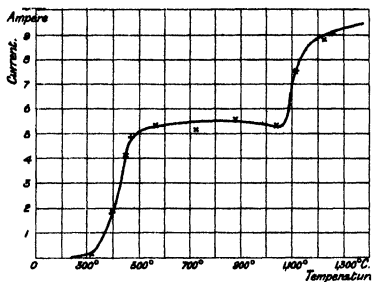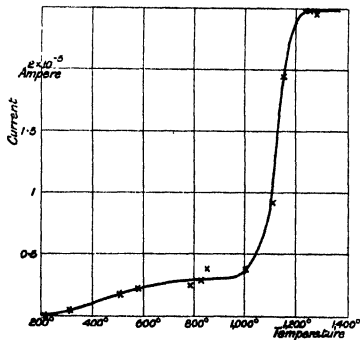

Diagram No. 13.

Diagram No. 14.

in the bulb and U-tube was counterbalanced by increasing the pressure of the air supply to the sprayer and the air pressure above the solution in the reservoir by an equal amount.

A solution containing 20 grammes of NaCl in a litre was sprayed for one hour, and the amount of NaCl in the plug determined by washing out with water and titrating against standard AgNO$_3$ solution. That the plug stopped all the spray was shown by passing the escaping air into a Bunsen flame when no coloration could be detected. With a shorter plug of glass-wool all the spray was not stopped. In this way it was found that 4·7 milligrammes of NaCl were stopped by the plug in one hour, this number being the mean of several concordant determinations.

In the later experiments a new sprayer was used, which sprayed 2·2 times as much salt as the old one. The ratio of the amounts of salt sprayed by the two sprayers was got by comparing the conductivities due to 1 per cent. KCl solutions when sprayed by the two sprayers under similar conditions at 1200°, and with an E.M.F. of 800 volts. Under these conditions the current is proportional to the electrochemical equivalent of the salt passing through the tube, as will be shown below. The amount of KI passing through the tube during the above measurements was therefore 1·46 × 10^{-3} milligramme per second.

To electrolyse this amount of KI in a solution requires a current of

$$\frac{1·46 \times 10^{-3} \times 39}{166 \times 0·405} = 8·5 \times 10^{-4} \text{ ampere.}$$

The current obtained at 1400° with 800 volts was 7 × 10^{-4} ampere, which is therefore nearly equal to the current required to electrolyse the KI used in a solution.

It is probable that at high temperatures the KI is converted into KOH by the water vapour present. The reaction being

$$KI + H_2O = KOH + HI.$$
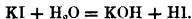
The view that all salts of the alkali metals are converted into hydrates at a high temperature in the presence of water vapour was used by ARRHENIUS to explain his results on the electric conductivity of flames, and this view is also in accord with the results of the investigation published in 1899, " On the Electric Conductivity and Luminosity of Flames containing Vaporised Salts."[*] According to this view the falling off in the current above 900° might be explained as being due to KOH being less ionized than KI at this temperature.

Thus up to about 900° the conductivity may be supposed entirely due to KI, but at this temperature KOH and HI begin to be formed, and conduct less well than KI, so that the current falls off. As the temperature rises, however, the KOH begins to dissociate into ions, so that at 1200° the current attains a maximum value when it is completely electrolysed.

The energy required to ionize a gramme molecular weight of KI can be calculated in the same way as was done for air by using the values of the current obtained at

low temperatures* where the current is small compared with that required to completely electrolyse the salt. At 250° C. a current of 2×10^{-7} ampere was obtained with 40 volts, and at 300° a current of 4×10^{-7}. Hence the energy required to ionize 1 gramme molecule of KI is about 15,000 calories, or about one quarter that necessary to ionize a gramme molecule of air.

Diagram No. 15 shows the variation of the current with the temperature for K_2CO_3, a solution containing 10 grammes in a litre, being sprayed with an E.M.F. of 400 volts.
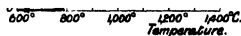

Diagram No. 15.

Diagram No. 16 shows the variation of the current with the temperature for KCl with an E.M.F. of 800 volts.
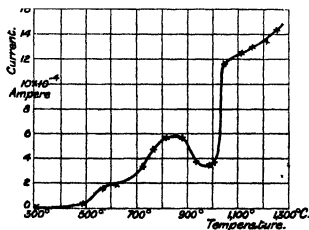

Diagram No. 16.

* At high temperatures the per cent. of the salt ionized is not small, so that the method used for calculating the energy breaks down.

3 K 2

The following table gives the maximum currents observed with KI, KCl, and K_2CO_3, and the currents required to electrolyse the same amount of each in a solution :—

Salt.	Current.	Current calculated.	Ratio.
KI	7×10^{-4}	$8 \cdot 5 \times 10^{-4}$	0·82
KCl	15×10^{-4}	$18 \cdot 9 \times 10^{-4}$	0·80
K_2CO_3 . . .	16×10^{-4}	$20 \cdot 4 \times 10^{-4}$	0·79

It thus appears that the maximum amount of current carried by the vapours is nearly equal to the maximum amount the same bodies could carry in a state of solution.* This agreement must be regarded as considerable evidence in favour of the view that the ions are of the same nature in the two cases.

5. *Summary of Results.*

(1.) The variation of the current between platinum electrodes in air with the temperature may be expressed approximately by a formula of the type $A\theta^n$, where θ is the absolute temperature and A and n constants. The value of n was about 17 when the E.M.F. used was 240 volts, and 13 with 40 volts.

(2.) The energy required to ionize 1 gramme molecular weight of air is about 60,000 calories between 1000° and 1300°.

(3.) The current through a halogen salt vapour in presence of air and water reaches a maximum value near 900°, and then falls as the temperature rises. Near 1100° it again rises rapidly with the temperature.

(4.) The energy required to ionize 1 gramme molecular weight of KI is about 15,000 calories at about 300°.

(5.) The maximum current carried by a definite amount of salt vapour is nearly equal to the current required to electrolyse the same amount of salt in an aqueous solution.*

In conclusion, I wish to say that my best thanks are due to Professor J. J. Thomson for much kindly interest and advice throughout the course of the work described in this paper.

---

* For similar results with nineteen different alkali salts see Appendix.

APPENDIX.

Added August 9, 1901.

SINCE writing the above paper 1 have made measurements with a slightly modified apparatus on a number of other salts of the alkali metals. The modifications made in the apparatus were not important. A separate galvanometer was provided for the thermo-couple, so that the temperature and conductivity could be observed simultaneously, and the length of the platinum tube was increased by 12 centims. to further insure that the current of air should have time to take up the temperature of the tube. The sprayer nozzle had to be readjusted, which considerably increased the amount of salt passing through the tube. With these exceptions the apparatus was practically identical with that described above.

Measurements were made with salts of the alkali metals besides those of potassium, so far only used.

At temperatures approaching 1300° the current E.M.F. curves for most salts resemble that given above for KI at 1200°. Below 1100° the current is almost always greater when the outer tube is negative, as already pointed out. However, above this temperature it has been found that the two-current E.M.F. curves cross each other, so that the current is then the greater at high E.M.F.'s when the outer tube is positive.

The following table gives the currents observed when a 1 per cent. solution of CsCl was being sprayed at a temperature of 1340° C.

P.D. Volts.	Current (amperes).	
	Outer tube −.	Outer tube +.
1200	$8·80 \times 10^{-4}$	$15·10 \times 10^{-4}$
800	8·65 ,,	13·10 ,,
600	8·80 ,,	9·30 ,,
400	7·62 ,,	4·25 ,,
200	2·34 ,,	1·14 ,,
120	0·93 ,,	0·46 ,,
40	0·23 ,,	0·12 ,,

These results are typical of the behaviour of most alkali metal salts.

It was found that with many salts on first applying the P.D. the current was often abnormally great for a short time, and then settled down rapidly to a steady value, just as it does for air. KI, KCl, and other easily volatile salts only show this effect very slightly, so that as these salts were mainly used in the experiments described above, this effect was not observed with salts until after the above paper was written.

It seems just possible that this effect is due to salt vapour (or, in the case of air, some gas) condensing on the cooler parts of the tube, and to the application of the P.D. causing some of this condensed vapour to evaporate, so that until equilibrium is again restored the amount of salt passing between the electrodes is abnormally large.

The main object of these further experiments was to test the conclusion that the maximum or saturation current carried by the vapour is equal to that required to electrolyse the same amount of the salt in a solution.

Using solutions containing 10 grammes in a litre it was not found possible to saturate the vapour, except in a few cases when the saturation current was found to be approximately that required by the above-mentioned conclusion.

Experiments were therefore tried with solutions containing 1 gramme in a litre, in the hope that with a smaller quantity of salt it would be easier to approach the saturation current. This device proved entirely successful, and the saturation current was in every case nearly inversely proportional to the electro-chemical equivalent of the salt used. It was also verified that the saturation current for a particular salt was proportional to the concentration of the solution sprayed.

The following table gives the currents observed at about 1350° C. with 840 volts with solutions containing 10 grammes in a litre, the outer tube being positively charged in every case.

Salt.	Electro-chemical equivalent. (E.)	Galvanometer deflection. (C.) $1 = 5 \cdot 2 \times 10^{-6}$ amperes.	EC.	Remarks.
CsCl . . .	168	290	$4 \cdot 9 \times 10^4$	Saturated.
RbI . . .	212	260	5·5 ,,	,,
Rb₂CO₃ . .	115	316	3·6 ,,	Not saturated.
KI	166	315	5·2 ,,	Saturated.
NaI . . .	150	315	4·7 ,,	Nearly saturated.
Na₂CO₃ . .	53	505	2·7 ,,	Not saturated.
LiI	134	330	4·4 ,,	,,
LiBr . . .	87	350	3·0 ,,	,,

With CsCl, RbI, KI, and NaI, which were all saturated, or nearly so, the product EC has nearly the same value, viz., $5 \cdot 1 \times 10^4$.

The following table gives the results obtained with solutions containing 1 gramme in a litre.

Salt.	Electro-chemical equivalent. (E.)	Galvanometer deflection. (C.) $1 = 5\cdot2 \times 10^{-6}$ ampere.	EC.
CsCl	168	31	$5\cdot2 \times 10^5$
Cs$_2$CO$_3$ . . .	163	31	5·1 ,,
RbI	212	24	5·1 ,,
RbCl	121	43	5·2 ,,
Rb$_2$CO$_3$ . . .	115	47	5·4 ,,
KI	166	32	5·3 ,,
KBr	119	41	4·9 ,,
KF	58	85	4·9 ,,
K$_2$CO$_3$ . . .	69	77	5·3 ,,
NaI	150	35	5·3 ,,
NaBr	103	47	4·9 ,,
NaCl	59	91	5·3 ,,
Na$_2$CO$_3$ . . .	53	91	4·8 ,,
LiI	134	39	5·2 ,,
LiBr	87	60	5·2 ,,
LiCl	43	120	5·2 ,,
Li$_2$CO$_3$ . . .	37	144	5·3 ,,

Mean value . . . $5\cdot14 \times 10^5$

It is clear from the above results that the saturation current is inversely proportional to the electro-chemical equivalent of the salt. The mean value of the product EC, $5\cdot14 \times 10^5$, is also very approximately one-tenth of the value $5\cdot1 \times 10^4$ obtained with solutions containing 10 grammes in a litre, which shows that the saturation current is proportional to the concentration of the solution sprayed.

The amount of salt passing through the tube was determined again by a modification of the method originally employed by ABRHENIUS to determine the amount of salt supplied by a sprayer to a flame.

A solution containing 40 grammes of lithium chloride per litre was sprayed and the air and spray mixed with coal-gas and the mixture then burnt from a brass tube, so as to form a Bunsen flame.

A Bunsen burner was adjusted so as to give another sensibly equal and similar flame, which was placed close beside the first. A weighed bead of fused LiCl was held in the axis of the second flame on a platinum wire loop, and its height in the flame adjusted till the tips of the two flames appeared equally brightly coloured.

Under these circumstances the rate of supply of salt to the two flames must be nearly the same, so that the loss of weight of the bead of LiCl measures the rate at which LiCl is supplied by the sprayer.

The loss of weight of the LiCl bead was found to be 7 milligrammes in 10 minutes in one experiment, and 6 milligrammes in 10 minutes in another experiment. This gives for the number of milligrammes of salt passing through the platinum tube per second when a solution containing 1 gramme in a litre is being sprayed—

Taking the amount of hydrogen liberated in electrolysis by 1 ampere in 1 second as $1 \cdot 04 \times 10^{-5}$ gramme, we can now calculate a theoretical value for the product EC given in the above tables of results. For solutions of n grammes in a litre

$$= 5n \times 10^{3}.$$

This agrees extremely well with the observed values of EC, viz.,

$$5 \cdot 1 \quad \times 10^{4} \quad \text{when} \quad n = 10,$$
$$\text{and } 5 \cdot 14 \times 10^{3} \quad \text{when} \quad n = 1.$$

It is evident that these results prove that FARADAY's laws for the passage of electricity through liquids apply also to alkali salts in the state of vapour. This must be regarded as very conclusive evidence in favour of the theory that the passage of electricity through salt vapours is a process very analogous to the electrolysis of salt solutions.

In a dilute solution of such a salt as KCl each molecule of the salt is believed to dissociate into two ions, $+K$ and $-Cl$. According to the corpuscular theory of electricity, if we denote a corpuscle by a then these ions are $K - a$ and $Cl + a$.

The results described in this paper prove that the amount of electricity which can be transported by salt in the form of vapour is equal to the amount required to electrolyse the same amount of salt in a solution.

This can be explained on the corpuscular theory in two ways, the first of which involves electrolysis of the salt, whereas the second does not. According to the first, the molecules of the salt vapour dissociate into ions, $K - a$ and $Cl + a$, exactly as in a solution. These ions then move to the electrodes and give up their charges, so becoming K and Cl, the K being at the negative electrode and the Cl at the positive electrode.

According to the other explanation a molecule of the salt vapour merely loses a corpuscle, thus forming two ions

$$KCl - a \quad \text{and} \quad a.$$

Then the KCl goes to the negative electrode and only corpuscles go to the positive

electrode, so that no separation of the two constituent atoms of the molecule takes place.

The high velocity of the negative ions, compared with that of the positive ions, seems to favour the latter view, but the known cases in which separation of the elements of a compound by electrolysis in gases appears to take place, strongly support the view that the ions are similar to those existing in solutions.

The present experiments do not show what happens to the ions after they have discharged on the electrodes, except that apparently they do not participate any further in the transport of the electricity. It is hoped that future experiments will throw more light on this question.

INDEX SLIP

XI. Mathematical Contributions to the Theory of Evolution.—X. Supplement to a Memoir on Skew Variation.*

By KARL PEARSON, F.R.S., University College, London.

(1.) In a memoir on Skew Variation published in the 'Phil. Trans.,' A, vol. 186, 1895, a series of frequency curves are discussed which are integrals of the differential equation

$$\frac{1}{y}\frac{dy}{dx} = \frac{-x}{c_1 + c_2 x + c_3 x^2} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (\text{i.}).$$

(See p. 381 of the memoir.)

The discussion of four main types is given in detail, and a brief reference is made to various sub-types which may occur. The types considered in that memoir covered at the time all the frequency series, and they were fairly numerous, that I had had occasion to deal with. In the course of the last few years, however, I have been somewhat puzzled by frequency distributions for which the criterion $2\beta_2 - 3\beta_1 - 6$ (see p. 378) was positive, and therefore à priori a curve of the type

$$y = y_0 \frac{1}{\left\{1 + \left(\frac{x}{a}\right)^2\right\}} e^{-\nu \tan^{-1} \frac{x}{a}}$$

was to be expected, but which on calculation gave ν imaginary. The frequency distributions in question arose† occasionally in sociological statistics, but also in

---

* 'Phil. Trans.,' A, vol. 186, p. 343.

† Some other frequency distributions, which on first investigation fell under Types V. and VI. of the present paper, were found with improved values for the moments to fall under types already discussed. Mr. W. F. SHEPPARD's values for the moments ('Lond. Math. Soc.,' vol. 29, p. 369, formula 30) should certainly be used in preference to those given by me ('Phil. Trans.,' A, vol. 186, p. 350) whenever we are calculating the moments of a curve from areas and not from true ordinates. I hope shortly to publish a paper on this point, which is one really of quadrature formulæ. Meanwhile for every true frequency curve with high contact at both terminals we ought to use

$$\mu_2 = \sigma^2(\nu_2' - \nu_1'^2 - \tfrac{1}{12}),$$
$$\mu_4 = \sigma^4(\nu_4' - 4\nu_1'\nu_3' + 6\nu_1'^2\nu_2' - 3\nu_1'^4 - \tfrac{1}{2}(\nu_2' - \nu_1'^2) + \tfrac{7}{240}),$$

instead of the values given on p. 350, μ_2 remaining unchanged.

biological investigations. It seemed, therefore, desirable to enter a little more fully into the analysis of the cases in which the criterion was positive but ν imaginary, and discover what types of frequency curves had escaped my attention.*

The key to the solution lies in the fact noted on p. 369 of the memoir, namely, that even if the criterion be positive, there will still be a solution akin to Type I. and not to Type IV. if ϵ be negative. No frequency series satisfying these conditions had at that time come under my notice, and later, when collecting data of floral variability, my own remark as to ϵ had slipped from my memory. It is the object of this supplement to obtain an improved criterion of type, to discuss the nature of the curves which fill the gap observed, and to illustrate by one or two examples the fitting of such curves to actual statistics.

(2.) *The Two Criteria.*

Throughout this supplement the notation of the previous memoir will be assumed to be familiar to the reader.

Turning to p. 378 of that memoir, we note that since β_1 and $r - 1$ are necessarily positive, z if positive must be $> r^2$. Hence ν can only become imaginary if z be negative, or

$$\frac{\beta_1 (r - 2)^2}{16 (r - 1)} > 1.$$

Substitute in this the value of r and it becomes

$$\frac{\beta_1 (\beta_2 + 3)^2}{4 (4\beta_2 - 3\beta_1)(2\beta_2 - 3\beta_1 - 6)} > 1 \quad . \quad . \quad . \quad . \quad . \quad \text{(ii.)}.$$

Hence the complete condition that a curve of Type IV. shall give the distribution of frequency is not only

$$\kappa_1 = 2\beta_2 - 3\beta_1 - 6 > 0,$$

but also

$$\kappa_2 = \frac{\beta_1 (\beta_2 + 3)^2}{4 (4\beta_2 - 3\beta_1)(2\beta_2 - 3\beta_1 - 6)} < 1.$$

Turning back to p. 369, we see that ϵ being positive the complete conditions for a curve of Type I. giving the distribution of frequency are

$$\kappa_1 = 2\beta_2 - 3\beta_1 - 6 < 0,$$

---

* I was very loath to adopt Professor EDGEWORTH's method of inventing new frequency curves by putting $x = f(x')$ in a normal frequency distribution, $y = y_0 e^{-x^2}$. Besides strong theoretical objections to this process, I had found Equation (i.) so sufficient for a great variety of cases that I felt confident it must cover the newly discovered outstanding cases, and this confidence seems justified by the result.

and
$$\kappa_2 = \frac{\beta_1(\beta_4 + 3)^2}{4(4\beta_2 - 3\beta_1)(2\beta_2 - 3\beta_1 - 6)} < 0.$$

The latter condition will be always satisfied since β_1 and $4\beta_2 - 3\beta_1$ are positive for any distribution whatever, and $2\beta_2 - 3\beta_1 - 6$ is negative by hypothesis.

Further, in the previous case κ_2 is seen to be essentially positive.

Hence the criteria written down cover all possible cases but those for which

$$\kappa_2 > 1.$$

Sub-cases which arise from transition curves just at the limits will, however, be likely to be of interest. What happens when $\kappa_2 = \infty$ and when $\kappa_2 = 1$? The only possibility for $\kappa_2 = \infty$ is $2\beta_2 - 3\beta_1 - 6$, or $\kappa_1 = 0$. But this curve has been fully treated under Type III. in the memoir.

We shall see later that $\kappa_2 = 1$ leads us up to a novel transition curve of considerable interest.

To ascertain something about the general case in which $\kappa_2 > 1$, let us return to the memoir again and examine the value of ϵ on p. 369. It can only be negative if

$$4 + \tfrac{1}{2}\beta_1(r + 2)^2/(r + 1) \text{ be } < 0,$$

where r is here
$$= \frac{6(\beta_2 - \beta_1 - 1)}{3\beta_1 - 2\beta_2 + 6}.$$

Substituting, we find at once

$$\kappa_2 > 1,$$

which in itself involves $\kappa_1 > 0$.

Hence the missing gap corresponds to those cases in which ϵ is negative.

It will be clear that κ_2, although in form giving a more complex criterion than κ_1, is really more effective, as covering all the possible cases. We have then the following scheme :—

Criterion κ_2.	Corresponding frequency curve.
$\kappa_2 = \infty$	Transition curve, Type III. (Memoir, p. 373).
$\kappa_2 > 1 \ \& \infty$	Type VI. (see p. 448 below).
$\kappa_2 = 1$	Transition curve, Type V. (see p. 446 below).
$\kappa_2 > 0 \ \& < 1$	Type IV. (Memoir, p. 376).
$\kappa_2 = 0, \beta_1 = 0, \beta_2 = 3$. .	Normal curve.
$\kappa_2 = 0, \beta_1 = 0, \beta_2$ not $= 3$	Type II. (Memoir, p. 372).
$\kappa_2 < 0$	Type I. (Memoir, p 367).

The object of this supplement is to discuss the calculation of curves of Type V., and to consider those of Type VI. somewhat more at length, they being only briefly referred to on p. 369 of the memoir. It will be seen that Type I. of the memoir has now broken up into two divisions. One portion is the old Type I. passing into the normal curve on one side and Type III. on the other. This Type III. separates the second portion, Type VI., of the old Type I. from the first portion. Type VI. passes from Type III. to the new transition curve Type V., which, like Type III., will be found to have a range limited in one direction only. Finally this new Type V. is the transition to the old Type IV. bounded on the other side by the sub-curve, the old Type II., and beyond that the normal curve. Thus we see that Types I. and IV. do not pass directly into each other through Type III., as might be supposed by the criterion $\kappa_1 >$ or < 0, but that there are a series of intervening curves, two of which, Types V. and VI., require further consideration, if we are to complete the whole round of frequency distributions embraced under the differential equation (i.).

(3.) On the Frequency Curve of Type V.

Returning to the fundamental differential equation (i.), let us consider what transformation takes place when the denominator on the right has *equal* roots.* We may then write it in the form

$$\frac{1}{y}\frac{dy}{dx} = \frac{-x}{c_0(c_1+x)^2} = \frac{c_1}{r_0}\frac{1}{(c_1+x)^2} - \frac{1}{r_0(c_1+x)}.$$

Hence

$$\log y = -\frac{c_1}{c_0}\frac{1}{(c_1+x)} - \frac{1}{c_0}\log(c_1+x) + \text{const.}$$

Thus

$$y = y_0\, e^{-\frac{\gamma}{c_1+x}}(c_1+x)^{-p},$$

where, y_0 is a constant, $\gamma = c_1/c_0$ and $p = 1/c_0$. Thus changing the origin we may write the curve :

$$y = y_0\, x^{-p}\, e^{-\gamma/x}. \qquad \qquad . \quad . \quad . \quad . \quad . \quad \text{(iii.),}$$

where $x_{m0} = \gamma/p$ gives the distance of the mode from the new origin.

To find the moments about this origin, we notice that, p and γ being positive, $y = 0$ when $x = 0$ and when $x = \infty$. Thus as in the curve of Type III. we have a range limited at one end only.

To find the moments we have, if a be the area,

$$a\,\mu'_n = \int_0^\infty y_0\, x^{-p+n}\, e^{-\gamma/x}\, dx \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iv.).}$$

---

* I owe to Miss AGNES KELLY, Ph.D., the suggestion that this type of frequency curve deserved fuller treatment.

Put $\gamma/x = z$, and we find

$$\alpha\,\mu'_n = y_0\,\gamma^{n-p+1}\int_0^\infty e^{-z}\,z^{p-n-2}\,dz$$

$$= y_0\,\gamma^{n-p+1}\,\Gamma\,(p-n-1) \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(v.)}.$$

Thus:

$$\alpha = y_0\,\gamma^{1-p}\,\Gamma\,(p-1) \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(vi)}.$$

$$\left.\begin{aligned}
\mu_1' &= \gamma/(p-2)\\
\mu_2' &= \gamma^2/(p-2)\,(p-3)\\
\mu_3'' &= \gamma^3/(p-2)\,(p-3)\,(p-4)\\
\mu_4' &= \gamma^4/(p-2)\,(p-3)\,(p-4)\,(p-5)
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad \text{(vii.)}$$

Transferring to the centroid we find

$$\left.\begin{aligned}
\mu_2 &= \frac{\gamma^2}{(p-2)^2(p-3)}\\[1mm]
\mu_3 &= \frac{4\gamma^3}{(p-2)^3(p-3)(p-4)}\\[1mm]
\mu_4 &= \frac{3(p+4)\,\gamma^4}{(p-2)^4(p-3)(p-4)(p-5)}
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad \text{(viii.)}.$$

$$\beta_1 = \mu_3^2/\mu_2^3 = \frac{16\,(p-3)}{(p-4)^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ix.)}.$$

$$\beta_2 = \mu_4/\mu_2^2 = \frac{3\,(p+4)\,(p-3)}{(p-5)(p-4)} \quad . \quad . \quad . \quad . \quad \text{(x.)}.$$

Eliminating p between β_1 and β_2 we find after some reductions :

$$\beta_1\,(\beta_2+3)^2 = 4\,(2\beta_2-3\beta_1-6)\,(4\beta_2-3\beta_1) \quad \cdot$$

or,

$$\kappa_2 = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xi.)}.$$

Clearly, since this is the condition for Type V., that transition curve is none other than the curve obtained by making the denominator of the right-hand side of the differential equation have equal roots. The curve is clearly of considerable interest, and its existence had not been noticed in the previous series of frequency curves.

The manner of fitting it is now easily described.

Equation (ix.) gives us a quadratic to find $p-4$:

$$(p-4)^2 - \frac{16}{\beta_1}\,(p-4) - \frac{16}{\beta_1} = 0 \quad . \quad . \quad . \quad . \quad . \quad \text{(xii.)}.$$

The positive root of this is the required solution.

γ is then found from the first of equations (viii.), or if σ be the standard deviation $= \sqrt{\mu_2}$, then :*

$$\gamma = \sigma (p - 2) . \sqrt{(p - 3)} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xiii.)}.$$

Then (vi.) gives :

$$\Gamma(p - 1) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xiv.)},$$

which determines y_0, the remaining constant for the shape of the curve.

For the position of the curve, we have for the distance from origin to mean, from the first of equations (vii.) :

$$\mu_1' = \gamma/(p - 2) = \sigma\sqrt{(p - 3)} \quad . \quad . \quad . \quad . \quad \text{(xv.)}.$$

If d be the distance from mode to mean we have :

$$d = \mu_1' - \gamma/p$$
$$= \frac{\gamma}{p} - \frac{\gamma}{p} = \frac{2\gamma}{p(p - 2)} \quad . \quad . \quad . \quad . \quad . \quad \text{(xvi.)}.$$

Further, the skewness :

$$\text{Sk.} = d/\sigma = \frac{2\sqrt{(p - 3)}}{p} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xvii.)}.$$

Thus the solution is completed.

(4.) On the Frequency Curve of Type VI.

Type VI., as we have seen, corresponds to the case in which Type I. of the memoir has its ϵ negative. Hence either m_1' or m_2' is negative and the curve transferring the origin takes the form

$$y = y_0 (x - a)^{m_1}/x^{m_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xviii.)}.$$

Now it is possible that this curve falls under the limited range type of a frequency from $x = 0$ to $x = a$, but as we see that the criterion places Type VI. between two curves of range limited in one direction only, we expect Type VI. also to be of that character, and a complete solution is obtained by taking the range from $x = a$ to $x = \infty$; this indeed fills up the gap for $\kappa_2 > 1$ and $< \infty$, and (xviii.) with this range is seen to pass into one or other of the two transition curves

$$y = y_0 x^p e^{-\gamma x},$$

or

$$y = y_0 x^{-p} e^{-\gamma/x},$$

according as we allow the first or second factor to approach a limit.†

---

* The sign of μ_3 will determine the sign of γ, or, what may be taken as the same thing, the direction of the axis of x.

† Write : $y = \text{const.} \times x^{-m_2} (1 - x/a)^{m_1}$ and make $m_1 = \infty$, and $a = \infty$ but m_1/a finite.

Or, $y = \text{const.} \times (1 - a/x)^{m_1}/x^{m_2 - m_1}$, and make $a = 0$, $m_1 = \infty$, and $a \times m_1$ together with $m_2 - m_1$ finite.

Accordingly we shall write Type VI. in the form

$$y = y_0 (x - a)^{q_2} / x^{q_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xix.)},$$

and take the range from a to ∞.

Differentiating to find the position of the mode we have

$$x_{mo} = \frac{a q_1}{q_1 - q_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xx.)}.$$

For the moments about the origin:

$$a \mu'_n = \int_a^\infty y_0 \frac{x^n (x - a)^{q_2}}{x^{q_1}} \, dx.$$

Put $a/x = z$, hence

$$a \mu'_n = \int_0^1 \frac{y_0}{a^{q_1 - q_2 - n - 1}} z^{q_1 - q_2 - n - 2} (1 - z)^{q_2} \, dz$$

$$= \frac{y_0}{a^{q_1 - q_2 - n - 1}} \, \mathrm{B} \, (q_1 - q_2 - n - 1, \, q_2 + 1)$$

$$= \frac{y_0}{a^{q_1 - q_2 - n - 1}} \frac{\Gamma (q_1 - q_2 - n - 1) \, \Gamma (q_2 + 1)}{\Gamma (q_1 - n)}.$$

Hence we deduce

$$a = \frac{y_0}{a^{q_1 - q_2 - 1}} \frac{\Gamma (q_1 - q_2 - 1) \, \Gamma (q_2 + 1)}{\Gamma (q_1)} \quad . \quad . \quad . \quad . \quad \text{(xxi.)}.$$

$$\left. \begin{aligned}
\mu_1' &= \frac{a (q_1 - 1)}{q_1 - q_2 - 2} \\[2mm]
\mu_2' &= \frac{a^2 (q_1 - 1)(q_1 - 2)}{(q_1 - q_2 - 2)(q_1 - q_2 - 3)} \\[2mm]
\mu_3' &= \frac{a^3 (q_1 - 1)(q_1 - 2)(q_1 - 3)}{(q_1 - q_2 - 2)(q_1 - q_2 - 3)(q_1 - q_2 - 4)} \\[2mm]
\mu_4' &= \frac{a^4 (q_1 - 1)(q_1 - 2)(q_1 - 3)(q_1 - 4)}{(q_1 - q_2 - 2)(q_1 - q_2 - 3)(q_1 - q_2 - 4)(q_1 - q_2 - 5)}
\end{aligned} \right\} \quad . \quad . \quad \text{(xxii.)}.$$

Now if we compare these results with those on p. 368 of the earlier memoir we see that the one set can be at once deduced from the other by writing $m_1 = - q_1$, $m_2 = q_2$. Thus with this interchange the whole of that solution holds, if we bear in mind that the range is now from $x = a$ to ∞.

We easily find:

$$r = - q_1 + q_2 + 2 \qquad \epsilon = 1 - q_1 + q_2 - q_1 q_2$$

and $1 - q_1$ and $q_2 + 1$ are the roots of

$$z^2 - rz + \epsilon = 0 . \qquad \qquad \text{(xxiii.)},$$

where r and ϵ are to be determined as in that memoir, pp. 368–369.

We have :

$$\mu_2 = \frac{a^2 (1 - q_1) (1 + q_2)}{r^2 (r + 1)} \qquad \qquad \text{(xxiv.)},$$

where $1 - q_1$ and r are both negative. This gives a.[*] Thus q_1, q_2 and a are known, and from Equation (xxi.)

$$y_0 = \frac{a \, a^{q_1 - q_2 - 1} \Gamma(q_1)}{\Gamma(q_1 - q_2 - 1) \Gamma(q_2 + 1)} \qquad \qquad \text{(xxv.)}$$

we find the remaining unknown constant for the shape of the curve, y_0. As before, various approximations may be used to the values of the Γ functions when either q_1 or q_2 or both are large.[†]

We easily obtain for the distance between mode and mean

$$d = \frac{a(q_1 + q_2)}{(q_1 - q_2)(q_1 - q_2 - 2)} \qquad \qquad \text{(xxvi.)},$$

and for the skewness :

$$\text{Sk.} = \frac{(q_1 + q_2) \sqrt{(q_1 - q_2 - 3)}}{(q_1 - q_2) \sqrt{\{(q_1 - 1)(q_2 + 1)\}}} . \qquad \qquad \text{(xxvii.)}.$$

(5.) A special case of some interest arises when the start of the curve is *à priori* known. Suppose its distance from the mean to be c and let (using moments about centroid)

$$\mu_2/c^2 = \gamma_2, \quad \mu_3/(2\mu_2 c) = \gamma_3 . \qquad \qquad \text{(xxviii.)}.$$

Then we easily find :

$$\gamma_2 = \frac{1 - q_1}{(1 + q_2)(-q_1 + q_2 + 3)}, \quad \gamma_3 = \frac{q_1 + q_2}{(1 + q_2)(q_1 - q_2 - 4)} .$$

---

[*] $1 - q_1$ being negative, ϵ is negative, and accordingly by what goes before κ_2 lies between 1 and ∞.

[†] The value of y_0 for curves of Type I., if m_1 be small but m_2 large ('Phil. Trans.,' A, vol. 186, p. 369, foot-note), is

$$y_0 = \frac{a}{b} (m_1 + m_2 + 1) \sqrt{\frac{m_1 + m_2}{m_2}} \, e^{\frac{1}{12} \left(\frac{1}{m_1 + m_2} - \frac{1}{m_1} \right)} \frac{m_2 \, m_1 e^{-m_1}}{\Gamma(m_1 + 1)},$$

and this can be easily modified to suit (xxv.) above. A very convenient and exact formula for $\Gamma(n + 1)$, if n is large, is that given by FORSYTH ('B.A. Report,' 1883, p. 47) :

$$\Gamma(n + 1) = \sqrt{2\pi} \left(\frac{\sqrt{n^2 + n} + \frac{1}{6}}{e} \right)^{n + \frac{1}{2}},$$

the error being less than $\dfrac{1}{240 n^3}$ of the whole.

Whence we deduce to determine q_1 and q_2 :

$$\gamma_2 - \gamma_3 \qquad \gamma_1 + \gamma_2 - (2\gamma_2 - \gamma_4 + \gamma_2\gamma_3)(\gamma_3 - \gamma_2) \qquad \text{(xxix.),}$$

and the solution proceeds as before.

(6.) *Illustrations.*—I propose to note a few distributions of frequency in which I have come across Types V. and VI.

(A.) *Statistics of Age of Bride at Marriage, the Bridegroom's Age being between 24 and 25 years.*[*]

The observations given in the table, p. 454, are taken from PEROZZO's memoir : "Nuove Applicazioni del Calcolo delle Probabilità . . . ," 'Reale Accademia dei Lincei,' Anno CCLXXIX., 1881–2, Tavola I.

The total number of recorded marriages is 28,454. The moments were calculated by using SHEPPARD's corrections ('London Math. Soc. Proc.,' vol. 29, p. 369), and are as follows :—

$$\text{Mean age of bride} = 22\cdot1877.$$

$$\mu_2 = 13\cdot3346$$
$$\mu_3 = 67\cdot8145$$
$$\mu_4 = 1224\cdot6342$$

Whence :
$$\beta_1 = 1\cdot9396$$
$$\beta_2 = 6\cdot8873$$
$$\kappa_1 = 1\cdot9558$$
$$\kappa_2 = 1\cdot1094$$

Thus by p. 445 we see that Type VI. is the frequency curve to be selected, but as κ_2 does not differ widely from unity, we shall probably get a good fit from Type V. as well.

Taking Type VI. first, we find :

$$r = -12\cdot11075, \qquad \epsilon = -317\cdot84987.$$

The quadratic (xxiii.) is accordingly :

$$z^2 + 12\cdot11075\,z - 317\cdot84987 = 0.$$

---

[*] I selected this example at random, as one out of several leading to the curve types it was my object to illustrate. There is so much tampering with statistics, however, whenever they refer to the ages of women, that it would probably have been better to have used the men.

Thus : $q_1 = 25\cdot88401$, $q_2 = 11\cdot77326$.

Hence by (xxiv.) $a = 8\cdot268,405$,

and by (xxv.) $\log y_0 = 24\cdot275,3032$.

We have accordingly for the equation to the curve :

$$y = 10^{24} \times 1\cdot884,965 \,\frac{(x - 8\cdot268,405)^{11\cdot77326}}{x^{25\cdot88401}}.$$

The distance from the origin to the mean is given by the first equation of (xxii.) :

$$\mu_1' = 16\cdot98913,$$

or, the theoretical range starts with brides of $5\cdot198,570 + 8\cdot268,405 = 13\cdot466,975$ years. This is an excellent underlimit to the age of women marrying men of 24 to 25 in a country like Italy. Our first group is at $15\cdot5$, and the above start is just two base units before this initial group.

The skewness $= \cdot498,953$, and the distance from mode to mean $= 1\cdot822,004$, or the mode is at $20\cdot3657$ years.

Turning now to Type V. we have the following results :—

$$16/\beta_1 = 8\cdot249,262.$$

Hence Equation (xii.) is :

$$(p - 4)^2 - 8\cdot249,262 \,(p - 4) - 8\cdot249,262 = 0.$$

Thus the positive value of p is :

$$p = 13\cdot150,747.$$

Equation (xiii.) gives :

$$\gamma = 129\cdot73081.$$

Then (xiv.) gives :

$$\log y_0 = 22\cdot367,6952.$$

Thus the equation to the curve is :

$$y = 10^{22} \times 2\cdot331,821 \, x^{-13\cdot150,747} \, e^{-129\cdot73081/x}.$$

To find the position of its start we have by (xv.) :

$$\mu_1' = 11\cdot6343,$$

or, since the mean age of brides is $22\cdot1877$, the youngest possible theoretical bride is $10\cdot5534$ years. This is probably a worse determination of the underlimit than in the case of Type VI. At the same time I notice that out of about 180,000 women, 101

were married between 14 and 15 years of age, and all the curves begin with a sensibly finite ordinate at 14·5; it is accordingly possible that a somewhat lower age than 13·5 actually occurs in Italy.

Equation (xvi.) gives us for the distance from mode to mean :

$$d = 1·7694,$$

or the modal age at marriage is 20·4183 years. This is only about ·053 of a year or about 19 days different from the modal age as given by Type VI., a most satisfactory agreement.

For the skewness we have from Equation (xvii.) :

$$Sk. = ·4845,$$

or, it differs by less than 3 per cent. from the skewness as given by Type (VI.).

The diagram (fig. i.) shows the two curves, and the table compares the results obtained from either with the observations.*

Italian Marriage.
Frequency of Brides of different Ages
with Bridegrooms 24-25.
(Fig. I).
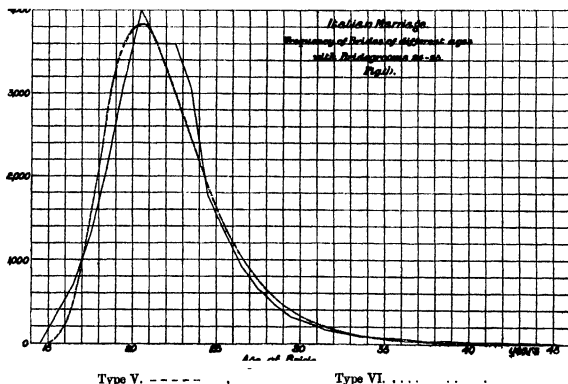
Type V. - - - - - ,   Type VI. ....

It is clear that for all practical purposes the curve of Type V. is as good as that of Type VI. Indeed, there is practically no difference between them except for the

* The observation data are really areas, while to save lengthy calculations we have compared both in diagram and table the ordinates of the theoretical curves. This is in general legitimate, if, as in this case, the number of groups is very large.

ages 15 to 17. The fit is, however, not a very good one, and although it is indefinitely better than a normal curve, and we see why in the absence of these types the statistics could not be fitted with any of the first series of skew curves, yet we are compelled to consider that there are causes other than chance at work very definitely affecting the frequency of the *recorded* ages. Thus the bridegrooms being 24 to 25, the desire of the bride to be recorded as younger than her husband probably fully accounts for the bulk of the preponderance of observation over theory

TABLE of Observed and Calculated Frequencies.

Age.	Observed frequency.	Calculated frequency.		Age.	Observed frequency.	Calculated frequency.	
		Type V.	Type VI.			Type V.	Type VI.
5–16	367	70	49	30–31	256	281	
6–17	717	514	489	31–32	164	201	
7–18	1294	1538	1560	32–33	134	148	146
8–19	2121	2751	2800	33–34	94	104	105
9–20	3156	3591	3622	34–35	77	75	76
10–21	4009	3830	3831	35–36	68	55	
11–22	3593	3577	3560	36–37	59	40	
12–23	3604	3055	3034	37–38	33	29	
23–24	3060	2456	2439	38–39	40	21	
24–25	1774	1894	1884	39–40	27	16	
25–26	1353	1419	1415	40–41	18	12	
26–27	936	1044	1043	41–42	21	9	
27–28	663	758	760	42–43	11	7	
28–29	468	546	549	43–44	14	5	
29–30	319	392	395	44–45	4	4	

in the frequency of the brides of 22 to 24. The defect of brides between 17 and 20 may be again due to the tendency to state the age as over 21, and so free the woman from the need for parental sanction.* These causes, giving a false displacement of age frequency, are probably in themselves sufficient to account for the theoretical defect in brides of 15 to 17.

(7.) (B.) *On the Variation in the Number of Lips of the Medusa* P. Pentata.

My data are the following, taken from a paper by ALFRED GOLDSBOROUGH MAYER: "The Variations of a Newly Arisen Species of Medusa," 'Science Bulletin of the Museum of the Brooklyn Institute,' vol. 1, p. 1, 1901.

---

* I have found in England the statement of the bride's age in the marriage licence is for the same reason occasionally not in accordance with the year of birth as shown by the parish register.

Frequency.	No. of lips
2	1
5	2
18	3
123	4
798	5
49	6
1	7
Total 996	

Mr. MAYER (p. 12) notes the failure of my curve of Type IV. I find for the constants :

$$\text{Mean} = 4\text{·}8685 \text{ lips.}$$
$$\mu_2 = \text{·}309,006, \quad \sigma = \text{·}55588$$
$$\mu_3 = - \text{·}350,697$$
$$\mu_4 = 1\text{·}181,718$$
$$\beta_1 = 4\text{·}16834$$
$$\beta_2 = 12\text{·}37598$$
$$\kappa_1 = 6\text{·}24694$$
$$\kappa_2 = 1\text{·}06594$$

Since κ_2 is so nearly unity we may use Type V.
Hence I find :

$$p = 8\text{·}66184 \qquad \gamma = - 8\text{·}811634$$

(γ must be negative since μ_3 is negative)

$$\mu_1' = 1\text{·}32270.$$

Thus the curve starts at 6·19118 lips. or the one medusa with seven lips is theoretically excluded. Here I have worked with the uncorrected moments because the lips are discontinuous variants. Working with SHEPPARD'S corrective terms the limit is about six lips, and with the corrective terms suggested in my memoir on skew variation the limit is 7·65. Further we have :

$$\log y_0 = 6\text{·}829,3633,$$
$$\text{distance from mean to mode} = \text{·}30541,$$
$$\text{Sk.} = \text{·}54941.$$

The mode is thus at 5·17889, in good agreement with observation.

The equation to the curve is, taking x *positive* from 6·19118 lips towards lesser values :

$$\log y = 6\text{·}829,3633 - 8\text{·}66184 \log x - \frac{3\text{·}826,8435}{x}.$$

This curve was drawn on a large scale and its areas read off with an integrator. The following theoretical frequencies were obtained:

No. of lips.	Observation.	Calculation.
6 and over	50	47
5	798	762
4	123	160·5
3	18	20
2	5	5
1	2	1·5

There would not be any serious divergence here, were it not for the group with four lips, which observation shows to be much under-represented. But it must be remembered that we have only seven groups, and that such a number is very insufficient for a good determination of the moments of a curve. Further, the variation is not really continuous, as indicated by the curve, but *discrete*. We have at present no clear statement as to how the moments of a discrete system of variation should be modified or corrected so as to give the best results for the moments of the continuous curve which is to theoretically represent the series. I am doubtful
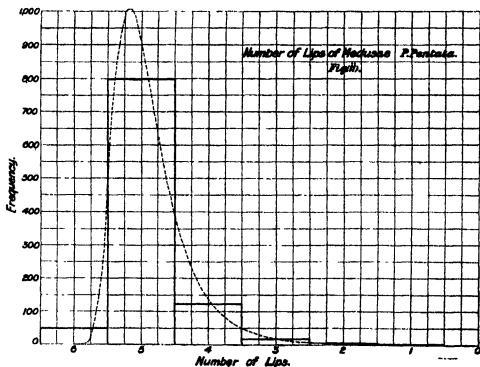
whether SHEPPARD's corrections—the best for continuous variation—are equally appropriate in this case. Above I have used merely the rough moments, but I

have found by considerable experience that in the case of discrete variables, to treat
the system as a polygon and correct, as in my memoir on Skew Variation ('Phil.
Trans.,' A, vol. 186, p. 350), appears to give the best results when the areas are
compared with the discrete groups. The point wants further investigation ; when
we have a large number of groups it is of little importance, but it makes a consider-
able difference in these excessively skew distributions of discrete variables when the
number of groups are small.*

Above all, the diagram (fig. ii.) shows how all important it is to compare *areas* and
not merely the *ordinates* of the frequency curve with the blocks representing the
discrete frequencies in such a case as this. The wide-spread custom among foreign
investigators of comparing merely the ordinates of the theoretical frequency curve
with the observed frequencies leads in such cases to most fallacious results.

(8.) (C.) *On the Distribution of Incidence of Scarlet Fever Cases with Age.*

It seems desirable to give an illustration of the method of dealing with a distri-
bution which falls under the class dealt with in Section (5) of this paper. Dr.
MACDONELL, in dealing with the intensity of incidence of different diseases at various
ages, has come across in scarlet fever a good illustration of curves of the types now
under consideration. The whole of the arithmetical work on the present example is
due to him, and I have to thank him very heartily for allowing me to use it here.

The statistics are taken from the 'Report of the Metropolitan Asylums Board'
(Statistical Part, 1899). They involve 39,253 *male* cases, distributed as follows :—

Year of life.	Frequency.	Year of life	Frequency
Under 1	443	20–25	926
1–2	1456	25–30	420
2–3	2631	30–35	215
3–4	3599	35–40	91
4–5	3862	40–45	45
5–10	15791	45–50	26
10–15	7359	50–55	17
15–20	2366	55–60	5
		60–65	1

The data being grouped partly in one and partly in five-year periods the moments
had to be calculated with caution, separating the material into two pieces. Taking
five years as the unit, Dr. MACDONELL found for the uncorrected moments :

* *E.g.*, petals of buttercups, teeth on the carapace of prawns, lips of medusæ, as compared with veins on
chestnut leaves, florets on ox-eyed daisy, &c.

Mean age of incidence, 8·60975 years.

$$\mu_2 = 1·369,345$$
$$\mu_3 = 3·233,194$$
$$\mu_4 = 19·143,575.$$

The moments were not modified by SHEPPARD'S corrections, for these suppose contact of a high order at both terminals of the curve, and it was quite apparent that the curve must rise at a finite angle on the birth side. The following additional constants were then determined :—

$$\beta_1 = 4·071,222, \qquad \beta_2 = 10·209,333,$$
$$\kappa_1 = 2·205,000, \qquad \kappa_2 = 2·813,783.$$

Thus κ_2 is > 1 and $< \infty$ and the distribution is of Type VI. Now let us suppose the incidence of scarlet fever to start with birth, although there might, as in the case of enteric fever, be really some antenatal cases.[*]

Turning to Section (5) we have :

$$c = \text{distance from birth to mean} = 8·60975 \text{ years} = 1·72195 \text{ units}.$$

Hence we deduce

$$\gamma_2 = ·461,819, \qquad \gamma_3 = ·685,596.$$

And so from (xxix.)

$$q_2 - q_1 = -10·532,485, \qquad q_2 + q_1 = 15·417,281 ;$$

or, $$q_1 = 12·974,883 \qquad\qquad q_2 = 2·442,398.$$

Then from

$$c = \mu_1' - a = a\,(q_2+1)/(q_1 - q_2 - 2)$$

we find

$$a = 4·268,104,$$

and, finally, after determining y_0 from (xxi.),

$$\log y_0 = 13·652,5078.$$

Thus the values of the frequency are given by

$$\log y = 13·652,5078 + 2·442,398 \log (x - 4·268,104) - 12·974,883 \log x.$$

The origin of the curve is thus 4·268,104 before birth. The mode is given by

$$x_{mo} = aq_1/(q_1 - q_2) = 5·257,842.$$

Thus : $x_{mo} - a = ·989,738 = 4·94869$ yrs.

---

[*] See 'Phil. Trans.,' A, vol. 186, p. 390. The remarkably sharp rise of the scarlet-fever distribution as compared with the enteric is, however, much against this.
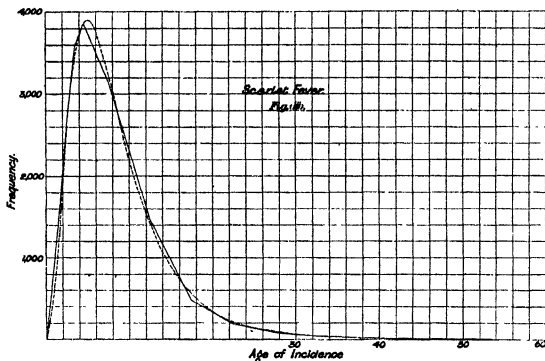
This gives for y_{mode} the value 3892.

Distance between mode and mean $= 3\cdot66106$ yrs.

Whence we find for skewness the value

$$\text{Sk.} = \cdot5347.$$

The diagram (fig. iii.) shows that the fit may be considered a good one.

(9.) The conclusions of this paper are, I think, of some interest from the general standpoint of scientific investigation. A certain number of frequency distributions had been found, not only by my co-workers and myself here, but by biologists in America, not to fit into the general system of skew distributions dealt with by me in my original memoir. The first conclusion was that however wide-reaching that system appeared to be, it was a failure for a few remarkably skew distributions. But on more careful investigation of the differential equation it appeared that two types of solution had been left out of consideration, and that these were precisely those needed in the recorded cases of failure.

I owe some apology to authors like Professor DAVENPORT and Dr. DUNCKER, who have recently issued text-books on the application of statistical methods to biological variation, because although we have known and used these curves for some years past, no account has hitherto been published of them, and, consequently, biological investigators[*] using their *résumés* of my methods have been, and I fear still may be, occasionally puzzled.

* *E.g.*, Mr. A. G. MAYER in the paper on Medusæ referred to above

XII. *Ellipsoidal Harmonic Analysis.*

By G. H. DARWIN, *F.R.S., Plumian Professor and Fellow of Trinity College, in the University of Cambridge.*

TABLE OF CONTENTS.

INTRODUCTION.

LAMÉ'S functions or ellipsoidal harmonics have been successfully used in many investigations, but the form in which they have been presented has always been such as to render numerical calculation so difficult as to be practically impossible. The object of the present investigation is to remove this imperfection in the method. I believe that I have now reduced these functions to such a form that numerical results will be accessible, although by the nature of the case the arithmetic will necessarily remain tedious.

Throughout my work on ellipsoidal harmonics I have enjoyed the immense advantage of frequent discussions with Mr. E. W. HOBSON. He has helped me freely from his great store of knowledge, and beginning, as I did, in almost complete ignorance of the subject, I could hardly have brought my attempt to a successful issue without his advice. In many cases the help derived from him has been of immense value, even where it is not possible to indicate a specific point as due to him. In other cases he has put me in the way of giving succinct proofs of propositions which I had only proved by clumsy and tedious methods, or where I merely felt sure of the truth of a result without rigorous proof. In particular, I should have been quite unable to carry out the investigation of § 19, unless he had shown me how the needed series were to be determined.

My original object in attacking this problem was the hope of being thereby enabled to obtain exact numerical results with respect to M. POINCARÉ's pear-shaped figure of equilibrium of a mass of liquid in rotation.* But I soon found that a partial investigation with one particular point in view was impracticable, and I was thus led on little by little to cover the whole field, in as far as it was necessary to do so for the purpose of practical application. This paper has then grown to such considerable dimensions that it seemed best that it should stand by itself, and that the discussion of the specific problem should be deferred.

A paper of this kind is hardly read even by the mathematician, unless he happens to be working at a cognate subject. It appears therefore to be useful to present a summary, which shall render it possible for the mathematical reader to understand the nature of the method and results, without having to pick it out from a long and complex train of analysis. Such a summary is given in Part III.

PART I.

FORMATION OF THE FUNCTIONS.

§ 1. *The Principles of Ellipsoidal Harmonic Analysis.*

The basis of this method of analysis is expounded in various works on the subject. I begin with a statement of results in my own notation.

* A paper giving the required result will be presented to the Society in the autumn.—[*July* 2, 1901.]

If u_1^2, u_2^2, u_3^2 denote the three roots of the cubic

$$\frac{x^2}{a^2 + u^2} + \frac{y^2}{b^2 + u^2} + \frac{z^2}{c^2 + u^2}$$

it may be proved that

$$x^2 = \frac{(a^2 + u_1^2)(a^2 + u_2^2)(a^2 + u_3^2)}{(b^2 - a^2)(c^2 - a^2)},$$

and y^2, z^2 may be written down by cyclical changes.

If for brevity we write

$$A_n^2 = u_n^2 + a^2, \; B_n^2 = u_n^2 + b^2, \; C_n^2 = u_n^2 + c^2, \; (n = 1, 2, 3),$$

LAPLACE's equation becomes

$$(u_2^2 - u_3^2)\left(A_1 B_1 C_1 \frac{d}{u_1 du_1}\right)^2 V_i + (u_3^2 - u_1^2)\left(A_2 B_2 C_2 \frac{d}{u_2 du_2}\right)^2 V_i$$
$$+ (u_1^2 - u_2^2)\left(A_3 B_3 C_3 \frac{d}{u_3 du_3}\right)^2 V_i = 0.$$

The solution is

$$V_i = U_1 U_2 U_3,$$

where U_1, U_2, U_3 are functions of u_1, u_2, u_3 respectively, and satisfy

$$\left(A_1 B_1 C_1 \frac{d}{u_1 du_1}\right)^2 U_1 = [i(i + 1)u_1^2 + \kappa^2]U_1,$$

and two other equations with suffixes 2 and 3, involving the same κ, a constant, and the same i, a positive integer.

If a, b, c are in ascending order of magnitude we may suppose u_1^2 to lie between $-a^2$ and ∞, u_2^2 between $-c^2$ and $-b^2$, and u_3^2 between $-b^2$ and $-a^2$.

If s_1, s_2, s_3 denote the three orthogonal arcs formed by the intersections of the three orthogonal quadrics,

and two other equations found by cyclical changes of suffixes.

§ 2. *Notation; limits of β so as to represent all Ellipsoids.*

I now change the notation, and let the three roots be defined thus :—

$$u_1^2 = k^2 \nu^2,$$
$$u_2^2 = k^2 \mu^2,$$
$$u_3^2 = k^2 \frac{1 - \beta \cos 2\phi}{1 - \beta},$$

where ν ranges from ∞ to 0, μ between ± 1, ϕ between 0 and 2π.

Let the axes of the fundamental ellipsoid of reference be

The ellipsoid defined by ν has its three axes a, b, c given by

$$a^2 = k^2\left(\nu^2 - \frac{1+\beta}{1-\beta}\right), \quad b^2 = k^2(\nu^2 - 1), \quad c^2 = k^2\nu^2, \ (a < b < c).$$

This mode of defining the axis is such as to indicate the relationship to the prolate ellipsoid $a = b < c$. But another hypothesis may be made which will bring the axes into relationship with those of the oblate ellipsoid $a = b > c$; for if we take a new k, numerically equal to the old one but imaginary, and replace ν^2 by $-\zeta^2$, we have

$$a^2 = k^2\left(\zeta^2 + \frac{1+\beta}{1-\beta}\right), \quad \beta^2 = k^2(\zeta^2 + 1), \quad c^2 = k^2\zeta^2, \ (a > b > c).$$

If β be made to range from 0 to ∞, all possible ellipsoids are comprised in either of these types. It will, however, now be shown that, by a proper choice of type, all ellipsoids may be included with the range of β from 0 to $\frac{1}{3}$.

Let us suppose the axes to be expressed in three forms, as follows :—

$$\text{(3.)}$$

$$
\begin{aligned}
&&&= k_2^2\nu_2^2, \\
b^2 \ &: k^2(\nu^2 - 1) &= k_1^2(\zeta_1^2 + 1) &= k_2^2\left(\nu_2^2 - \frac{1+\beta_2}{1-\beta_2}\right), \\
c^2 &= k^2\nu^2 &= k_1^2\left(\zeta_1^2 + \frac{1+\beta_1}{1-\beta_1}\right) &= k_2^2(\nu_2^2 - 1).
\end{aligned}
$$

Then we have

$$b^2 - a^2 = \frac{2k^2\beta}{1-\beta} = \quad k_1^2 \quad - - \quad k_2^2\frac{1+\beta_2}{1-\beta_2}$$

$$c^2 - b^2 = \quad k^2 \quad = \frac{2k_1^2\beta_1}{1-\beta_1} = \frac{2k_2^2\beta_2}{1-\beta_2}.$$

Therefore

$$\frac{b^2 - a^2}{c^2 - b^2} = \frac{2\beta}{1-\beta} = \frac{1-\beta_1}{2\beta_1} = -\frac{1+\beta_2}{2\beta_2}$$

$$\frac{c^2 - a^2}{c^2 - b^2} = \frac{1+\beta}{1-\beta} = \frac{1+\beta_1}{2\beta_1} = -\frac{1-\beta_2}{2\beta_2}$$

And

$$\frac{b^2 - a^2}{2c^2 - a^2 - b^2} = \quad \beta = \frac{1-\beta_1}{1+3\beta_1} = \frac{1+\beta_2}{1-3\beta_2}$$

Now let β increase from 0 to ∞.

As β passes from 0 to $\frac{1}{3}$, form (1) is appropriate.

As β passes from $\frac{1}{3}$ to 1, β_1 decreases from $\frac{1}{3}$ to 0, so that form (2) is appropriate.

Lastly, as β passes from 1 to ∞, β_2 increases from 0 to $\frac{1}{3}$, so that form (3) is appropriate.

But we might equally well have written forms (1) and (3) so as to involve ζ, and form (2) so as to involve ν, and it follows that all possible ellipsoids are comprised in the range of β from 0 to $\frac{1}{3}$, provided that the type be appropriately chosen.

The developments in this paper are made in powers of β. It will, therefore, be well to show that there is a class of ellipsoids, analogous to ellipsoids of revolution, which might form the basis of developments similar to those carried out below.

Ellipsoids of revolution are defined by the condition

$$a^2 - c^2 = b^2 - c^2, \text{ or } a^2 = b^2.$$

In the class to which I refer

$$a^2 - c^2 = c^2 - b^2, \text{ or } c^2 = \tfrac{1}{2}(a^2 + b^2).$$

Ellipsoids of this kind are given by $\beta = \beta_1 = -\beta_2 = \frac{1}{3}$: for in this case $b^2 = \frac{1}{2}(a^2 + c^2)$. They are also given by

$$\beta = \infty, \ -\beta_1 = \beta_2 = \tfrac{1}{3}; \text{ for then } c^2 = \tfrac{1}{2}(a^2 + b^2).$$

Hence if we only allow β to range from 0 to $\frac{1}{3}$, $\beta = 0$ corresponds with ellipsoids of revolution, to which spheroidal harmonic analysis is applicable; and $\beta = \frac{1}{3}$ corresponds with this new class for which the corresponding analysis has not yet been worked out.

We shall see below that the solid harmonic for this case where $\beta = \frac{1}{3}$ will be of the form $B(\nu) B(\mu) E(\phi)$, where B and E satisfy the equations

$$(\nu^2 + 1)(\nu^2 - 1)\frac{d^2 B}{d\nu^2} + 2\nu^3 \frac{dB}{d\nu} - i(i+1)\nu^2 B + s^2 B = 0,$$

$$\cos 2\phi \frac{d^2 E}{d\phi^2} - \sin 2\phi \frac{dE}{d\phi} + i(i+1) E \cos 2\phi - s^2 E = 0.$$

I am not clear whether or not it would be advisable to proceed *ab initio* from these equations, but at any rate I shall show hereafter how the B- and E-functions may be determined from the analysis of the present paper with any degree of accuracy desirable.

If it were proposed to use the functions corresponding to $\beta = \frac{1}{3}$ as a basis for the development of general ellipsoidal harmonics, we should have to assume

$$a^2 = k^2\nu'^2, \ b^2 = k^2(\nu'^2 - 1), \ c^2 = k'^2\left(\nu'^2 - \frac{2}{1 - \eta}\right);$$

or else $$a^2 = k'^2\zeta'^2, \ b^2 = k^2(\zeta'^2 + 1), \ c^2 = k'^2\left(\zeta'^2 + \frac{2}{1 - \eta}\right).$$

The developments would then proceed by powers of η.

In order to discover what is the greatest value of η which must be used so as to comprise all ellipsoids, when we proceed from both bases of development, a comparison must be made between this assumption and the previous one.   Suppose in fact that

$$a^2 = k^2\left(\nu^2 - \frac{1 + \beta}{1 - \beta}\right) = k^2\zeta'^2; \ b^2 = k^2(\nu^2 - 1) = k'^2(\zeta'^2 + 1);$$

$$c^2 = k^2\nu^2 = k^2\left(\zeta'^2 + \frac{2}{1 - \eta}\right).$$

Then $$b^2 - a^2 = \frac{2k^2\beta}{1 - \beta} = k'^2; \ c^2 - b^2 = k^2 = k'^2\frac{1 + \eta}{1 - \eta},$$

and therefore $$\frac{2\beta}{1 - \beta} = \frac{1 - \eta}{1 + \eta}, \text{ or } \eta = \frac{1 - 3\beta}{1 + 3\beta}.$$

When η and β are both equally great, they must each equal the positive root of $\beta = \dfrac{1 - 3\beta}{1 + \beta}$.   This root is $\sqrt{5} - 2$ or $\cdot236$.   Thus the greatest values will be

$$\beta = \eta = \frac{1}{\sqrt{5} + 2} = \frac{1}{4\cdot236}.$$

In this case $\eta^3 = \beta^3 = \frac{1}{75}$ very nearly, whereas when $\beta = \frac{1}{3}, \beta^2 = \frac{1}{9}$.   Thus if the developments were to stop with β^2 we should double the accuracy of the result. However, I do not at present propose to carry out the process suggested.

§ 3. *The Differential Equations.*

We now put $u_1^2 = k^2\nu^2, \ u_2^2 = k^2\mu^2, \ u_3^2 = k^2\dfrac{1 - \beta\cos 2\phi}{1 - \beta}$;

$$a^2 = -k^2\frac{1 + \beta}{1 - \beta}, \quad b^2 = -k^2, \quad c^2 = 0;$$

and find from the formulæ of § 1,

$$\left.\begin{aligned}
\frac{x^2}{k^2} &= -\frac{1 - \beta}{1 + \beta}\left(\nu^2 - \frac{1 + \beta}{1 - \beta}\right)\left(\mu^2 - \frac{1 + \beta}{1 - \beta}\right)\cos^2\phi, \\
\frac{y^2}{k^2} &= -(\nu^2 - 1)(\mu^2 - 1)\sin^2\phi, \\
\frac{z^2}{k^2} &= \nu^2\mu^2\frac{1 - \beta\cos 2\phi}{1 + \beta}
\end{aligned}\right\} \quad . \quad . \ (1).$$

It will be observed that y is independent of β, and that it has the same form as in spheroidal harmonic analysis when β vanishes. Since μ^2 is less than 1 and ν^2 greater than $\dfrac{1+\beta}{1-\beta}$, x and y are real.

In all the earlier portion of this paper I always write $\mu^2 - 1$ and not $1 - \mu^2$, so as to maintain perfect symmetry with respect to ν and μ.

We now have

$$A_1{}^2 = k^2\left(\nu^2 - \frac{1+\beta}{1-\beta}\right), \qquad B_1{}^2 = k^2(\nu^2 - 1), \qquad C_1{}^2 = k^2\nu^2\,;$$

$$A_2{}^2 = k^2\left(\mu^2 - \frac{1+\beta}{1-\beta}\right), \qquad B_2{}^2 = k^2(\mu^2 - 1), \qquad C_2{}^2 = k^2\mu^2\,;$$

$$A_3{}^2 = \frac{-2k^2\beta\cos^2\phi}{1-\beta}, \qquad B_3{}^2 = \frac{2k^2\beta\sin^2\phi}{1-\beta}, \qquad C_3{}^2 = k^2\frac{1-\beta\cos2\phi}{1-\beta}.$$

Let us denote the differential operators involved in our equations, thus:—

$$D_1 = (1-\beta)^{\frac12}\frac{A_1 B_1 C_1}{k u_1}\frac{d}{du_1}, \qquad D_2 = (1-\beta)^{\frac12}\frac{A_2 B_2 C_2}{k u_2}\frac{d}{du_2},$$

$$D_3 = -\sqrt{-1}\cdot(1-\beta)^{\frac12}\frac{A_3 B_3 C_3}{k u_3}\frac{d}{du_3}.$$

Then

$$\left.\begin{aligned}
D_1 &= (1-\beta)^{\frac12}\left(\nu^2 - \frac{1+\beta}{1-\beta}\right)^{\frac12}(\nu^2-1)^{\frac12}\frac{d}{d\nu},\\[4pt]
D_2 &= (1-\beta)^{\frac12}\left(\mu^2 - \frac{1+\beta}{1-\beta}\right)^{\frac12}(\mu^2-1)^{\frac12}\frac{d}{d\mu},\\[4pt]
D_3 &= (1-\beta\cos2\phi)^{\frac12}\frac{d}{d\phi}
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (2).$$

$$A_1 B_1 C_1\frac{d}{u_1 du_1} = \frac{k D_1}{(1-\beta)^{\frac12}}, \qquad A_2 B_2 C_2\frac{d}{u_2 du_2} = \frac{k D_2}{(1-\beta)^{\frac12}},$$

$$A_3 B_3 C_3\frac{d}{u_3 du_3} = \sqrt{-1}\cdot\frac{k D_3}{(1-\beta)^{\frac12}}.$$

Hence our differential equations are

$$\frac{D_1{}^2 U_1}{1-\beta} = \left[i(i+1)\nu^2 + \frac{\kappa^2}{k^2}\right]U_1,\ \text{a similar equation with suffix 2, and}$$

$$\frac{-D_3{}^2 U_3}{1-\beta} = \left[i(i+1)\frac{1-\beta\cos2\phi}{1-\beta} + \frac{\kappa^2}{k^2}\right]U_3.$$

Let us replace κ^2 by another constant such that

$$\left(i(i+1)\nu^2 + \frac{\kappa^2}{k^2}\right)(1-\beta) = i(i+1)[\nu^2(1-\beta)-1] + s^2 - \beta\sigma,$$

so that

$$\frac{\kappa^2}{k^2} = -\frac{i(i+1)-s^2+\beta\sigma}{1-\beta}$$

In this formula s is a constant integer and σ a constant to be determined.

Our equations are now

$$[D_1{}^2 - i(i+1)[\nu^2(1-\beta)-1] - s^2 + \beta\sigma]\,U_1 = 0,$$

a similar equation for μ

and

$$[D_3{}^2 - i(i+1)\beta\cos 2\phi + s^2 - \beta\sigma]\,U_3 = 0 \qquad \left.\begin{array}{c} \\ \\ \end{array}\right\} \quad . \quad (3).$$

And LAPLACE's equation is

$$\left[\left(\mu^2 - \frac{1-\beta\cos 2\phi}{1-\beta}\right)D_1{}^2 + \left(\frac{1-\beta\cos 2\phi}{1-\beta} - \nu^2\right)D_2{}^2 - (\nu^2-\mu^2)\,D_3{}^2\right]U_1U_2U_3 = 0.$$

$$. \quad . \quad . \quad . \quad (4).$$

LAPLACE's operator ∇^2 is equal to the differential operator in (4), divided by

$$- k^2(\nu^2-\mu^2)\left(\nu^2 - \frac{1-\beta\cos 2\phi}{1-\beta}\right)\left(\frac{1-\beta\cos 2\phi}{1-\beta} - \mu^2\right).$$

It is well known that in spheroidal harmonic analysis there are two kinds of functions of ν and μ which satisfy the differential equation, and they are usually denoted $P_i{}^s$, $Q_i{}^s$. The Q-functions of the variable μ have no significance, so that virtually there are P- and Q-functions of ν, but only P-functions of μ. The like is true in the present case, however, with the additional complication that each of the functions may assume one of two alternative forms. I adopt a parallel notation and write for U_1 and U_2 either $\mathfrak{P}_i{}^s$, $\mathfrak{Q}_i{}^s$, or $P_i{}^s$, $Q_i{}^s$, as the case may be. Since ν and μ enter in the first two equations in exactly the same way, we need only consider one of them, and we may usually write simply (for example) $\mathfrak{P}_i{}^s$ where the full notation would be $\mathfrak{P}_i{}^s(\nu$ or $\mu)$. In the early part of the investigation I shall only refer to the P-functions, and the Q-functions will be considered later.

In spheroidal harmonic analysis the third function is a cosine or sine of $s\phi$. So here also we find functions of two kinds associated with cosines and sines, which I shall denote $\mathfrak{C}_i{}^s$, $\mathfrak{S}_i{}^s$, $C_i{}^s$, $S_i{}^s$, the variable ϕ being understood.

Throughout the greater part of this paper the functions will be of degree denoted by i, and it seems useless to print the subscript i hundreds of times. I shall accordingly drop the subscript i except where it shall be necessary or advisable to retain it; for example, $\mathfrak{P}^s$ will be the abridged notation for $\mathfrak{P}_i{}^s(\nu)$.

The operators involved in the differential equations (3) will occur so frequently that an abridged notation seems justifiable. I therefore write

$$\psi_s = D_1{}^2 - i(i+1)[\nu^2(1-\beta)-1] - s^2 + \beta\sigma,$$

$$\chi_s = D_3{}^2 - i(i+1)\beta\cos 2\phi + s^2 - \beta\sigma,$$

where

$$D_1 = (1-\beta)^{\frac{1}{2}}\left(\nu^2 - \frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}(\nu^2-1)^{\frac{1}{2}}\frac{d}{d\nu}, \qquad \left.\begin{array}{c} \\ \\ \end{array}\right| \quad . \quad . \quad . \quad (5).$$

$$D_3 = (1-\beta\cos 2\phi)^{\frac{1}{2}}\frac{d}{d\phi}$$

The equations are then

$$\psi_s \,(\mathbf{P}' \text{ or } \mathbf{P}^s) = 0 \,,$$
$$\chi_s \,(\mathbf{C}' \text{ or } \mathbf{S}' \text{ or } \mathbf{C}^s \text{ or } \mathbf{S}^s) = 0 \quad\Big\}\quad . \quad . \quad . \quad . \quad (5).$$

§ 4. *The Forms of the Functions.*

It is well known that the function U_i is a linear function of u_1 of degree i made up in one of the eight following ways :—

1. When i is even, a linear function of $u_1{}^2$ of degree $\frac{1}{2}i$.
2, 3, 4. When i is odd, a linear function of $u_1{}^2$ of degree $\frac{1}{2}(i-1)$, multiplied by A_1, or B_1, or C_1.
5, 6, 7. When i is even, a linear function of $u_1{}^2$ of degree $\frac{1}{2}(i-2)$, multiplied by $B_1 C_1$, or $C_1 A_1$, or $A_1 B_1$.
8. When i is odd, a linear function of $u_1{}^2$ of degree $\frac{1}{2}(i-3)$, multiplied by $A_1 B_1 C_1$.

These eight classes may be conveniently specified by the initials O, A, B, C, BC, CA, AB, ABC, but it is better to rearrange them according as they are associated with the evenness or oddness of i and s, and with the cosine or sine functions. This new grouping may be defined by a shorthand notation involving the initials E, O and C or S, which shall denote successively the evenness or oddness of i and s, and cosine or sine.

We shall see below that this arrangement is as follows : —

> O or EEC ; i even, s even, cosine.
>
> AB or EES ; i even, s even, sine.
>
> A or OOC ; i odd, s odd, cosine.
>
> B or OOS ; i odd, s odd, sine.
>
> C or OEC ; i odd, s even, cosine.
>
> ABC or OES ; i odd, s even, sine.
>
> CA or EOC ; i even, s odd, cosine.
>
> CB or EOS ; i even, s odd, sine.

Since the several functions are linear in $u_1{}^2$, they are in the new notation functions of ν^2 or μ^2, or of $\nu^2 - 1$ and $\mu^2 - 1$.

Hence $\mathbf{P}'(\nu)$ and $\mathbf{P}^s(\nu)$ involve linear functions of $\nu^2 - 1$ of various degrees multiplied by various factors ; and the same is true of the functions of μ.

In the case of the third root the linear function of powers of $\cos 2\phi$ may be replaced by a series of cosines of even multiples of ϕ. Further, in forming the $\mathbf{C}$, $\mathbf{S}$, $\mathbf{C}$, $\mathbf{S}$ functions we may regard A_3 as being $\cos \phi$, B_3 as $\sin \phi$, and C_3 as $(1 - \beta \cos 2\phi)^{\frac{1}{2}}$,

since this only amounts to dropping constant factors which may be deemed to be included in the, as yet, undetermined coefficients of the several series.

I will now consider in detail the forms of the several P-functions of ν (those for μ following by symmetry), and at the same time indicate more precisely the nature of the notation adopted.

In the following series, indicated by Σ, the variable t is supposed *to proceed from the lower to the upper limit by 2 at a time.* The reader will be able to perceive the manner of the formation of the functions when he bears in mind that

$$A_1 = k\left(\nu^2 - \frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}, \quad B_1 = k(\nu^2-1)^{\frac{1}{2}}, \quad C_1 = k\nu.$$

Type O or EEC , $\quad \mathfrak{P}^t = \overset{.}{\underset{0}{\Sigma}}\alpha_t(\nu^2-1)^{\frac{1}{2}t}.$

Type AB or EES ; $\quad P^t = \overset{.}{\underset{2}{\Sigma}}\alpha_t(\nu^2-1)^{\frac{1}{2}(t-2)}\left(\nu^2-\frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}.$

Type A or OOC ; $\quad P^t = \overset{.}{\underset{1}{\Sigma}}\alpha_t(\nu^2-1)^{\frac{1}{2}(t-1)}\left(\nu^2-\frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}.$

Type B or OOS ; $\quad \mathfrak{P}^t = \overset{.}{\underset{1}{\Sigma}}\alpha_t(\nu^2-1)^{\frac{1}{2}t}.$

Type C or OEC ; $\quad \mathfrak{P}^t = \overset{4}{\underset{1}{\Sigma}}\alpha_t\nu(\nu^2-1)^{\frac{1}{2}(t-1)}.$

Type ABC or OES ; $\quad P^t = \overset{.}{\underset{3}{\Sigma}}\alpha_t\nu(\nu^2-1)^{\frac{1}{2}(t-3)}\left(\nu^2-\frac{1+\beta}{1-\beta}\right).$

Type CA or EOC ; $\quad P^t = \overset{.}{\underset{2}{\Sigma}}\alpha_t\nu(\nu^2-1)^{\frac{1}{2}(t-2)}\left(\nu^2-\frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}.$

Type CB or EOS ; $\quad \mathfrak{P}^t = \overset{4}{\underset{2}{\Sigma}}\alpha_t\nu(\nu^2-1)^{\frac{1}{2}(t-2)}.$

Observe that P is always associated with $\left(\nu^2-\frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}$, and that, each form being repeated twice, there are two forms of function of each kind. Moreover, a cosine and a sine function are always associated with different kinds. It is obvious that the $\mathfrak{P}$-functions are expressible in terms of the ordinary P-functions of spherical harmonic analysis, and that if we take out the factor $\left(\frac{\nu^2-\frac{1+\beta}{1-\beta}}{\nu^2-1}\right)^{\frac{1}{2}}$ the P-functions are similarly expressible. This factor will occur so frequently that I write

$$\Omega(\nu) = \left(\frac{\nu^2-\frac{1+\beta}{1-\beta}}{\nu^2-1}\right)^{\frac{1}{2}},$$

and as elsewhere commonly put Ω to denote $\Omega(\nu)$.

We assume then the following forms for the functions :—

For EEC, OEC, OOS, EOS

$$\mathfrak{P}^i = q_s \mathrm{P}^i + \Sigma \beta^n q_{s-2n} \mathrm{P}^{i-2n} + \Sigma \beta^n q_{s+2n} \mathrm{P}^{i+2n}.$$

For EES, OES, OOC, EOC

$$\mathrm{P}^i = \Omega \{ q'_s \mathrm{P}^i + \Sigma \beta^n q'_{s-2n} \mathrm{P}^{i-2n} + \Sigma \beta^n q'_{s+2n} \mathrm{P}^{i+2n} \}$$

$$\left. \right\} \quad . \quad . \quad . \quad (6).$$

In these series n proceeds by intervals of one at a time, beginning from a lower limit of unity. In both forms the upper limit of the first Σ is $\frac{1}{2}s$ or $\frac{1}{2}(s-1)$ according as s is even or odd; and the upper limit of the second Σ is $\frac{1}{2}(i-s)$ or $\frac{1}{2}(i-s-1)$ according as i and s agree or do not agree in evenness or in oddness.

The factor Ω contains $(\nu^2 - 1)^{\frac{1}{2}}$ in the denominator, but P^i does not become infinite when $\nu = \pm 1$, because when s is not zero P^i is divisible by $(\nu^2 - 1)^{\frac{1}{2}}$ and we shall see that q'_0 is zero.* When s is zero there is no function of the P type.

It may be noted that the limits of the series are such that neither q nor q' can ever have a negative suffix.

We shall ultimately make q_s and q'_s equal to unity, and this will be justifiable because there must be one arbitrary constant.

We have now to consider the forms of the cosine and sine functions. They may be derived at once from the preceding results, for we have only to read $(\nu^2 - 1)^{\frac{1}{2}i}$ as $\cos t\phi$ where t is even; $(\nu^2 - 1)^{\frac{1}{2}}$ as $\sin \phi$, $\left(\nu^2 - \dfrac{1+\beta}{1-\beta} \right)^{\frac{1}{2}}$ as $\cos \phi$, and ν as $(1 - \beta \cos 2\phi)^{\frac{1}{2}}$.

The factor $(1 - \beta \cos 2\phi)^{\frac{1}{2}}$ will occur frequently, and I write

$$\Phi(\phi) = (1 - \beta \cos 2\phi)^{\frac{1}{2}},$$

and as elsewhere I commonly write Φ to denote $\Phi(\phi)$.

The following are the results :—

Type O or EEC ; $\mathfrak{C}^i = \overset{i}{\underset{0}{\Sigma}} \gamma_t \cos t\phi$.

Type AB or EES ; $\mathfrak{S}^i = \overset{i}{\underset{2}{\Sigma}} \gamma_t \sin t\phi$.

It is clear that we may equally well regard the lower limit in the latter as zero.

Type A, or OOC ; each term is of type $\cos(t-1)\phi \cos\phi$ or $\cos(t-2)\phi + \cos t\phi$.

Hence $\mathfrak{C}^i = \overset{i}{\underset{1}{\Sigma}} \gamma_t \cos t\phi$.

---

* This also follows from the fact that the series for P^i begins with $\Omega a_s(\nu^2 - 1)$ in the case of EES, and with $\Omega a_s \nu (\nu^2 - 1)$ in the case of OES. Thus in the former case there is no term Ωa_0 and in the latter no term $\Omega a_1 \nu$

Type B, or OOS ; since we now have $\cos(t-1)\phi\sin\phi$,

$$\mathbf{S}^{\prime} = \overset{i}{\underset{1}{\Sigma}}\,\gamma_t\sin t\phi\,.$$

Type C. or OEC ; $\mathbf{C}^{\prime} = \Phi\overset{i}{\underset{1}{\Sigma}}\,\gamma_t\cos(t-1)\phi\,.$

Type ABC, or OES ; each term is of type $\Phi\cos(t-1)\phi\sin\phi\cos\phi$, which gives $[\sin(t+1)\phi - \sin(t-3)\phi]\,\Phi$. Hence

$$\mathbf{S}^{\prime} = \Phi\overset{i}{\underset{2}{\Sigma}}\,\gamma_t\sin(t-1)\phi\,.$$

It is clear that we may equally well regard the lower limit as unity.

Type CA, or EOC ; each term is of type $\Phi\cos(t-2)\phi\cos\phi$.   Hence

$$\mathbf{C}^{\prime} = \Phi\overset{i}{\underset{2}{\Sigma}}\,\gamma_t\cos(t-1)\phi\,.$$

Type CB, or EOS ; each term is of type $\Phi\cos(t-2)\phi\sin\phi$.   Hence

$$\mathbf{S}^{\prime} = \Phi\overset{i}{\underset{2}{\Sigma}}\,\gamma_t\sin(t-1)\phi\,.$$

When i and s agree as to evenness or oddness we have the forms independent of Φ, when they differ in this respect the factor Φ occurs.

Therefore (in alternative form) for EEC, EES, OOC, OOS

$$\left.\begin{array}{l}\begin{Bmatrix}\mathbf{C}^{\prime}\\\mathbf{S}^{\prime}\end{Bmatrix} = p_s\Big\{{\cos\atop\sin}\,s\phi + \Sigma\beta^n p_{s-2n}\Big\{{\cos\atop\sin}\,(s-2n)\,\phi + \Sigma\beta^n p_{s+2n}\Big\{{\cos\atop\sin}\,(s+2n)\,\phi\\[6pt]\text{and for OEC. OES, EOC, EOS}\\[4pt]\begin{Bmatrix}\mathbf{C}^{\prime}\\\mathbf{S}^{\prime}\end{Bmatrix} = \Phi\Big[p^{\prime}_s\Big\{{\cos\atop\sin}\,s\phi + \Sigma\beta^n p^{\prime}_{s-2n}\Big\{{\cos\atop\sin}\,(s-2n)\,\phi + \Sigma\beta^n p^{\prime}_{s+2n}\Big\{{\cos\atop\sin}\,(s+2n)\,\phi\Big]\end{array}\right\} \quad (7).$$

In these series n proceeds by intervals of one at a time, beginning with unity.   In both forms the upper limit of the first Σ is $\frac{1}{2}s$ or $\frac{1}{2}(s-1)$ according as s is even or odd. In the first form the upper limit of the second Σ is $\frac{1}{2}(i-s)$, and in the second form it is $\frac{1}{2}(i-s-1)$.

We shall ultimately put p_s and $p^{\prime}_s$, which may be regarded as arbitrary constants, equal to unity.

§ 5. *Preparation for determination of the Functions.*

In order to determine the coefficients q, $q^{\prime}$, p, $p^{\prime}$ and σ, we have to substitute these assumed forms in the differential equations.

Where the functions involve Ω and Φ as factors, the forms already given for the

differential equations are perhaps the most convenient, but in the other cases a reduction seems desirable.

By considering the forms of D_1 and D_3 in (3) it is easy to show that

$$\psi_i = \left[(\nu^2 - 1)\frac{d}{d\nu}\right]^2 - i(i+1)(\nu^2 - 1) - s^2$$
$$- \beta\left[(\nu^2 - 1)(\nu^2 + 1)\frac{d^2}{d\nu^2} + 2\nu^3\frac{d}{d\nu} - i(i+1)\nu^2 - \sigma\right] \quad . \quad (8),$$

$$\chi_i = \frac{d^2}{d\phi^2} + s^2 - \beta\left[\cos 2\phi\frac{d^2}{d\phi^2} - \sin 2\phi\frac{d}{d\phi} + i(i+1)\cos 2\phi + \sigma\right] \quad . \quad (9).$$

By making β vanish we reduce these operators to the forms appropriate to spheroidal harmonic analysis. By making β infinite we obtain the differential equations specified in § 2 as appropriate to ellipsoids of the class $c^2 = \frac{1}{2}(a^2 + b^2)$.

It is now necessary to perform the operation ψ_i on typical terms P^i and ΩP^i, and χ_i on typical terms $\left\{\begin{matrix}\cos\\\sin\end{matrix}\right\} t\phi$ and $\Phi\left\{\begin{matrix}\cos\\\sin\end{matrix}\right\} t\phi$.

(a.) To find $\psi_i(P^i)$.

The form (8) for ψ_i is here convenient.

It is clear that

$$\left\{\left[(\nu^2 - 1)\frac{d}{d\nu}\right]^2 - i(i+1)(\nu^2 - 1) - s^2\right\} P^i = (t^2 - s^2) P^i,$$

because P^i is the solution of the differential equation found by erasing the term $- s^2 P^i$ from each side.

Again we have from the same differential equation

$$(\nu^2 - 1)\frac{d^2}{d\nu^2} P^i = - 2\nu\frac{dP^i}{d\nu} + i(i+1) P^i + \frac{t^2}{\nu^2 - 1} P^i.$$

It may be noted in passing that this is equally true when the subject of operation is Q^i, the function of the other form.

Therefore

$$\left[(\nu^2 - 1)(\nu^2 + 1)\frac{d^2}{d\nu^2} + 2\nu^3\frac{d}{d\nu} - i(i+1)\nu^2 - \sigma\right] P^i$$
$$= \left[- 2\nu\frac{d}{d\nu} + i(i+1) + t^2\frac{\nu^2 + 1}{\nu^2 - 1} - \sigma\right] P^i.$$

Hence

$$\psi_i(P^i) = (t^2 - s^2) P^i - \beta\left\{- 2\nu\frac{d}{d\nu} + i(i+1) + t^2\frac{\nu^2 + 1}{\nu^2 - 1} - \sigma\right\} P^i.$$

We have now to eliminate $\nu \dfrac{dP^\mu}{d\nu}$ and $\dfrac{\nu^2 + 1}{\nu^2 - 1} P^\mu$.

It is known that $\qquad P = \dfrac{1}{2^i\,i\,!}\left(\dfrac{d}{d\nu}\right)^i (\nu^2 - 1)^i,$

and $\qquad\qquad P^\mu = (\nu^2 - 1)^{\mu}\left(\dfrac{d}{d\nu}\right)^t P.$

The differential equation satisfied by P^μ involves t in the form t^2. Hence $(\nu^2 - 1)^{-\mu}\left(\dfrac{d}{d\nu}\right)^{-t} P$ can only differ from P^μ by a constant factor. In order to find that factor suppose ν to be infinitely large ;

then $\qquad\qquad P = \dfrac{2i\,!}{2^i\,(i\,!)^2}\,\nu^i,$

and

Also $\qquad (\nu^2 - 1)^{-\mu}\left(\dfrac{d}{d\nu}\right)^{-t} P = \nu^{-t}\dfrac{2i\,!}{2^i\,(i\,!)^2}\cdot\dfrac{i\,!}{i + t\,!}\,\nu^{i+t} = \dfrac{2i\,!}{2^i\,i\,!}\cdot\dfrac{\nu^i}{i + t\,!}\cdot$

Therefore the factor is $\dfrac{i + t\,!}{i - t\,!}$, and

$$P^\mu = (\nu^2 - 1)^{\mu}\left(\dfrac{d}{d\nu}\right)^t P = \dfrac{i + t\,!}{i - t\,!}(\nu^2 - 1)^{-\mu}\left(\dfrac{d}{d\nu}\right)^{-t} P.$$

It will be convenient to pause here and obtain the corresponding formulæ for the Q-functions. Various writers have adopted various conventions as to the factors involved in these functions. I write

$$Q = P\int_\nu^\infty \dfrac{d\nu}{(\nu^2 - 1)(P)^2},$$

and $\qquad\qquad Q^\mu = (\nu^2 - 1)^{\mu}\left(\dfrac{d}{d\nu}\right)^t Q.$

As in the case of P^μ we may change the sign of t, if we introduce a constant factor, and this may be found by making ν infinitely great. In that case it is easy to show that

$$Q = \dfrac{2^i\,(i\,!)^2}{2i + 1\,!}\cdot\dfrac{1}{\nu^{i+1}}\cdot$$

By performing $\left(\dfrac{d}{d\nu}\right)^t$ and $\left(\dfrac{d}{d\nu}\right)^{-t}$ on Q it follows that the constant factor is the same as before, and that the alternative forms for Q^μ are exactly the same as for P^μ.

Hence the transformations which follow for the P-functions are equally applicable to the Q-functions.

If we differentiate P^i in its two forms we find

$$\frac{dP^i}{d\nu} = t\nu(\nu^2-1)^{\frac{1}{2}(t-2)}\left(\frac{d}{d\nu}\right)^t P + (\nu^2-1)^{\frac{1}{2}t}\left(\frac{d}{d\nu}\right)^{t+1} P = \frac{t\nu}{\nu^2-1}P^i + \frac{P^{i+1}}{(\nu^2-1)^{\frac{1}{2}}}.$$

And $$\frac{dP^i}{d\nu} = \frac{i+t!}{i-t!}\left\{ -t\nu(\nu^2-1)^{-\frac{1}{2}(t+2)}\left(\frac{d}{d\nu}\right)^{-t} P + (\nu^2-1)^{-\frac{1}{2}t}\left(\frac{d}{d\nu}\right)^{-(t-1)} P \right\}$$

$$= -\frac{t\nu}{\nu^2-1}P^i + \frac{i+t!}{i-t!}\cdot\frac{i-t+1!}{i+t-1!}\cdot\frac{P^{i-1}}{(\nu^2-1)^{\frac{1}{2}}}.$$

I now write

$$\{i, t\} = (i+t)(i-t+1) = i(i+1) - t(t-1).$$

It is clear that

$$\{i, -t\} = \{i, t+1\}, \text{ and } \{i, 0\} = \{i, 1\} = i(i+1).$$

Now since $\frac{i+t!}{i-t!}\frac{i-t+1!}{i+t-1!} = \{i, t\}$, by taking the sum and difference of the two forms of $\frac{dP^i}{d\nu}$, we have

$$\frac{dP^i}{d\nu} = \frac{1}{2(\nu^2-1)^{\frac{1}{2}}}[P^{i+1} + \{i, t\}P^{i-1}],$$

$$\frac{\nu P^i}{(\nu^2-1)^{\frac{1}{2}}} = \frac{1}{2}\left[-\frac{1}{t}P^{i+1} + \frac{\{i, t\}}{t}P^{i-1}\right] \qquad . . . . . (10).$$

It is easy to verify, by means of the relationship $P^{-t} = \frac{i-t!}{i+t!}P^t$, that these equations are true when t is negative. They are also true when $t = 0$, although the second equation then becomes nugatory.

Multiply the first of (10) by ν and the second by $\frac{\nu}{(\nu^2-1)^{\frac{1}{2}}}$, and apply them a second time.

Then since $$\frac{\{i, t+1\}}{t+1} - \frac{\{i, t\}}{t-1} = -\frac{2i(i+1)}{t^2-1},$$

$$\frac{\{i, t+1\}}{t(t+1)} + \frac{\{i, t\}}{t(t-1)} = 2\left(\frac{i(i+1)}{t^2-1} - 1\right),$$

$$\frac{\nu^2}{\nu^2-1} = \frac{1}{\nu^2-1}$$

$$2\nu\frac{dP^i}{d\nu} = \frac{1}{2}\left[-\frac{P^{i+2}}{t+1} - \frac{2i(i+1)}{t^2-1}P^i + \frac{\{i, t\}\{i, t-1\}}{t-1}P^{i-2}\right],$$

$$\frac{\nu^2+1}{\nu^2-1}P^i = \frac{1}{2}\left[\frac{P^{i+2}}{t(t+1)} - \frac{2i(i+1)}{t^2-1}P^i + \frac{\{i, t\}\{i, t-1\}}{t(t-1)}P^{i-2}\right] \qquad . . (11).$$

These equations are always true although for $t = \pm 1$ and 0 they become nugatory.

Then

$$- 2\nu \frac{d\mathrm{P}^\mu}{d\nu} + t^2 \frac{\nu^2 + 1}{\nu^2 - 1} \mathrm{P}^\mu = \tfrac{1}{2} \left[\mathrm{P}^{\mu+2} + \{i, t\}\{i, t-1\}\mathrm{P}^{\mu-2} \right] - i(i+1)\mathrm{P}^\mu.$$

Hence

$$\psi_s(\mathrm{P}^\mu) = - \tfrac{1}{2}\beta \left[\frac{2(\nu^2 - t^2)}{\beta} \mathrm{P}^\mu + \mathrm{P}^{\mu+2} - 2\sigma\mathrm{P}^\mu + \{i, t\}\{i, t-1\}\mathrm{P}^{\mu-2} \right]. \quad (12)$$

(β.) *To find* $\psi_s(\Omega\,\mathrm{P}^\mu)$.

It is now best to use ψ_s in the form (5), where D_1 is defined by (2).

Now

$$\mathrm{D}_1(\Omega\mathrm{P}^\mu) = \frac{1}{(1 - \beta)^{\frac{1}{2}}} \left\{ [(\nu^2 - 1)(1 - \beta) - 2\beta] \frac{d\mathrm{P}^\mu}{d\nu} + \frac{2\beta\nu}{\nu^2 - 1} \mathrm{P}^\mu \right\},$$

and

$$\mathrm{D}_1{}^2(\Omega\mathrm{P}^\mu) = (\nu^2 - 1)\,\Omega \left\{ [(\nu^2 - 1)(1 - \beta) - 2\beta] \frac{d^2\mathrm{P}^\mu}{d\nu^2} \right.$$
$$\left. + \left(1 - \beta + \frac{\beta}{\nu^2 - 1}\right) 2\nu \frac{d\mathrm{P}^\mu}{d\nu} - \frac{2\beta\mathrm{P}^\mu}{\nu^2 - 1}\left(\frac{\nu^2 + 1}{\nu^2 - 1}\right) \right\}.$$

The latter terms of ψ_s contribute

$$\Omega \left\{ - i(i+1)[(\nu^2 - 1)(1 - \beta) - \beta]\mathrm{P}^\mu - (\nu^2 - \beta\sigma)\mathrm{P}^\mu \right\}.$$

Therefore

$$\psi_s(\Omega\mathrm{P}^\mu) = \Omega \left\{ (\nu^2 - 1)[(\nu^2 - 1)(1 - \beta) - 2\beta] \frac{d^2\mathrm{P}^\mu}{d\nu^2} \right.$$
$$+ [(\nu^2 - 1)(1 - \beta) + \beta]\, 2\nu \frac{d\mathrm{P}^\mu}{d\nu} - i(i+1)(1 - \beta)(\nu^2 - 1)\mathrm{P}^\mu$$
$$\left. - \nu^2\mathrm{P}^\mu + \beta i(i+1)\mathrm{P}^\mu + \beta\sigma\mathrm{P}^\mu - 2\beta\mathrm{P}^\mu \frac{\nu^2 + 1}{\nu^2 - 1} \right\}.$$

But $(\nu^2 - 1)\dfrac{d^2\mathrm{P}^\mu}{d\nu^2} = - 2\nu \dfrac{d\mathrm{P}^\mu}{d\nu} + i(i+1)\mathrm{P}^\mu + \dfrac{t^2}{\nu^2 - 1}\mathrm{P}^\mu$, and we find on reduction that

$$\psi_s(\Omega\mathrm{P}^\mu) = \Omega\mathrm{P}^\mu(t^2 - \nu^2) - \beta\Omega \left[- 6\nu \frac{d\mathrm{P}^\mu}{d\nu} + i(i+1)\mathrm{P}^\mu + \frac{\nu^2 + 1}{\nu^2 - 1}(t^2 + 2)\mathrm{P}^\mu - \sigma\mathrm{P}^\mu \right].$$

On substituting for $\dfrac{\nu d\mathrm{P}^\mu}{d\nu}$ and $\dfrac{\nu^2 + 1}{\nu^2 - 1}\mathrm{P}^\mu$ their values, we have

$$\psi_s(\Omega\mathrm{P}^\mu) = - \tfrac{1}{2}\beta\Omega \left[\frac{2(\nu^2 - t^2)}{\beta}\mathrm{P}^\mu + \frac{t+2}{t}\mathrm{P}^{\mu+2} - 2\sigma\mathrm{P}^\mu \right.$$
$$\left. + \frac{t-2}{t}\{i, t\}\{i, t-1\}\mathrm{P}^{\mu-2} \right]. \quad . \quad . \quad . \quad (13).$$

(γ.) *To find* $\chi_s\left(\begin{Bmatrix} \cos \\ \sin \end{Bmatrix} t\phi\right).$

In this case the most convenient form for χ_s is that in (9), and we easily find

$$\chi_s\left(\begin{Bmatrix}\cos\\\sin\end{Bmatrix}t\phi\right) = -\tfrac{1}{2}\beta\left\{-\frac{2(s^2-t^2)}{\beta}\begin{Bmatrix}\cos\\\sin\end{Bmatrix}t\phi + \{i, t+1\}\begin{Bmatrix}\cos\\\sin\end{Bmatrix}(t+2)\phi\right.$$
$$\left. + 2\sigma\begin{Bmatrix}\cos\\\sin\end{Bmatrix}t\phi + \{i, t\}\begin{Bmatrix}\cos\\\sin\end{Bmatrix}(t-2)\phi\right\}. \quad . (14).$$

(δ.) *To find* $\chi_s\left(\Phi\begin{Bmatrix}\cos\\\sin\end{Bmatrix}t\phi\right).$

I now use the form χ_s as defined in (5), where D_s is given in (2), so that $D_s = (1 - \beta\cos 2\phi)^{\frac{1}{2}}\dfrac{d}{d\phi} = \Phi\dfrac{d}{d\phi}.$

We have

$$D_s\left(\Phi\begin{Bmatrix}\cos\\\sin\end{Bmatrix}t\phi\right) = \mp t\begin{Bmatrix}\sin\\\cos\end{Bmatrix}t\phi \pm \tfrac{1}{2}\beta(t+1)\begin{Bmatrix}\sin\\\cos\end{Bmatrix}(t+2)\phi \pm \tfrac{1}{2}\beta(t-1)\begin{Bmatrix}\sin\\\cos\end{Bmatrix}(t-2)\phi$$

and

$$D_s^2\left(\Phi\begin{Bmatrix}\cos\\\sin\end{Bmatrix}t\phi\right) = \Phi\left[-t^2\begin{Bmatrix}\cos\\\sin\end{Bmatrix}t\phi + \tfrac{1}{2}\beta(t+1)(t+2)\begin{Bmatrix}\cos\\\sin\end{Bmatrix}(t+2)\phi\right.$$
$$\left. + \tfrac{1}{2}\beta(t-1)(t-2)\begin{Bmatrix}\cos\\\sin\end{Bmatrix}(t-2)\phi\right].$$

The latter terms of χ_s contribute

$$\Phi\left\{(s^2-\beta\sigma)\begin{Bmatrix}\cos\\\sin\end{Bmatrix}t\phi - \tfrac{1}{2}\beta i(i+1)\left[\begin{Bmatrix}\cos\\\sin\end{Bmatrix}(t+2)\phi + \begin{Bmatrix}\cos\\\sin\end{Bmatrix}(t-2)\phi\right]\right\}$$

Therefore

$$\chi_s\left(\Phi\begin{Bmatrix}\cos\\\sin\end{Bmatrix}t\phi\right) = -\tfrac{1}{2}\beta\Phi\left\{-\frac{2(s^2-t^2)}{\beta}\begin{Bmatrix}\cos\\\sin\end{Bmatrix}t\phi + \{i, t+2\}\begin{Bmatrix}\cos\\\sin\end{Bmatrix}(t+2)\phi\right.$$
$$\left. + 2\sigma\begin{Bmatrix}\cos\\\sin\end{Bmatrix}t\phi + \{i, t-1\}\begin{Bmatrix}\cos\\\sin\end{Bmatrix}(t-2)\phi\right\} \quad . (15).$$

§ 6. *Determination of the Coefficients in the Functions.*

In this section I use successively the four results (12) (13) (14) (15) obtained in the last section under the headings (α), (β), (γ), (δ).

(α) $\mathfrak{P}^s = q_s P^s + \Sigma\beta^s q_{s-2s} P^{s-2s} + \Sigma\beta^s q_{s+2s} P^{s+2s}.$

The limits of the first Σ are 1 to $\frac{1}{2}s$ or $\frac{1}{2}(s-1)$, and of the second 1 to $\frac{1}{2}(i-s)$ or $\frac{1}{2}(i-s-1)$.

Applying the operation ψ_s to $\mathfrak{P}^s$ and equating $-\dfrac{2}{\beta}\psi_s(\mathfrak{P}^s)$ to zero, we have

$$\Sigma 8n(s-n)\beta^{n-2n}q_{s-2n}P^{s-2n} - \Sigma 8n(s+n)\beta^{n-1}q_{s+2n}P^{s+2n}$$
$$+ q_s[P^{s+2} - 2\sigma P^s + \{i, s\}\{i, s-1\}P^{s-2}]$$
$$+ \Sigma\beta^n q_{s-2n}[P^{s-2n+2} - 2\sigma P^{s-2n} + \{i, s-2n\}\{i, s-2n-1\}P^{s-2n-2}]$$
$$+ \Sigma\beta^n q_{s+2n}[P^{s+2n+2} - 2\sigma P^{s+2n} + \{i, s+2n\}\{i, s+2n-1\}P^{s+2n-2}] = 0.$$

The coefficients of the P's must vanish separately. This gives from the coefficients of P^{s-2n} and P^{s+2n} the following :—

$$2[4n(s-n) - \beta\sigma]q_{s-2n} + \beta^2 q_{s-2n-2}$$
$$+ \{i, s-2n+2\}\{i, s-2n+1\}q_{s-2n+2} = 0,$$

$$-2[4n(s+n) + \beta\sigma]q_{s+2n} + q_{s+2n-2}$$
$$+ \beta^2\{i, s+2n+2\}\{i, s+2n+1\}q_{s+2n+2} = 0.$$

These equations may be written in the form

$$\frac{2q_{s-2n}}{q_{s-2n+2}} = \frac{-\{i, s-2n+2\}\{i, s-2n+1\}}{4n(s-n) - \beta\sigma + \frac{1}{2}\beta^2\left(\dfrac{2q_{s-2n-2}}{q_{s-2n}}\right)},$$

$$\frac{2q_{s+2n}}{q_{s+2n-2}} = \frac{1}{4n(s+n) + \beta\sigma - \frac{1}{2}\beta^2\{i, s+2n+2\}\{i, s+2n+1\}\left(\dfrac{2q_{s+2n+2}}{q_{s+2n}}\right)},$$

. . (16).

Whence by continued application, the continued fractions

$$\frac{2q_{s-2n}}{q_{-2n+2}} = -\frac{\{i, s-2n+2\}\{i, s-2n+1\}}{4n(s-n) - \beta\sigma -} \quad \frac{\frac{1}{4}\beta^2\{i, s-2n\}\{i, s-2n-1\}}{4(n+1)(s-n-1) - \ldots}$$
$$-2r+2\}\{s, s-2n-2r+1\}$$

$$\frac{2q_{s+2n}}{q_{s+2n-2}} = \frac{1}{4n(s+n) + \beta\sigma -} \quad \frac{\frac{1}{4}\beta^2\{s, s+2n+2\}\{i, s+2n+1\}}{4(n+1)(s+n+1) + \beta\sigma - \ldots}$$
$$-\frac{1}{4}\beta^2\{i, s+2n+2r\}\{i, s+2n+2r-1\}$$

We must now consider what I may call the middle of the series, which corresponds with $n = 0$. In this case each of the Σ's contributes one term and the q_s term gives another. The result is

$$-2\sigma q_s + \beta q_{s-2} + \beta q_{s+2}\{i, s+2\}\{i, s+1\} = 0,$$

or

$$\beta\sigma = \frac{1}{2}\beta^2\left(\frac{2q_{s-2}}{q_s}\right) + \frac{1}{2}\beta^2\{i, s+2\}\{i, s+1\}\left(\frac{2q_{s+2}}{q_s}\right).$$

Since $2q_{s-2}/q_s$ and $2q_{s+2}/q_s$ are expressible as continued fractions, we have an equation for $\beta\sigma$, if the continued fractions terminate.

We shall now consider those terminations.

First, suppose that s is even, corresponding to types EEC, OEC.

The first continued fraction depends only on the first Σ. The condition to be satisfied is

$$2s^2\beta^{\frac{1}{2}(i-2)}q_0P + 2(s^2 - 4)\beta^{\frac{1}{2}(i-4)}q_2P^3 + \dots$$
$$+ \beta^{\frac{1}{2}i}q_0[P^2 - 2\sigma P + \{i, 0\}\{i, -1\}P^{-2}]$$
$$+ \beta^{\frac{1}{2}(i-2)}q_2[P^4 - 2\sigma P^3 + \{i, 2\}\{i, 1\}P]$$
$$+ \beta^{\frac{1}{2}(i-4)}q_4[P^6 - 2\sigma P^5 + \{i, 4\}\{i, 3\}P^3] + \dots = 0.$$

Since $\{i, 0\}\{i, -1\}P^{-2} = P^2$, we have, by equating to zero the coefficients of P and P^3, results which may be written

$$\frac{2q_0}{q_2} = \frac{-\{i, 2\}\{i, 1\}}{s^2 - \beta\sigma}, \qquad \frac{2q_2}{q_4} = \frac{-\{i, 4\}\{i, 3\}}{s^2 - 4 - \beta\sigma + \frac{1}{2}\beta^2\left(\frac{2q_0}{q_2}\right)}.$$

Hence the q's disappear from the first continued fraction, which terminates with

$$\frac{-\frac{1}{2}\beta^2\{i, 2\}\{i, 1\}}{s^2 - \beta\sigma}.$$

In this last term the $\frac{1}{4}\beta^2$ which prevails elsewhere is replaced by $\frac{1}{2}\beta^2$.

Observe that when $s = 2$ the first continued fraction is replaced by a simple fraction, so that the equation for $\beta\sigma$ becomes

$$\beta\sigma = \frac{-\frac{1}{2}\beta^2\{i, 2\}\{i, 1\}}{4 - \beta\sigma} + \frac{1}{4}\beta^2\{i, 4\}\{i, 3\}\left(\frac{2q_4}{q_2}\right).$$

Secondly, suppose that s is odd, corresponding to the types OOS, EOS.

The condition to be satisfied is now

$$2(s^2 - 1)\beta^{\frac{1}{2}(i-3)}q_1P^1 + 2(s^2 - 9)\beta^{\frac{1}{2}(i-5)}q_3P^3 + \dots$$
$$+ \beta^{\frac{1}{2}(i-3)}q_1[P^3 - 2\sigma P^1 + \{i, 1\}\{i, 0\}P^{-1}]$$
$$+ \beta^{\frac{1}{2}(i-3)}q_3[P^5 - 2\sigma P^3 + \{i, 3\}\{i, 2\}P^1]$$
$$+ \beta^{\frac{1}{2}(i-5)}q_5[P^7 - 2\sigma P^5 + \{i, 5\}\{i, 4\}P^3] + \dots = 0.$$

Now $\{i, 1\}\{i, 0\}P^{-1} = i(i+1)\frac{i+1}{i-1}P^{-1} = i(i+1)P^1$, and if we equate to zero the coefficients of P^1 and P^3 we obtain results which may be written

$$\frac{2q_1}{q_3} = \frac{-\{i, 3\}\{i, 2\}}{s^2 - 1 - \beta\sigma + \frac{1}{2}\beta i(i+1)},$$

$$\frac{2q_3}{q_5} = \frac{-\{i, 5\}\{i, 4\}}{s^2 - 9 - \beta\sigma + \frac{1}{2}\beta^2\left(\frac{2q_1}{q_3}\right)}.$$

Thus the q's again disappear, and the first continued fraction ends with

$$\frac{-\frac{1}{4}\beta^2\{i, 3\}\{i, 2\}}{s^2 - 1 - \beta\sigma + \frac{1}{2}\beta i(i+1)}.$$

Observe that when $s = 3$, the continued fraction reduces to a simple fraction, and the equation for $\beta\sigma$ becomes

$$\beta\sigma = \frac{-\frac{1}{4}\beta^2\{i, 3\}\{i, 2\}}{8 - \beta\sigma + \frac{1}{2}\beta i(i+1)} + \frac{1}{4}\beta^2\{i, 5\}\{i, 4\}\left(\frac{2q_5}{q_3}\right).$$

The case of $s = 1$ must be considered separately.

We have next to consider the termination of the second fraction, which depends only on the second Σ.

First, when i and s are either both even or both odd, the types are EEC and OOS, and the limits are $\frac{1}{2}(i - s)$ to 1. The condition to be satisfied is

$$- 2(i^2 - s^2)\beta^{\frac{1}{2}(i-s-2)}q_i P^i - 2[(i-2)^2 - s^2]\beta^{\frac{1}{2}(i-s-4)}q_{i-2}P^{i-2} - \ldots$$
$$+ \beta^{\frac{1}{2}(i-s)}q_i[P^{i+2} - 2\sigma P^i + \{i, i\}\{i, i-1\}P^{i-2}]$$
$$+ \beta^{\frac{1}{2}(i-s-2)}q_{i-2}[P^i - 2\sigma P^{i-2} + \{i, i-2\}\{i, i-3\}P^{i-4}]$$
$$+ \beta^{\frac{1}{2}(i-s-4)}q_{i-4}[P^{i-2} - 2\sigma P^{i-4} + \{i, i-4\}\{i, i-5\}P^{i-6}] + \ldots = 0.$$

Now P^{i+2} is zero, and equating the coefficients of P^i and P^{i-2} to zero we obtain results which may be written

$$\frac{2q_i}{q_{i-2}} = \frac{1}{i^2 - s^2 + \beta\sigma},$$

$$\frac{2q_{i-2}}{q_{i-4}} = \frac{1}{(i-2)^2 - s^2 + \beta\sigma - \frac{1}{4}\beta^2\{i, i\}\{i, i-1\}\left(\frac{2q_i}{q_{i-2}}\right)}.$$

Hence this continued fraction ends with

$$\frac{-\frac{1}{4}\beta^2\{i, i\}\{i, i-1\}}{i^2 - s^2 + \beta\sigma}.$$

Secondly, when i and s differ as to evenness or oddness, the types are OEC and EOS, and the limits are $\frac{1}{2}(i-s-1)$ to 1. The same investigation applies again when i is changed into $i - 1$.

Hence the continued fraction ends with

$$\frac{-\frac{1}{4}\beta^2\{i, i-1\}\{i, i-2\}}{(i-1)^2 - s^2 + \beta\sigma}.$$

The cases of $s = 0$, $s = 1$ must be considered by themselves.

When $s = 0$, the types are EEC and OEC. The "middle" of the series is now also an end, and the condition is

$$- 8q_2 P^2 - 8 \cdot 2^2 \beta q_4 P^4 - \ldots + q_0 [P^2 - 2\sigma P + \{i, 0\} \{i, -1\} P^{-2}]$$
$$+ \beta q_2 [P^4 - 2\sigma P^2 + \{i, 2\} \{i, 1\} P] + \beta^2 q_4 [P^6 - 2\sigma P^4 + \{i, 4\} \{i, 3\} P^2] + \ldots = 0.$$

Writing P^2 for $\{i, 0\} \{i, -1\} P^{-2}$ and equating the coefficients of P and P^2 to zero, we have

$$\beta\sigma = \tfrac{1}{2}\beta^2 \{i, 1\} \{i, 2\} \left(\frac{q_2}{q_0}\right),$$

$$\frac{q_2}{q_0} = \frac{1}{4 + \beta\sigma - \tfrac{1}{4}\beta^2 \{i, 3\} \{i, 4\} \left(\dfrac{2q_4}{q_2}\right)}.$$

Therefore
$$\beta\sigma = \frac{\tfrac{1}{2}\beta^2 \{i, 1\} \{i, 2\}}{4 \cdot 1^2 + \beta\sigma -} \quad \frac{\tfrac{1}{4}\beta^2 \{i, 3\} \{i, 4\}}{4 \cdot 2^2 + \beta\sigma -} \quad \frac{-\tfrac{1}{4}\beta^2 \{i, 5\} \{i, 6\}}{4 \cdot 3^2 + \beta\sigma - \ldots},$$

ending with $\dfrac{-\tfrac{1}{4}\beta^2 \{i, i\} \{i, i - 1\}}{i^2 + \beta\sigma}$ for EEC, and with $\dfrac{-\tfrac{1}{4}\beta^2 \{i, i - 1\} \{i, i - 2\}}{(i - 1)^2 + \beta\sigma}$ for OEC.

Next when $s = 1$ the types are OOS, EOS; the "middle" is again an end, and the condition is

$$- 8 \cdot 1 \cdot 2q_3 P^3 - 8 \cdot 2 \cdot 3\beta q_5 P^5 - \ldots + q_1 [P^3 - 2\sigma P^1 + \{i, 1\} \{i, 0\} P^{-1}]$$
$$+ \beta q_3 [P^5 - 2\sigma P^3 + \{i, 3\} \{i, 2\} P^1] + \beta^2 q_5 [P^7 - 2\sigma P^5 + \{i, 5\} \{i, 4\} P^3] + \ldots = 0.$$

Writing $i(i + 1) P^1$ for $\{i, 1\} \{i, 0\} P^{-1}$ and equating to zero the coefficients of P^1 and P^3, we have

$$\beta\sigma - \tfrac{1}{2}\beta i(i + 1) = \tfrac{1}{4}\beta^2 \{i, 3\} \{i, 2\} \left(\frac{2q_3}{q_1}\right),$$

$$\frac{2q_3}{q_1} = \frac{1}{4 \cdot 1 \cdot 2 + \beta\sigma - \tfrac{1}{4}\beta^2 \{i, 5\} \{i, 4\} \left(\dfrac{2}{q_3}\right)}.$$

Therefore
$$\beta\sigma - \tfrac{1}{2}\beta i(i + 1) = \frac{\tfrac{1}{4}\beta^2 \{i, 3\} \{i, 2\}}{4 \cdot 1 \cdot 2 + \beta\sigma -} \quad \frac{\tfrac{1}{4}\beta^2 \{i, 5\} \{i, 4\}}{4 \cdot 2 \cdot 3 + \beta\sigma -} \quad \frac{\tfrac{1}{4}\beta^2 \{i, 7\} \{i, 6\}}{4 \cdot 3 \cdot 4 + \beta\sigma - \ldots}$$

ending with $\dfrac{-\tfrac{1}{4}\beta^2 \{i, i\} \{i, i - 1\}}{i^2 + \beta\sigma}$ for OOS, and with $\dfrac{-\tfrac{1}{4}\beta^2 \{i, i - 1\} \{i, i - 2\}}{(i - 1)^2 + \beta\sigma}$ for EOS.

(β.) We have next to consider the other form of P-function for types EES, OES, OOC, EOC, namely,

$$P^s = \Omega \left[q'_s P^s + \Sigma \beta^n q'_{s-2n} P^{s-2n} + \Sigma \beta^n q'_{s+2n} P^{s+2n} \right],$$

where
$$\Omega = \left(\frac{P^2 - \dfrac{1 + \beta}{1 - \beta}}{P^2 - 1} \right)^{\frac{1}{2}}.$$

Let us write $q'_{s \pm 2n} = (s \pm 2n) q_{s \pm 2n}$. The q's are not now the actual coefficients of any P-function, but we shall see that they are determinable by almost the same relationships as those already found, and therefore the notation is convenient.

We now have

$$\mathbf{P}^i \doteq \Omega \left[q_s {}_s \mathbf{P}^i + \Sigma \beta^n q_{s-2n} (s-2n) \mathbf{P}^{i-2n} + \Sigma \beta^n q_{s+2n} (s+2n) \mathbf{P}^{i+2n} \right].$$

Applying the operation ψ_s to $\mathbf{P}^i$ and equating $-\dfrac{2}{\Omega \beta} \psi_s (\mathbf{P}^i)$ to zero, we have

$$\Sigma 8n (s-n)(s-2n) q_{s-2n} \mathbf{P}^{i-2n} \Sigma 8n (s+n)(s+2n) q_{s+2n} \mathbf{P}^{i+2n}$$

$$+ q_s [(s+2) \mathbf{P}^{i+2} - 2\sigma s \mathbf{P}^i + \{i, s\} \{i, s-1\} (s-2) \mathbf{P}^{i-2}]$$

$$+ \Sigma \beta^n q_{s-2n} [(s-2n+2) \mathbf{P}^{i-2n+2} - 2\sigma (s-2n) \mathbf{P}^{i-2n}$$
$$+ \{i, s-2n\} \{i, s-2n-1\} (s-2n-2) \mathbf{P}^{i-2n-2}]$$

$$+ \Sigma \beta^n q_{s+2n} [(s+2n+2) \mathbf{P}^{i+2n+2} - 2\sigma (s+2n) \mathbf{P}^{i+2n}$$
$$+ \{i, s+2n\} \{i, s+2n-1\} (s+2n-2) \mathbf{P}^{i+2n-2}] = 0.$$

This is the same equation as before, if we replace $t\mathbf{P}^i$ by $\mathbf{P}^i$. As we may equate coefficients of $t\mathbf{P}^i$ to zero (instead of coefficients of $\mathbf{P}^i$), we obtain the same equations for the q's as before.

A certain change must, however, be noted with respect to the beginning of the first series, which determines the end of the first continued fraction.

We previously wrote $\mathbf{P}^2$ for $\{i, 0\} \{i, -1\} \mathbf{P}^{-2}$ and $i(i+1) \mathbf{P}^1$ for $\{i, 1\} \{i, 0\} \mathbf{P}^{-1}$. But the corresponding terms will now be $\{i, 0\} \{i, -1\} (-2) \mathbf{P}^{-2}$ and $\{i, 1\} \{i, 0\} (-1) \mathbf{P}^{-1}$, and these are equal to $-(2\mathbf{P}^2)$ and $-(1 \cdot \mathbf{P}^1)$.

Hence it follows that when s is even (EES, OES)

$$\frac{2q_2}{q_0} = \frac{-\{i, 2\} \{i, 1\}}{s^2 - \beta \sigma}, \qquad \frac{2q_3}{q_4} = \frac{-\{i, 4\} \{i, 3\}}{s^2 - 4 - \beta \sigma}.$$

The q_0 term has disappeared from the latter of these, and thus the continued fraction is independent of q_0. This is correct, since whatever value (short of infinity) q_0 may have q'_0, being equal to $0 q_0$, vanishes. Hence the continued fraction is docked of one term and ends with

$$\frac{-\tfrac{1}{2} \beta^2 \{i, 4\} \{i, 3\}}{s^2 - 4 - \beta \sigma}.$$

It is important to note the deficiency of one term in the fraction, since it indicates that when $s = 2$ the first continued fraction entirely disappears.

When $s = 0$ there is no function of the $\mathbf{P}$ form, so the question of interpretation does not arise.

When s is odd (OOC, EOC) the only change is that $i(i+1)$ enters with the opposite sign, so that the first fraction ends with

$$\frac{-\tfrac{1}{8}\beta^2\{i,3\}\{i,2\}}{s^2-1-\beta\sigma-\tfrac{1}{8}\beta i(i+1)}.$$

When $s=1$, we have $\beta\sigma+\tfrac{1}{8}\beta i(i+1)$ equal to the same fraction as before.

When the q's are determined we have $q'_i = tq_i$. But it is desired that in the case (α) q_s should be unity, and that in the case (β) q' should be unity. This condition will be satisfied in the present case if we determine all the q's, put q_s equal to unity, and finally take

$$q'_{s \pm 2n} = \frac{s \pm 2n}{s} q_{s \pm 2n}.$$

Thus in both (α) and (β) we put q_s equal to unity, and in (β) determine the q''s by the above equation.

$(\gamma.)$ We now have to consider the cosine and sine functions.
For EEC, EES, OOC, OOS

$$\begin{Bmatrix}\mathfrak{C}^s_i\\\mathfrak{S}^s_i\end{Bmatrix} = p_s \begin{Bmatrix}\cos\\\sin\end{Bmatrix} s\phi + \Sigma\beta^n p_{s-2n} \begin{Bmatrix}\cos\\\sin\end{Bmatrix}(s-2n)\phi + \Sigma\beta^n p_{s+2n} \begin{Bmatrix}\cos\\\sin\end{Bmatrix}(s+2n)\phi.$$

The first Σ has limits $\tfrac{1}{2}s$ or $\tfrac{1}{2}(s-1)$ to 1, the second $\tfrac{1}{2}(i-s)$ to 1.

Apply the operation χ_s and equate $-\dfrac{2}{\beta}\chi_s\left(\begin{Bmatrix}\mathfrak{C}^s_i\\\mathfrak{S}^s_i\end{Bmatrix}\right)$ to zero; then

$$-\Sigma 8n(s-n)\beta^{n-1}p_{s-2n}\begin{Bmatrix}\cos\\\sin\end{Bmatrix}(s-2n)\phi + \Sigma 8n(s+n)\beta^{n-1}p_{s+2n}\begin{Bmatrix}\cos\\\sin\end{Bmatrix}(s+2n)\phi$$

$$+ p_s\left[\{i,s+1\}\begin{Bmatrix}\cos\\\sin\end{Bmatrix}(s+2)\phi + 2\sigma\begin{Bmatrix}\cos\\\sin\end{Bmatrix}s\phi + \{i,s\}\begin{Bmatrix}\cos\\\sin\end{Bmatrix}(s-2)\phi\right]$$

$$+ \Sigma\beta^n p_{-2n}\left[\{i,s-2n+1\}\begin{Bmatrix}\cos\\\sin\end{Bmatrix}(s-2n+2)\phi + 2\sigma\begin{Bmatrix}\cos\\\sin\end{Bmatrix}(s-2n)\phi\right.$$
$$\left. + \{i,s-2n\}\begin{Bmatrix}\cos\\\sin\end{Bmatrix}(s-2n-2)\phi\right]$$

$$+ \Sigma\beta^n p_{+2n}\left[\{i,s+2n+1\}\begin{Bmatrix}\cos\\\sin\end{Bmatrix}(s+2n+2)\phi + 2\sigma\begin{Bmatrix}\cos\\\sin\end{Bmatrix}(s+2n)\phi\right.$$
$$\left. + \{i,s+2n\}\begin{Bmatrix}\cos\\\sin\end{Bmatrix}(s+2n-2)\phi\right] = 0.$$

It we equate to zero the coefficients of $\left\{ \begin{array}{c} \cos \\ \sin \end{array} \right\} (s \pm 2n) \phi$, we find

$$\frac{2p_{s-2n}}{p_{s-2n+2}} = \frac{\{i, s - 2n + 2\}}{4n(s - n) - \beta\sigma - \frac{1}{4}\beta^2 \{i, s - 2n - 1\} \left(\dfrac{2p_{s-2n-2}}{p_{s-2n}} \right)},$$

$$\frac{2p_{s+2n}}{p_{s+2n-2}} = \frac{-\{i, s + 2n - 1\}}{4n(s + n) + \beta\sigma + \frac{1}{4}\beta^2 \{i, s + 2n + 2\} \left(\dfrac{2p_{s+2n+2}}{p_{s+2n}} \right)}.$$

These will, as before, lead to continued fractions, and by elimination of the p's to an equation for $\beta\sigma$. The equation will agree with our former result, for it can of course make no difference from which equation we determine σ.[*] It follows then by comparison with the previous result (16) that

$$\frac{p_{s-2n}}{p_{s-2n+2}} = -\frac{1}{\{i, s - 2n + 1\}} \frac{q_{s-2n}}{q_{s-2n+2}},$$

$$\frac{p_{s+2n}}{p_{s+2n-2}} = -\{i, s + 2n - 1\} \frac{q_{s+2n}}{q_{s+2n-2}}.$$

Hence when the q's are found, the p's follow at once.

(δ.) For OEC, OES, EOC, EOS

$$\left\{ \begin{array}{c} \mathbf{C}' \\ \mathbf{S}' \end{array} \right\} = \Phi \left[p_s' \left\{ \begin{array}{c} \cos \\ \sin \end{array} \right\} s\phi + \Sigma\beta'p'_{s-2n} \left\{ \begin{array}{c} \cos \\ \sin \end{array} \right\} (s - 2n)\phi + \Sigma\beta'p'_{s+2n} \left\{ \begin{array}{c} \cos \\ \sin \end{array} \right\} (s + 2n)\phi \right],$$

where $\Phi = (1 - \beta \cos 2\phi)^{\frac{1}{2}}$.

The limits of the first Σ are $\frac{1}{2}s$ or $\frac{1}{2}(s - 1)$ to 1, of the second $\frac{1}{2}(i - s - 1)$ to 1. Proceeding exactly as before we find

$$\frac{2p'_{s-2n}}{p'_{s-2n+2}} = \frac{\{i, s - 2n + 1\}}{4n(s - n) - \beta\sigma - \frac{1}{4}\beta^2 \{i, s - 2n\} \left(\dfrac{2p'_{s-2n-2}}{p'_{s-2n}} \right)},$$

$$\frac{}{p'_{s+2n-2}} = \frac{-\{i, s + 2n\}}{4n(s + n) + \beta\sigma + \frac{1}{4}\beta^2 \{i, s + 2n + 1\} \left(\dfrac{2p'_{s+2n}}{p'_{s+2n}} \right)}.$$

By comparison with (16) we see that

$$\frac{p_{s+2n}}{p'_{s+2n-2}} = -\{i, s + 2n\} \frac{q_{s+2n}}{q_{s+2n-2}}.$$

Therefore when the q's are found, the p's follow at once.

We may now summarise our results, as follows :—

In the general case where s is neither 0 nor 1, $\beta\sigma$ is the root which nearly vanishes of the equation

$$\beta\sigma = \frac{-\tfrac{1}{4}\beta^2\{i, s\}\{i, s-1\}}{4.1(s-1)-\beta\sigma-...} \qquad \frac{\tfrac{1}{4}\beta^2\{i, s-2\}\{i, s-3\}}{4.2(s-2)-\beta\sigma-...}$$
$$+ \frac{\tfrac{1}{4}\beta^2\{i, s+1\}\{i, s+2\}}{4.1(s+1)+\beta\sigma-...} \qquad \frac{\tfrac{1}{4}\beta^2\{i, s+3\}\{i, s+4\}}{4.2(s+2)+\beta\sigma-...}.$$

The continued fractions terminate variously for the various types of function. The end of the first continued fraction is as follows :—

For EEC $\dfrac{-\tfrac{1}{4}\beta^2\{i, 1\}\{i, 2\}}{s^2-\beta\sigma}$; and when $s = 2$ this is the whole fraction.

For EES $\dfrac{-\tfrac{1}{4}\beta^2\{i, 3\}\{i, 4\}}{s^2-4-\beta\sigma}$; and when $s = 2$ the fraction disappears.

For OOC $\dfrac{-\tfrac{1}{4}\beta^2\{i, 2\}\{i, 3\}}{s^2-1-\beta\sigma-\tfrac{1}{4}\beta i(i+1)}$; and when $s = 3$ this is the whole fraction.

For OOS $\dfrac{-\tfrac{1}{4}\beta^2\{i, 2\}\{i, 3\}}{s^2-1-\beta\sigma+\tfrac{1}{4}\beta i(i+1)}$; and when $s = 3$ this is the whole fraction.

For OEC $\dfrac{-\tfrac{1}{4}\beta^2\{i, 1\}\{i, 2\}}{s^2-\beta\sigma}$; and when $s = 2$ this is the whole fraction.

For OES $\dfrac{-\tfrac{1}{4}\beta^2\{i, 3\}\{i, 4\}}{s^2-4-\beta\sigma}$; and when $s = 2$ the fraction disappears.

For EOC $\dfrac{-\tfrac{1}{4}\beta^2\{i, 2\}\{i, 3\}}{s^2-1-\beta\sigma-\tfrac{1}{4}\beta i(i+1)}$; and when $s = 3$ this is the whole fraction.

For EOS $\dfrac{-\tfrac{1}{4}\beta^2\{i, 2\}\{i, 3\}}{s^2-1-\beta\sigma+\tfrac{1}{4}\beta i(i+1)}$; and when $s = 3$ this is the whole fraction.

For the first four of these types, viz., EEC, EES, OOC, OOS, the second continued fraction ends with

$$\frac{-\tfrac{1}{4}\beta^2\{i, i\}\{i, i-1\}}{i^2-s^2+\beta\sigma} \; ; \quad \text{and when } s = i \text{ this is the whole fraction, but with the}$$
$$\text{sign changed.}$$

For the last four, viz., OEC, OES, EOC, EOS, it ends with

$$\frac{-\tfrac{1}{4}\beta^2\{i, i-1\}\{i, i-2\}}{(i-1)^2-s^2+\beta\sigma} \; ; \quad \text{and when } s = i-1 \text{ this is the whole fraction, but}$$
$$\text{with the sign changed.}$$

When $s = 0$, the equation becomes

$$\beta\sigma = \frac{\frac{1}{4}\beta^2\{i, 1\}\{i, 2\}}{4 \cdot 1^2 + \beta\sigma -} \quad \frac{\frac{1}{4}\beta^2\{i, 3\}\{i, 4\}}{4 \cdot 2^2 + \beta\sigma - \dots} ,$$

ending when i is even (EEC) with $\quad \dfrac{-\frac{1}{4}\beta^2\{i, i\}\{i, i-1\}}{i^2 + \beta\sigma}$;

and when i is odd (OEC) with $\quad \dfrac{-\frac{1}{4}\beta^2\{i, i-1\}\{i, i-2\}}{(i-1)^2 + \beta\sigma}$.

When $s = 1$ the equation has two forms, which may, however, be written together. If the upper sign refers to cosines (OOC, EOC) and the lower to sines (OOS, EOS), the equations are :—

$$\beta\sigma \pm \frac{1}{2}\beta i (i + 1) = \frac{\frac{1}{4}\beta^2\{i, 2\}\{i, 3\}}{4 \cdot 1.2 + \beta\sigma -} \quad \frac{\frac{1}{4}\beta^2\{i, 4\}\{i, 5\}}{4 \cdot 2.3 + \beta\sigma - \dots} ,$$

ending when i is even (EOC, EOS) with $\quad \dfrac{-\frac{1}{4}\beta^2\{i, i-1\}\{i, i-2\}}{(i-1)^2 - 1 + \beta\sigma}$;

and when i is odd (OOC, OOS) with $\quad \dfrac{-\frac{1}{4}\beta^2\{i, i\}\{i, i-1\}}{i^2 - 1 + \beta\sigma}$.

It might appear at first sight that a difficulty will arise in the interpretation of these results when i is small, for the numbers in the denominators of the fractions increase, and yet it is possible that the number at the end should be smaller than that at the beginning; thus apparently the fraction ends before it begins. But this difficulty does not really arise, because in such cases the numerator will always be found to vanish, and thus the whole fraction disappears. For example, in the last case specified, if $s = 1, i = 2$ the denominators, according to the formula, begin with $8 + \beta\sigma$ and end with $0 + \beta\sigma$, but the fraction has for numerator $\{2, 2\}\{2, 3\}$, which vanishes.

When $\beta\sigma$ has been determined we find the q's by the formulæ—

$$\frac{2q_{s-2n}}{q_{s-2n+2}} = \frac{-\{i, s-2n+2\}\{i, s-2n+1\}}{4n(s-n)-\beta\sigma-} \quad \frac{\frac{1}{4}\beta^2\{i, s-2n\}\{i, s-2n-1\}}{4(n+1)(s-n-1)-\beta\sigma-\dots} ,$$

$$\frac{2q_{s+2n}}{q_{s+2n-2}} = \frac{1}{4n(s+n)+\beta\sigma-} \quad \frac{\frac{1}{4}\beta^2\{i, s+2n+2\}\{i, s+2n+1\}}{4(n+1)(s+n+1)+\beta\sigma-\dots} .$$

The terminations of the continued fractions are as specified above in the equation for $\beta\sigma$.

By forming continued products of ratios of successive q's, we can find all the q's as multiples of q_s, and $q_s = 1$.

In the cases EEC, OEC, OOS, EOS, these are the required coefficients for $\mathfrak{P}$.

In the cases EES, OES, OOC, EOS we put $q'_{s \pm 2n} = \frac{s \pm 2n}{s} q_{s \pm 2n}$, and thus find the coefficients for P'.

The coefficients for $\mathfrak{C}$, $\mathfrak{S}$ in EEC, EES, OOC, OOS are determined by

$$\frac{p_{s-2n}}{p_{s-2n+2}} = -\frac{1}{\{i, s-2n+1\}} \frac{q_{s-2n}}{q_{s-2n+2}},$$

$$\frac{p_{s+2n}}{p_{s+2n-2}} = -\{i, s+2n-1\} \frac{q_{s+2n}}{q_{s+2n-2}}.$$

The coefficients for C, S in OEC, OES, EOC, EOS are determined by

$$p'_{s-2n+2}$$

$$\frac{p'_{s+2n}}{p'_{s+2n-2}} = -\{i, s+2n\} \frac{q_{s+2n}}{q_{s+2n-2}}.$$

It follows that if we put $q_s = 1$ and $p_s = 1$

$$p_{s-2n} = (-)^n \frac{1}{\{i, s-2n+1\}\{i, s-2n+3\} \ldots \{i, s-1\}} q_{s-2n},$$

$$p_{s+2n} = (-)^n \{i, s+2n-1\} \{i, s+2n-3\} \ldots \{i, s+1\} q_{s+2n},$$

$$p'_{s-2n} = (-)^n \frac{1}{\{i, s-2n+2\}\{i, s-2n+4\} \ldots \{i, s\}} q_{s-2n},$$

$$p'_{s+2n} = (-)^n \{i, s+2n\} \{i, s+2n-2\} \ldots \{i, s+2\} q_{s+2n}.$$

When $s = 0$, q_2/q_0 is equal to that which would be given by the general formula for $\frac{2q_{s+2n}}{q_{s+2n-2}}$ when we put in it $n = 1$, $s = 0$. Hence it follows that the q's for $s = 0$ have double the values given by the general formula.

If we change the sign of s, the two continued fractions in the equation for $\beta\sigma$ are simply interchanged. Hence $\beta\sigma$ is unchanged when s changes sign. Also, since $\{i, t\}$ is equal to $\{-i-1, t\}$, $\beta\sigma$ is unchanged when $-i-1$ is written for i. A consideration of the forms of the q's and p's shows that $q_{-s+2t} \, \mathrm{P}^{-s+2t}$ is equal to $\frac{s-s!}{i+s!} q_{s-2t} \, \mathrm{P}^{s-2t}$, and therefore

$$\left\{ \begin{matrix} \mathfrak{P}_i^s \\ \mathsf{P}_i^s \end{matrix} \right\} = \left\{ \begin{matrix} \mathfrak{P}_{-i-1}^s \\ \mathsf{P}_{-i-1}^s \end{matrix} \right\}.$$

§ 7. *Rigorous determination of the Functions of the second degree.*

If a numerical value be attributed to β it is obviously possible to obtain the rigorous expressions for the several functions. Thus, if β were $\frac{1}{3}$ we could determine the harmonics of the ellipsoids of the class $c^2 = \frac{1}{2}(a^2 + b^2)$. But I do not think it is possible to obtain rigorous results in algebraic form when i is greater than 3. In order, however, to show how our formulæ lead to the required result I will determine the five functions corresponding to $i = 2$, but I will not work out the case of $i = 3$, although it is easy to do so.

When $s = 0$

$$\beta\sigma = \frac{\frac{1}{2}\beta^2\{2, 1\}\{2, 2\}}{4 + \beta\sigma} = \frac{12\beta^2}{4 + \beta\sigma}.$$

Therefore $\beta\sigma = -2 + 2(1 + 3\beta^2)^{\frac{1}{2}}$, or writing $B^2 = 1 + 3\beta^2$ for brevity, $\beta\sigma = 2(B - 1)$. Then putting $q_0 = 1$, and remembering that the value of q_4 is twice that given by the general formula,

$$q_2 = \frac{1}{4 + \beta\sigma} = \frac{B - 1}{6\beta^2}.$$

Therefore
$$\mathfrak{P}_2 = P_2 + \frac{B - 1}{6\beta} P_2^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (17),$$

where
$$P_2 = \tfrac{3}{2}\nu^2 - \tfrac{1}{2}, \qquad P_2^2 = 3(\nu^2 - 1).$$

The coefficient of the cosine function is given by

$$p_2 = -\{2, 1\} q_2 = -6q_2.$$

Therefore
$$\mathfrak{C}_2 = 1 - \frac{B - 1}{\beta} \cos 2\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad (18).$$

$s = 1$, cosines; EOC type.

The continued fraction has $\frac{1}{2}\beta^2\{i, 2\}\{i, 3\}$ in the numerator, and vanishes because $\{2, 3\} = 0$. Therefore

$$\beta\sigma + \tfrac{1}{2}\beta i(i + 1) = 0, \text{ where } i = 2.$$

Therefore
$$\sigma = -3.$$

But the coefficient is independent of β, for

$$\mathsf{P}_2{}^1 = \Omega[q_1' P_2{}^1], \text{ and } q_1' = 1.$$

Therefore
$$\mathsf{P}_2{}^1 = \sqrt{\frac{\nu^2 - \frac{1 + \beta}{1 - \beta}}{\nu^2 - 1}} \, P_2{}^1 = 3\nu\left(\nu^2 - \frac{1 + \beta}{1 - \beta}\right)^{\frac{1}{2}} \qquad . \quad (19).$$

Clearly
$$\mathsf{C}_2{}^1 = \cos\phi \, (1 - \beta\cos 2\phi)^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (20).$$

$s = 1$, sine; EOS type.

The continued fraction again vanishes and $\sigma = 3$, but it is not needed to express the functions. Putting $q_1' = 1$,

$$\mathfrak{P}_3{}^1 = q_1' \, P_1{}^1 = P_1{}^1 = 3\nu (\nu^2 - 1)^{\frac{1}{2}} \quad \cdots \cdots \cdots (21),$$

$$\mathfrak{S}_3{}^1 = \sin \phi \, (1 - \beta \cos 2\phi)^{\frac{1}{2}} \quad \cdots \cdots \cdots \cdots (22).$$

$s = 2$, cosine; EEC type.

The second continued fraction vanishes because it contains $\{2, 3\}$ in the numerator. The equation is then

$$\beta\sigma = \frac{-\frac{1}{2}\beta^2\{2, 1\}\,\{2, 2\}}{4 - \beta\sigma} = \frac{-12\beta^2}{4 - \beta\sigma}.$$

Therefore $\beta\sigma = 2 - 2\,(1 + 3\beta^2)^{\frac{1}{2}} = 2\,(1 - B)$. Then putting $q_2 = 1$,

$$q_0 = -\frac{1}{2}\frac{\{2, 2\}\{2, 1\}}{4 - \beta\sigma} = \frac{-2\,(B - 1)}{\beta^2}.$$

Therefore

$$\mathfrak{P}_3{}^2 = \frac{-2\,(B - 1)}{\beta}\, P_2 + P_3{}^2 \quad \cdots \cdots \cdots (23),$$

where

$$P_2 = \tfrac{3}{2}\nu^2 - \tfrac{1}{2},$$
$$P_3{}^2 = 3\,(\nu^2 - 1).$$

Then

$$p_0 = -\frac{1}{\{2, 1\}}\,q_0 = -\tfrac{1}{6}q_0, \text{ and}$$

$$\mathfrak{C}_3{}^2 = \frac{B - 1}{3\beta} + \cos 2\phi \quad \cdots \cdots \cdots \cdots (24).$$

$s = 2$, sine; EES type.

Both fractions disappear and σ vanishes, but is not needed for determining the functions. Noting that $q_0' = 0$, and $q_2' = 1$,

$$\mathbf{P}_3{}^2 = \Omega \left[q_2' P_2{}^2\right] = 3\left(\nu^2 - \frac{1 + \beta}{1 - \beta}\right)^{\frac{1}{2}} (\nu^2 - 1)^{\frac{1}{2}} \quad \cdots \cdots (25),$$

$$\mathbf{S}_3{}^2 = \sin 2\phi \quad \cdots \cdots \cdots \cdots \cdots \cdots (26).$$

We can write down the functions of μ by symmetry, and the products of the three functions give rigorously the five solid harmonic solutions of LAPLACE'S equation of the second degree. As I have remarked above, the seven harmonics of the third degree may be obtained rigorously by a parallel process.

§ 8. *Approximate Form of the Functions.*

It is clear that the first approximation to $\beta\sigma$ is zero, and that the second approximation, in the general case, is

$$\beta\sigma = -\tfrac{1}{16}\beta^2 \frac{\{i,s\}\{i,s-1\}}{s-1} + \tfrac{1}{16}\beta^2 \frac{\{i,s+1\}\{i,s+2\}}{s+1}$$

$$= \tfrac{1}{8}\beta^2\left(3s^2 - 2i(i+1) - \frac{i^2(i+1)^2}{s^2-1}\right).$$

If this expression were inserted in $\frac{q_{s\pm2}}{q_s}$ we should obtain $q_{s\pm2}$ correct to β^2. But since the next approximation would only introduce β^4, it follows that $q_{s\pm2}$ would be correct to β^3 inclusive. Now $q_{s\pm2}$ enters in the functions with a factor β, and therefore this approximation would give results correct to β^4 inclusive. Since the similar operation could be applied with equal ease in all the cases in which the continued fractions assume special forms, it follows that this degree of accuracy is very easily attainable. However, the forms of the coefficients would be rather complicated, and it would render the subsequent algebra so tedious that I do not propose at present to carry the approximation beyond β^2.

It now suffices to put $\sigma = 0$ in the denominators of all the continued fractions, whereby the coefficients are determined, except in the cases of $s = 1$, $s = 3$, where we put $\sigma = \pm \tfrac{1}{2}i(i+1)$.

In the general case we have

$$\left.\begin{aligned}
q_{s-2} &= -\tfrac{1}{8}\frac{\{i,s\}\{i,s-1\}}{s-1}, & q_{s+2} &= \frac{1}{8(s+1)}, \\[2mm]
q_{s-4} &= \frac{\{i,s\}\{i,s-1\}\{i,s-2\}\{i,s-3\}}{128(s-1)(s-2)}, & q_{s+4} &= \frac{1}{128(s+1)(s+2)}, \\[2mm]
q'_{s-2} &= \frac{s-2}{s}\ q_{s-2}, & q'_{s+2} &= \frac{s+2}{s}\ q_{s+2}, \\[2mm]
q'_{s-4} &= \frac{s-4}{s}\ q_{s-4}, & q'_{s+4} &= \frac{s+4}{s}\ q_{s+4}, \\[2mm]
p_{s-2} &= \frac{\{i,s\}}{8(s-1)}, & p_{s+2} &= \frac{-\{i,s+1\}}{8(s+1)}, \\[2mm]
p_{s-4} &= \frac{\{i,s\}\{i,s-2\}}{128(s-1)(s-2)}, & p_{s+4} &= \frac{\{i,s+1\}\{i,s+3\}}{128(s+1)(s+2)}, \\[2mm]
p'_{s-2} &= \frac{\{i,s-1\}}{8(s-1)}, & p'_{s+2} &= \frac{-\{i,s+2\}}{8(s+1)}, \\[2mm]
p'_{s-4} &= \frac{\{i,s-1\}\{i,s-3\}}{128(s-1)(s-2)}, & p'_{s+4} &= \frac{\{i,s+2\}\{i,s+4\}}{128(s+1)(s+2)}
\end{aligned}\right\} \quad . \ (27).$$

When $s = 0$, we double the results given by the general formula and find

$$\left.\begin{aligned}
&q_2 = \tfrac{1}{4},\ q_4 = \tfrac{1}{148},\ p_2 = -\tfrac{1}{4}\{i,1\},\ p_4 = \tfrac{1}{148}\{i,1\}\{i,3\}. \\
\text{There are no}\quad &q'_2,\ q'_4,\ \text{and}\ p'_2 = -\tfrac{1}{4}\{i,2\},\ p'_4 = \tfrac{1}{148}\{i,2\}\{i,4\}
\end{aligned}\right\} \quad . \ (28).$$

When $s = 1$,

$$q_8 = \frac{1}{16 \mp \beta i (i + 1)} = \tfrac{1}{16}(1 \pm \tfrac{1}{16}\beta i(i + 1)), \text{ with upper sign for cosines}$$

(EOC, OOC) and lower sign for sines (OOS, EOS).

$q_5 = \tfrac{1}{128.8.8} = \tfrac{1}{768}$ for all cases.

But for OOS. EOS we use the $\mathfrak{P}$ form, and for EOC, OOC the P form; and for the latter $\dfrac{s + 2}{s} = 3$, $\dfrac{s + 4}{s} = 5$.

Therefore for OOS, EOS (sines)

$$q_3 = \tfrac{1}{16}(1 - \tfrac{1}{16}\beta i(i + 1)), \quad q_5 = \tfrac{1}{768}, \; \Big|$$

and for EOC, OOC (cosines)

$$q_3{}' = \tfrac{1}{16}(1 + \tfrac{1}{16}\beta i(i + 1)), \quad q_5{}' = \tfrac{1}{768} \; \Big|$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad . \quad . \quad . \quad . \; (29).$

For OOC, OOS, with upper sign for cosine and lower sign for sine,

$$p_3 = -\tfrac{1}{16}\{i, 2\}\,[1 \pm \tfrac{1}{16}\beta i(i + 1)], \quad p_5 = \tfrac{1}{768}\{i, 2\}\{i, 4\} \quad . \quad . \; (29).$$

For EOC, EOS, with upper sign for cosine and lower sign for sine,

$$p_3{}' = -\tfrac{1}{16}\{i, 3\}\,[1 \pm \tfrac{1}{16}\beta i(i + 1)], \quad p_5{}' = \tfrac{1}{768}\{i, 3\}\{i, 5\} \quad . \quad . \; (29).$$

When $s = 2$ the coefficients may be derived from the general formula.
When $s = 3$

$$q_1 = \frac{-\tfrac{1}{8}\{i, 3\}\{i, 2\}}{8 \mp \tfrac{1}{2}\beta i(i + 1)} = -\tfrac{1}{16}\{i, 2\}\{i, 3\}\,[1 \pm \tfrac{1}{16}\beta i(i + 1)],$$

the upper sign applying to cosines (OOC, EOC) the lower to sines (OOS, EOS);

$$q_5 = \tfrac{1}{84}, \quad q_7 = \tfrac{1}{3840}.$$

But for OOS, EOS the $\mathfrak{P}$ form applies, and for OOC, EOC the P form applies.

Also with $s = 3$, $\dfrac{s - 2}{s} = \tfrac{1}{3}$, $\dfrac{s + 2}{s} = \tfrac{5}{3}$, $\dfrac{s + 4}{s} = \tfrac{7}{3}$.

Therefore for OOS, EOS

$$q_1 = -\tfrac{1}{16}\{i, 2\}\{i, 3\}[1 - \tfrac{1}{16}\beta i(i+1)], \; q_5 = \tfrac{1}{21}, \; q_7 = \tfrac{1}{2160}.$$

For OOC, EOC

$$q_1' = -\tfrac{1}{48}\{i, 2\}\{i, 3\}[1 + \tfrac{1}{16}\beta i(i+1)]', \; q_5' = \tfrac{1}{60}, \; q_7' = \tfrac{7}{16800}.$$

For OOC, OOS, with upper sign for cosine and lower for sine,

$$p_1 = \tfrac{1}{16}\{i, 3\}[1 \pm \tfrac{1}{16}\beta i(i+1)], \; p_5 = -\tfrac{1}{36}\{i, 4\},$$
$$p_7 = \tfrac{1}{2160}\{i, 4\}\{i, 6\}. \qquad \left.\right\} \quad (30).$$

For EOC, EOS, with upper sign for cosine and lower for sine,

$$p_1' = \tfrac{1}{16}\{i, 2\}[1 \pm \tfrac{1}{16}\beta i(i+1)], \; p_5' = -\tfrac{1}{36}\{i, 5\},$$
$$p_7' = \tfrac{1}{2160}\{i, 5\}\{i, 7\}.$$

It will save much trouble to note that if we were to admit negative suffixes to the q's, the general formula would give us the term $\beta^3 q_{-1} \, \mathrm{P}^{-1}$, where

$$q_{-1} = \frac{\{i\,3\}\{i, 2\}\{i, 1\}\{i, 0\}}{128 \cdot 2 \cdot 1}.$$

Thus this term is $\dfrac{1}{(16)^3}\beta^3 i\,(i+1) \cdot \{i, 3\}\{i, 2\}\mathrm{P}^1$. But this is exactly that part of the term in (30) which arises from $\beta q_1 \mathrm{P}^1$, but which is not included in the general formula.

Similarly the general formula gives for q'_{-1}, p_{-1}, p'_{-1} those parts of the terms arising from q_1', p_1, p_1' which are not included in the general formula.

It follows that in much of the subsequent work we need not devote special consideration to the case of $s = 3$.

§ 9. *Factors of Transformation between the two forms of P-function and C- or S-function.*

The rigorous expressions $\mathfrak{P}^s$ and P^s always differ from one another, but approximately they are the same up to a certain power of β, provided that s is greater than a certain quantity.

Since $\Omega = \left(\dfrac{\nu^3 - \frac{1}{2}\frac{1+\beta}{1-\beta}}{\nu^2 - 1}\right)^{\frac{1}{4}} = \left(1 + \dfrac{2\,\beta}{(\nu^2 - 1)(1 - \beta)}\right)^{\frac{1}{4}}$, it is legitimate to develop Ω in powers of $1/(\nu^2 - 1)$ up to a certain power, say t, provided that it is to be multiplied by a function involving at least $(\nu^2 - 1)^t$ as a factor; for this condition insures that there shall be no infinite terms when $\nu = \pm 1$. At present, I limit the development to β^2, so that

$$\Omega = 1 - \frac{\beta + \beta^2}{\nu^2 - 1} - \frac{\frac{1}{2}\beta^2}{(\nu^2 - 1)^2}.$$

Therefore

$$\mathbf{P}^{\prime} = \left(1 - \frac{\beta + \beta^2}{\nu^2 - 1} - \frac{\tfrac{1}{2}\beta^2}{(\nu^2 - 1)^2}\right)\mathbf{P}^{\prime} + \left(1 - \frac{\beta}{\nu^2 - 1}\right)(\beta q'_{s-2}\mathbf{P}^{\prime-2} + \beta q'_{s+2}\mathbf{P}^{\prime+2})$$
$$+ \beta^2 q'_{s-4}\mathbf{P}^{\prime-4} + \beta^2 q'_{s+4}\mathbf{P}^{\prime+4}.$$

It is obvious on inspection that we cannot rely on this development if s is less than 4.

If then s is equal to, or greater than 4, this value of $\mathbf{P}^{\prime}$, when properly developed, to the adopted order of approximation can only differ from $\mathfrak{P}^{\prime}$ by a constant factor, say $C_s^{\prime}$ or shortly $C^{\prime}$; so that

and we have to determine the constant $C^{\prime}$.

We might develop the above expression for $\mathbf{P}^{\prime}$ completely and compare it with $\mathfrak{P}^{\prime}$, but this is unnecessary since the comparison of a single term suffices.

I now write

$$\Sigma_i^{\prime} = \frac{i(i+1)}{\kappa^2 - 1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32),$$

or shortly Σ. This notation is introduced because this function occurs very frequently hereafter.

We have seen in (11) (slightly modified) that

$$\frac{\mathbf{P}^{\prime}}{\nu^2 - 1} = \tfrac{1}{4}\left\{\frac{\mathbf{P}^{\prime+2}}{s(s+1)} - 2(\Sigma+1)\mathbf{P}^{\prime} + \right.$$

We may write this

$$\frac{\mathbf{P}^{\prime}}{\nu^2 - 1} = \alpha_s \mathbf{P}^{\prime+2} + \beta_s \mathbf{P}^{\prime+1} + \gamma_s \mathbf{P}^{\prime-2},$$

where

$$\alpha_s = \frac{1}{4s(s+1)}, \quad \beta_s = -\tfrac{1}{2}(\Sigma+1), \quad \gamma_s = \frac{\{i,s\}\{i,s-1\}}{4s(s-1)}.$$

Then

$$\frac{\mathbf{P}^{\prime}}{(\nu^2 - 1)^2} = \alpha_s(\alpha_{s+2}\mathbf{P}^{\prime+4} + \beta_{s+2}\mathbf{P}^{\prime+2} + \gamma_{s+2}\mathbf{P}^{\prime})$$
$$+ \beta_s(\alpha_s\mathbf{P}^{\prime+2} + \beta_s\mathbf{P}^{\prime} + \gamma_s\mathbf{P}^{\prime-2})$$
$$+ \gamma_s(\alpha_{s-2}\mathbf{P}^{\prime} + \beta_{s-2}\mathbf{P}^{\prime-2} + \gamma_{s-2}\mathbf{P}^{\prime-4}).$$

Therefore the coefficient of $\mathbf{P}^{\prime}$ is $\alpha_s\gamma_{s+2} + (\beta_s)^2 + \alpha_{s-2}\gamma_s$, or

$$\tfrac{1}{16}\left[\frac{\{i,s+2\}\{i,s+1\}}{s(s+1)^2(s+2)} + 4(\Sigma+1)^2 + \frac{\{i,s\}\{i,s-1\}}{s(s-1)^2(s-2)}\right].$$

I now introduce a further abridgement and write

$$T_i' = \frac{(i-1)\,i\,(i+1)\,(i+2)}{s^2-4} \quad\ldots\ldots\quad (32),$$

or shortly T.

Then, after reduction, I find

$$\frac{P'}{(s^2-1)^2} = k\big[-\Sigma^2 s^2 + \Sigma^2 + 4\Sigma + 3 + T\big]P' + \ldots$$

Accordingly the coefficient of P' in $\mathbf{P}'$ is

$$1 + \tfrac{1}{2}\beta(\Sigma+1) + \tfrac{1}{2}\beta^2(\Sigma+1) - \tfrac{1}{16}\beta^2\big[-\Sigma^2 s^2 + \Sigma^2 + 4\Sigma + 3 + T\big]$$
$$- \tfrac{1}{4}\beta^2\Big[\frac{q'_{s-2}}{(s-1)(s-2)} + q'_{s+2}\frac{\{i,\,s+1\}\{i,\,s+2\}}{(s+1)(s+2)}\Big].$$

But $q'_{s-2} = -\dfrac{(s-2)\{i,\,s\}\{i,\,s-1\}}{8s(s-1)}$, $q'_{s+2} = \dfrac{s+2}{8s(s+1)}$, and the last term in the above expression will be found to be equal to $+\tfrac{1}{8}\beta^2(\Sigma^2-1)$. Thus the coefficient of P' in the development of $\mathbf{P}'$ is

$$1 + \tfrac{1}{2}\beta(\Sigma+1) + \tfrac{1}{16}\beta^2(\Sigma^2 s^2 + \Sigma^2 + 4\Sigma + 3 - T);$$

but the same coefficient in $\mathfrak{P}'$ is unity.

Therefore

$$\frac{1}{C_i'} = 1 + \tfrac{1}{2}\beta(\Sigma+1) + \tfrac{1}{16}\beta^2(\Sigma^2 s^2 + \Sigma^2 + 4\Sigma + 3 - T)$$
$$C_i' = 1 - \tfrac{1}{2}\beta(\Sigma+1) + \tfrac{1}{16}\beta^2(-\Sigma^2 s^2 + 3\Sigma^2 + 4\Sigma + 1 + T) \quad\Bigg\}\quad .(33).$$
$$(C_i')^2 = 1 + \beta(\Sigma+1) + \tfrac{1}{8}\beta^2(\Sigma^2 s^2 + 3\Sigma^2 + 8\Sigma + 5 - T)$$
$$(C_i')^2 = 1 - \beta(\Sigma+1) + \tfrac{1}{8}\beta^2(-\Sigma^2 s^2 + 5\Sigma^2 + 8\Sigma + 3 + T)$$

The squares of this constant and of its reciprocal are given because they will be needed at a later stage.

We next consider the cosine and sine functions.

$$\begin{Bmatrix}\mathbf{C}'\\\mathbf{S}'\end{Bmatrix} = \Phi\Big[\begin{Bmatrix}\cos\\\sin\end{Bmatrix}s\phi + \beta p'_{s-2}\begin{Bmatrix}\cos\\\sin\end{Bmatrix}(s-2)\phi + \ldots\Big].$$

As far as β^2

$$\Phi = (1 - \beta\cos 2\phi)^{\frac{1}{2}} = 1 - \tfrac{1}{2}\beta\cos 2\phi - \tfrac{1}{16}\beta^2(1+\cos 4\phi).$$

Therefore

$$\begin{Bmatrix} \mathbf{C}' \\ \mathbf{S}' \end{Bmatrix} = [1 - \tfrac{1}{4}\beta\cos 2\phi - \tfrac{1}{18}\beta^2(1 + \cos 4\phi)]\begin{Bmatrix} \cos \\ \sin \end{Bmatrix}s\phi$$

$$+ \beta\left[p'_{s-2}\begin{Bmatrix} \cos \\ \sin \end{Bmatrix}(s - 2)\phi + p'_{s+2}\begin{Bmatrix} \cos \\ \sin \end{Bmatrix}(s + 2)\phi \right][1 - \tfrac{1}{2}\beta\cos 2\phi]$$

$$+ \beta^2 p'_{s-4}\begin{Bmatrix} \cos \\ \sin \end{Bmatrix}(s - 4)\phi + \beta^2 p'_{s+4}\begin{Bmatrix} \cos \\ \sin \end{Bmatrix}(s + 4)\phi.$$

This expression, when developed, must lead to $\mathbf{C}'$ or $\mathbf{S}'$ multiplied by a constant factor.

Let

$$\begin{Bmatrix} \mathbf{C}' \\ \mathbf{S}' \end{Bmatrix} = \mathbf{D}_s' \begin{Bmatrix} \mathbf{C}' \\ \mathbf{S}' \end{Bmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34).$$

Then $\mathbf{D}_s'$ or $\mathbf{D}^s$ may be found by considering only the coefficient of $\begin{Bmatrix} \cos \\ \sin \end{Bmatrix}s\phi$. Hence

$$\frac{1}{\mathbf{D}^s} = 1 - \tfrac{1}{16}\beta^2 - \tfrac{1}{2}\beta^2 p'_{s-2} - \tfrac{1}{2}\beta^2 p'_{s+2}.$$

But $\qquad p'_{s-2} = \dfrac{\{s, s-1\}}{8(s-1)}, \qquad p'_{s+2} = \dfrac{-\{s, s+2\}}{8(s+1)}, \qquad$ and

$$p'_{s-2} + p'_{s+2} = \tfrac{1}{4}(\Sigma + 2).$$

Therefore

$$\frac{1}{\mathbf{D}_s'} = 1 - \tfrac{1}{16}\beta^2(\Sigma + 3), \left.\begin{matrix} \\ \\ \end{matrix}\right\}$$

$$\frac{1}{(\mathbf{D}_s')^2} = 1 - \tfrac{1}{8}\beta^2(\Sigma + 3) \quad \left.\begin{matrix} \\ \end{matrix}\right\} \quad . \quad . \quad . \quad . \quad . \quad (35).$$

The reciprocals may clearly be written down at once.

There are no factors by which $\mathfrak{P}^3$, $\mathfrak{P}^2$, $\mathfrak{P}^1$ can be converted into $\mathbf{P}^3$, $\mathbf{P}^2$, $\mathbf{P}^1$; but this is not true of the cosine and sine functions.

In the case of $s = 3$, it will be found that the general formula holds good for the factor whereby $\mathbf{C}^3$, $\mathbf{S}^3$ are convertible into $\mathbf{C}^3$, $\mathbf{S}^3$.

When $s = 2$,

$$\begin{Bmatrix} \mathbf{C}^2 \\ \mathbf{S}^2 \end{Bmatrix} = [1 - \tfrac{1}{4}\beta\cos 2\phi - \tfrac{1}{16}\beta^2(1 + \cos 4\phi)]\begin{Bmatrix} \cos \\ \sin \end{Bmatrix}2\phi + \beta p'_0(1 - \tfrac{1}{2}\beta\cos 2\phi)\begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

$$+ \beta p'_4(1 - \tfrac{1}{2}\beta\cos 2\phi)\begin{Bmatrix} \cos \\ \sin \end{Bmatrix}4\phi + \beta^2 p'_6\begin{Bmatrix} \cos \\ \sin \end{Bmatrix}6\phi.$$

Then

$$\mathbf{C}^2 = [1 - (\tfrac{3}{32} + \tfrac{1}{4}p'_0 + \tfrac{1}{4}p'_4)\beta^2]\cos 2\phi + \dots$$

$$\mathbf{S}^2 = [1 - (\tfrac{3}{32} + \tfrac{1}{4}p'_4)\beta^2]\sin 2\phi + \dots$$

But $p'_0 = \frac{1}{8}\{i, 1\}$, $p'_4 = -\frac{1}{14}\{i, 4\}$, $p'_6 = \frac{1}{15 \cdot 16}\{i, 4\}\{i, 6\}$, and

$$\frac{3}{2 \cdot 4} + \frac{1}{2}p'_0 + \frac{1}{4}p'_4 = \frac{1}{3 \cdot 2}(5\Sigma + 7),$$

$$\frac{3}{2 \cdot 4} + \frac{1}{4}p'_4 \qquad = -\frac{1}{3 \cdot 2}(\Sigma - 5), \quad \text{where } \Sigma = \frac{i(i+1)}{2^2 - 1} = \frac{1}{3}i(i+1).$$

Therefore the factors are

$$\frac{1}{D_i^2}(\cos) = 1 - \frac{1}{3 \cdot 2}\beta^2(5\Sigma^2 + 7),$$

$$\frac{1}{D_i^2}(\sin) = 1 + \frac{1}{3 \cdot 2}\beta^2(\Sigma^2 - 5)$$

(36).

It is easy to verify that the other coefficients of $\mathfrak{C}^2$ and $\mathfrak{S}^2$ are in fact reproduced.

The notation adopted here and below for distinguishing the cosine and sine factors is perhaps rather clumsy, but I have not thought it worth while to take distinctive symbols for the factors in these cases, because they will not be of frequent occurrence. When $s = 1$,

$$\begin{Bmatrix}\mathfrak{C}^1 \\ \mathfrak{S}^1\end{Bmatrix} = [1 - \tfrac{1}{2}\beta \cos 2\phi - \tfrac{1}{16}\beta^2(1 + \cos 4\phi)]\begin{Bmatrix}\cos \\ \sin\end{Bmatrix}\phi + \beta p'_3(1 - \tfrac{1}{2}\beta \cos 2\phi)\begin{Bmatrix}\cos \\ \sin\end{Bmatrix}3\phi$$

$$+ \beta^2 p'_5 \begin{Bmatrix}\cos \\ \sin\end{Bmatrix}5\phi$$

$$= [1 \mp \tfrac{1}{4}\beta - \tfrac{1}{4}\beta^2(p'_3 + \tfrac{1}{4})]\begin{Bmatrix}\cos \\ \sin\end{Bmatrix}\phi + \dots$$

This must be equal to $\dfrac{1}{D_i^1}\begin{Bmatrix}\mathfrak{C}^1 \\ \mathfrak{S}^1\end{Bmatrix}$.

Now, with upper sign for cosine and lower for sine,

$$p'_3 = -\frac{1}{16}\{i, 3\}[1 \pm \tfrac{1}{16}\beta i(i+1)], \quad p'_5 = \frac{1}{7 \cdot 8 \cdot 5}\{i, 3\}\{i, 5\}.$$

Substituting for p'_3 its values, we find with the upper sign

$$\frac{1}{D_i^1}(\cos) = 1 - \tfrac{1}{4}\beta - \tfrac{1}{4}\beta^2(p'_3 + \tfrac{1}{4}) = 1 - \tfrac{1}{4}\beta + \tfrac{1}{6 \cdot 4}\beta^2[i(i+1) - 10].$$

And with the lower sign

(37).

$$\frac{1}{D_i^1}(\sin) = 1 + \tfrac{1}{4}\beta - \tfrac{1}{4}\beta^2(p'_3 + \tfrac{1}{4}) = 1 + \tfrac{1}{4}\beta + \tfrac{1}{6 \cdot 4}\beta^2[i(i+1) - 10]$$

It follows that

$$D_i^1(\cos) = 1 + \tfrac{1}{4}\beta - \tfrac{1}{6 \cdot 4}\beta^2[i(i+1) - 14]$$

$$D_i^1(\sin) = 1 - \tfrac{1}{4}\beta - \tfrac{1}{6 \cdot 4}\beta^2[i(i+1) - 14]$$

$$\left[\frac{1}{D_i^1}(\cos)\right]^2 = 1 - \tfrac{1}{2}\beta + \tfrac{1}{3 \cdot 2}\beta^2[i(i+1) - 8]$$

$$[D_i^1(\cos)]^2 = 1 + \tfrac{1}{2}\beta - \tfrac{1}{3 \cdot 2}\beta^2[i(i+1) - 16]$$

$$\left[\frac{1}{D_i^1}(\sin)\right]^2 = 1 + \tfrac{1}{2}\beta + \tfrac{1}{3 \cdot 2}\beta^2[i(i+1) - 8]$$

$$[D_i^1(\sin)]^2 = 1 - \tfrac{1}{2}\beta - \tfrac{1}{3 \cdot 2}\beta^2[i(i+1) - 16]$$

. (37).

We cannot in the present case use $\Sigma_i{}^1$ as an abridged notation, because it is infinite as involving $s^2 - 1$ in the denominator.

It is easy to verify that the other coefficients of $\mathbb{C}^1$ and $\mathbb{S}^1$ are, in fact, reproduced in the transformation.

Lastly when $s = 0$, we have only cosine functions.   As before

$$\mathbb{C} = 1 - \tfrac{1}{2}\beta\cos 2\phi - \tfrac{1}{16}\beta^2(1 + \cos 4\phi) + \beta p'_2(1 - \tfrac{1}{2}\beta\cos 2\phi)\cos 2\phi \\ + \beta^2 p'_4 \cos 4\phi$$

$$= 1 - \tfrac{1}{16}\beta^2 - \tfrac{1}{2}\beta^2 p'_2 + \ldots \quad .$$

This must be equal to $\dfrac{\mathbb{C}}{\mathrm{D}_i}$, and therefore $\dfrac{1}{\mathrm{D}_i} = 1 - \tfrac{1}{16}\beta^2 - \tfrac{1}{2}\beta^2 p'_2$.

Now $\qquad\qquad p'_2 = -\tfrac{1}{2}\{i, 2\}, \quad p'_4 = \tfrac{1}{16}\{i, 2\}\{i, 4\}$.

Hence $\qquad\qquad \dfrac{1}{\mathrm{D}_i} = 1 + \tfrac{1}{16}\beta^2[\iota(i+1) - 3] \quad . \quad . \quad . \quad . \quad . \quad . \quad (38).$

Since in this case $\Sigma_i = \dfrac{i(\iota+1)}{-1}$,

$$\dfrac{1}{\mathrm{D}_i} = 1 - \tfrac{1}{16}\beta^2[\Sigma_i + 3] . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38).$$

Thus the general formula again holds good.

It is easy to verify that the other coefficients of $\mathbb{C}$ are in fact reproduced.

The principal use of the transforming factors, determined in this section, is that it will enable us to avoid some tedious analysis hereafter.

§ 10. *The Functions of the Second Kind.*

The second continued fraction of § 6 terminates because

$$\{i, s + 2n + 2\}\{i, s + 2n + 1\} = 0$$

when $n = \tfrac{1}{2}(i - s)$ or $\tfrac{1}{2}(i - s - 1)$, since one of the two factors then assumes the form $\{i, i + 1\}$.

Hence it follows that the equation for determining σ is the same as before; but we cannot on that account assume that the q coefficients vanish when their suffixes are greater than i.

In considering the P-functions it was immaterial whether or not we regarded them as vanishing, because P^t vanishes if t is greater than i.   But the Q-functions do not vanish in this case, and therefore we must postulate the existence of q's with suffix greater than i.

In fact, whilst we have as before, when i and s are both odd or both even,

$$\frac{2q_i}{q_{i-2}} = \frac{1}{i^2 - s^2 + \beta\sigma},$$

we also have

$$\frac{2q_{i+2}}{q_i} = \frac{1}{(i+2)^2 - s^2 + \beta\sigma - \tfrac{1}{4}\beta^2\{i, i+4\}\{i, i+3\}\left(\dfrac{2q_{i+4}}{q_{i+2}}\right)},$$

and similarly a fraction for $\dfrac{2q_{i+4}}{q_{i+2}}$, and so forth.

It follows therefore that while the q's with suffixes less than or equal to i depend on finite continued fractions, those with suffixes greater than i depend on infinite continued fractions.

It thus appears that while the first series in the expression for $\mathfrak{Q}_i^s$ or for Q_i^s has limits 1 to $\tfrac{1}{2}s$ or $\tfrac{1}{2}(s-1)$, as before, the limits of the second series are 1 to ∞.

Thus we have found an expansion for this class of functions in powers of β.

In the limited case in which the coefficients have been actually evaluated, namely, where the development is only carried as far as the squares of β, we have

$$q_{s+2} = \frac{1}{8(s+1)}, \qquad q_{s+4} = \frac{1}{128(s+1)(s+2)},$$

$$q'_{s+2} = \frac{s+2}{8s(s+1)}, \qquad q'_{s+4} = \frac{s+4}{128s(s+1)(s+2)}.$$

These coefficients do not vanish when $s+2$ or $s+4$ are greater than i, and this confirms the conclusion already arrived at.

In spherical harmonic analysis there is no occasion to consider the value of Q_i^s when s is greater than i, and the values are therefore not familiar. I will therefore now determine them.

It is known[*] that

$$\mathrm{Q}_i = \frac{2^i (i!)^2}{2i+1}\left[\frac{1}{\nu^{i+1}} + \frac{i+2!}{2 \cdot 1! \, i!} \cdot \frac{1}{(2i+3)\nu^{i+3}} + \frac{i+4!}{2^2 \cdot 2! \, i!} \cdot \frac{1}{(2i+3)(2i+5)\nu^{i+5}} + \cdots\right].$$

Therefore differentiating

$$\left(\frac{d}{d\nu}\right)^{i+1}\mathrm{Q}_i = (-)^{i+1}2^i \cdot i!\left[\frac{1}{\nu^{2i+2}} + \frac{i+1}{1!}\frac{1}{\nu^{2i+4}} + \frac{(i+1)(i+2)}{2!} \cdot \frac{1}{\nu^{2i+6}} + \cdots\right],$$

And

$$\left(\frac{d}{d\nu}\right)^{i+2}\mathrm{Q}_i = (-)^i\frac{2^{i+1} \cdot s + 1!}{(\nu^2 - 1)^{i+2}}\nu.$$

* BRYAN, 'Camb. Phil. Soc. Proc.,' vol. 6, 1888, p. 293.

$$\left(\frac{d}{d\nu}\right)^{i+2} Q_i = (-)^i 2^{i+1} \cdot i + 1! \left[-\frac{2(i+2)\nu^2}{(\nu^2-1)^{i+2}} + \frac{1}{(\nu^2-1)^{i+2}}\right],$$

$$= (-)^{i+1} \left[\frac{2^{i+2} \cdot i + 2!}{(\nu^2-1)^{i+2}} + \frac{2^{i+1}(2i+3) \cdot i + 1!}{(\nu^2-1)^{i+2}}\right].$$

$$\left(\frac{d}{d\nu}\right)^{i+4} Q_i = (-)^i \left[\frac{2^{i+2} \cdot i + 3!}{(\nu^2-1)^{i+4}} + \frac{2^{i+2}(2i+3) \cdot i + 2!}{(\nu^2-1)^{i+3}}\right]\nu.$$

But $\qquad Q_i^s = (\nu^2-1)^\mu \left(\frac{d}{d\nu}\right)^s Q_i,$

therefore $\qquad Q_i^{i+1} = (-)^{i+1} \dfrac{2^i \cdot i!}{(\nu^2-1)^{\frac{1}{2}(i+1)}}$

$$Q_i^{i+2} = (-)^{i+1} \frac{2^{i+1} \cdot i+1!}{(\nu^2-1)^{\frac{1}{2}(i+2)}} [2i+4 + (2i+3)(\nu^2-1)] \right\} \quad . \quad (39).$$

$$Q_i^{i+4} = (-)^i \frac{2^{i+2} \cdot i+2! \nu}{(\nu^2-1)^{\frac{1}{2}(i+6)}} [2i+6 + (2i+3)(\nu^2-1)]$$

These are all the functions which can be needed for the expression as far as β^3 of $\mathfrak{Q}_i^s$ or of Q_i^s when s is less or equal to i. If s is equal to i, we shall have terms $\beta^2 q_{i+4} Q^{i+4}$ or $\Omega\beta^2 q'_{i+4} Q'^{i+4}$, and these are the furthest.

But it is well known that there is another expression for these functions of the second kind.

The differential equation is

$$\left[(\nu^2-1)^2 \frac{d^2}{d\nu^2} + 2\nu(\nu^2-1)\frac{d}{d\nu} - i(i+1)(\nu^2-1) - s^2\right]\mathfrak{Q}_i^s$$

$$- \beta\left[(\nu^2-1)(\nu^2+1)\frac{d^2}{d\nu^2} + 2\nu^3\frac{d}{d\nu} - i(i+1)\nu^3 - \sigma\right]\mathfrak{Q}_i^s = 0,$$

where $\mathfrak{Q}_i^s$ may be interpreted as meaning also Q_i^s.

Let us assume that

$$\mathfrak{Q}_i^s = \mathfrak{P}_i^s \int_\nu^\infty V d\nu$$

is a solution, where $\mathfrak{Q}_i^s$, $\mathfrak{P}_i^s$ may be interpreted as meaning also Q_i^s, P_i^s.

Then since $\mathfrak{P}_i^s$ is a solution of the differential equation, we have

$$(\nu^2-1)^2 \left[2V\frac{d\mathfrak{P}_i^s}{d\nu} + \mathfrak{P}_i^s\frac{dV}{d\nu}\right] + 2\nu(\nu^2-1)\mathfrak{P}_i^s V$$

$$- \beta\left[(\nu^2-1)(\nu^2+1)\left(2V\frac{d\mathfrak{P}_i^s}{d\nu} + \mathfrak{P}_i^s\frac{dV}{d\nu}\right) + 2\nu^3\mathfrak{P}_i^s V\right] = 0.$$

This is eas'ly reducible to

$$\overset{i}{\underset{\nu}{}}\log\left[V\left(\mathfrak{P}^s_i\right)^2(\nu^2 - 1)^{\frac{1}{2}}\left(\nu^2 - \frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}\right] = 0,$$

whence $V = \dfrac{\mathfrak{E}^s_i}{(\mathfrak{P}^s_i)^2(\nu^2 - 1)^{\frac{1}{2}}(\nu^2 - \frac{1+\beta}{1-\beta})^{\frac{1}{2}}}$ where $\mathfrak{E}^s_i$ is a constant.

Hence

$$\left. \begin{aligned}
\mathfrak{Q}^s_i &= \mathfrak{E}^s_i \mathfrak{P}^s_i \int_\nu^\infty \frac{d\nu}{(\mathfrak{P}^s_i)^2(\nu^2 - 1)^{\frac{1}{2}}(\nu^2 - \frac{1+\beta}{1-\beta})^{\frac{1}{2}}} \\
Q^s_i &= E^s_i P^s_i \int_\nu^\infty \frac{d\nu}{(P^s_i)^2(\nu^2 - 1)^{\frac{1}{2}}(\nu^2 - \frac{1+\beta}{1-\beta})^{\frac{1}{2}}}
\end{aligned} \right\} \quad \cdots \cdots (40).$$

The general solution of the differential equation must be

$$\alpha \mathfrak{P}^s_i + \gamma \mathfrak{Q}^s_i,$$

and we have already found both $\mathfrak{P}^s_i$ and $\mathfrak{Q}^s_i$. Hence the two $\mathfrak{Q}^s_i$'s must be different expressions for the same thing, for the form of $\mathfrak{Q}^s_i$ as a series negatives the hypothesis that it involves $\mathfrak{P}^s_i$ in the form $\gamma_1\mathfrak{P}^s_i + \gamma_2\mathfrak{Q}^s_i$.

Having then two forms of $\mathfrak{Q}^s_i$ or of Q^s_i, it remains to evaluate the coefficients $\mathfrak{E}^s_i$, E^s_i, which are involved in the equations (40). In order to do this it will suffice to consider the case where ν is very great, so that

$$P^s_i = \frac{2i!}{2^i i!} \cdot \frac{\nu^i}{i - t!}, \qquad Q^s_i = (-)^i \frac{2i! \cdot i!}{2i + 1!} \cdot \frac{i + t!}{\nu^{i+1}}.$$

As far as concerns the first term in the series

$$\mathfrak{P}^s_i = \frac{2i!}{2^i i!} \cdot \frac{\nu^i}{i - s!}\left[1 + \beta q_{i-2} \frac{i - s!}{i - s + 2!} + \beta q_{i+2} \frac{i - s!}{i - s - 2!} \right.$$
$$\left. + \beta^2 q_{i-4} \frac{i - s!}{i - s + 4!} + \beta^2 q_{i+4} \frac{i - s!}{i - s - 4!} \right]$$

$$\mathfrak{Q}^s_i = (-)^i \frac{2i \cdot s!}{2i + 1!} \cdot \frac{i + s!}{\nu^{i+1}}\left[1 + \beta q_{i-2} \frac{i + s - 2!}{i + s!} + \beta q_{i+2} \frac{i + s + 2!}{i + s!} \right.$$
$$\left. + \beta^2 q_{i-4} \frac{i + s - 4!}{i + s!} + \beta^2 q_{i+4} \frac{i + s + 4!}{i + s!} \right].$$

It will be observed that if s is equal to i or $i - 1$ the terms in $\mathfrak{P}^s_i$ in q_{i+2} and q_{i+4} disappear; and if s is equal to $i - 2$ or $i - 3$ that in q_{i+4} disappears. This agrees, as it should do, with the vanishing of P^{i+2} and P^{i+4} when the order is greater than the degree.

If we write $\mathfrak{P}' = a\nu'$ and $\mathfrak{Q}' = \dfrac{\gamma}{\nu'^{i+1}}$, the first of our equations (40) becomes, when ν is very large,

$$\frac{\gamma}{\nu'^{i+1}} = \mathfrak{Q}'a\nu'\int_\nu^\infty \frac{d\nu}{\nu^2(a\nu')^2} = \mathfrak{Q}'\frac{\nu'}{a}\int_\nu^\infty \frac{d\nu}{\nu^{i+2}}$$

$$= \frac{\mathfrak{Q}'}{a}\cdot\frac{1}{\nu'^{i+1}}\cdot\frac{1}{2i+1}.$$

Therefore $\mathfrak{Q}' = (2i+1)\,a\gamma$, and since the a, γ in the case of the P', Q' only differ from these in the accenting of the q's we have

$$\mathfrak{Q}' = (-)^{\frac{i+s!}{i-s!}}\left[1+\beta q_{i-2}\frac{i-s!}{i-s+2!}+\dots\right]\left[1+\beta q_{i-2}\frac{i+s-2!}{i+s!}+\dots\right].$$

$\mathsf{E}' = $ the same with accented q's.

Effecting the multiplication of the series

$$\mathfrak{Q}' = (-)^r\frac{i+s!}{i-s!}\Big[1+\beta\Big(q_{i-2}\frac{i-s!}{i-s+2!}+q_{i-2}\frac{i+s-2!}{i+s!}$$
$$+q_{i+2}\frac{i-s!}{i-s-2!}+q_{i+2}\frac{i+s+2!}{i+s!}\Big)$$
$$+\beta^2\Big(q_{i-4}\frac{i-s!}{i-s+4!}+q_{i-4}\frac{i+s-4!}{i+s!}+q_{i+4}\frac{i-s!}{i-s-4!}$$
$$+q_{i+4}\frac{i+s+4!}{i+s!}+q_{i-2}q_{i-2}\frac{i-s!}{i+s!}\frac{i+s-2!}{i-s+2!}$$
$$+q_{i+2}q_{i+2}\frac{i-s!}{i+s!}\frac{i+s+2!}{i-s-2!}+q_{i-2}q_{i+2}\frac{i-s!}{i+s!}\frac{i+s+2!}{i-s+2!}$$
$$+q_{i+2}q_{i-2}\frac{i-s!}{i+s!}\frac{i+s-2!}{i-s-2!}\Big)\Big],$$

$\mathsf{E}' = $ the same with accented q's.

If we substitute for the q's their values, the coefficient of β inside $[\quad]$ in the expression for $\mathfrak{Q}'$ is

$$\tfrac{1}{8}\left[-\frac{(i+s)(i+s-1)}{s-1}-\frac{(i-s+1)(i-s+2)}{s-1}+\frac{(i-s)(i-s-1)}{s+1}\right.$$
$$\left.+\frac{(i+s+1)(i+s+2)}{s+1}\right].$$

In the expression for E' the first pair of these terms are multiplied by $\dfrac{s-2}{s}$, and the second pair by $\dfrac{s+2}{s}$

The coefficient of β^2 in the expression for $\mathbb{C}^i$ is

$$\tfrac{1}{128}\left[\frac{(i+s)(i+s-1)(i+s-2)(i+s-3)}{(s-1)(s-2)} + \frac{(i-s+1)(i-s+2)(i-s+3)(i-s+4)}{(s-1)(s-2)}\right.$$

$$+ \frac{(i-s)(i-s-1)(i-s-2)(i-s-3)}{(s+1)(s+2)} + \frac{(i+s+1)(i+s+2)(i+s+3)(i+s+4)}{(s+1)(s+2)}$$

$$+ \frac{2(i+s)(i+s-1)(i-s+1)(i-s+2)}{(s-1)^2} + \frac{2(i-s)(s-1)(i+s+1)(i+s+2)}{(s+1)^2}$$

$$\left.- \frac{2(i+s-1)(i+s)(i+s+1)(i+s+2)}{(s-1)(s+1)} - \frac{2(i-s-1)(i-s)(i-s+1)(i-s+2)}{(s-1)(s+1)}\right].$$

In the expression for E^i the first pair of these terms are multiplied by $\dfrac{s-\tfrac{4}{s}}{s}$; the second pair by $\dfrac{s+4}{s}$; the first of the third pair by $\left(\dfrac{s-2}{s}\right)^2$, and the second by $\left(\dfrac{s+2}{s}\right)^2$; and the last pair by $\dfrac{s^2-4}{s^2}$.

The reduction of terms such as these will occur frequently hereafter, and I will therefore say a word on the most convenient way of carrying it out. It is obvious that the coefficient of β may be arranged in the form

$$A i (i + 1) + B (2i + 1) + C.$$

The coefficient A is equal to the coefficient of i^2 in the original expression, and if we put $i = 0$ we have $B + C$, and with $i = -1$, $-B + C$. Hence A, B, C may be easily determined.

Again the coefficient of β^2 may be arranged in the form

$$A i^2 (i + 1)^2 + B (2i + 1) i (i + 1) + C i (i + 1) + D (2i + 1) + E.$$

This may be written

$$A i^4 + 2 (A + B) i^3 + (A + 3B + C) i^2 + (B + C + D) i + D + E.$$

It is easy to pick out the coefficients of i^4, i^3, i^2, and we thus obtain A, B, C. Then putting i successively equal to 0 and -1 we have $D + E$ and $-D + E$.

In order to express the results succinctly I use as before the notation

$$\Sigma^i = \frac{i(i+1)}{s^2-1}, \qquad \mathrm{T}_i^s = \frac{(i-1)i(i+1)(i+2)}{\ }.$$

and I usually omit the superscript and subscript s and i.

Proceeding in this way I find

$$
\begin{aligned}
\mathfrak{C}_i^s = (-)^s \frac{i+s!}{i-s!} \{1 &- \tfrac{1}{2}\beta(\Sigma-1) + \tfrac{1}{3\cdot8}\beta^2[-s^2(\Sigma^2+2\Sigma-1) \\
&+ 3(\Sigma^2-2\Sigma+2) + 2\mathrm{T}]\} \\
\mathsf{E}_i^s = (-)^s \frac{i+s!}{i-s!} \{1 &+ \tfrac{1}{2}\beta(\Sigma+3) + \tfrac{1}{3\cdot8}\beta^2[s^2(3\Sigma^2-2\Sigma+1) \\
&- (\Sigma^2-26\Sigma-42) - 2\mathrm{T}]\}
\end{aligned}
\tag{41}
$$

These results may be verified, for if we multiply $\mathfrak{C}^s$ by $\frac{1}{(C')^s}$, as given in (33), we ought to find E^s; and this is so.

The formulæ apparently fail when $s = 0, 1, 2, 3$; but when $s = 3$ they still hold good because, as remarked above, the general formula for $s = 3$ gives correct results when properly interpreted. Thus it only remains to consider $s = 0, 1, 2$.

When $s = 2$ the coefficients of β remain as in (41). In the coefficients of β^2

$$
q_{i-4} = 0, \quad q_{i-2} = -\tfrac{1}{8}\{i,2\}\{i,1\}, \quad q_{i+2} = \tfrac{1}{8\cdot3}, \quad q_{i+4} = \tfrac{1}{13\cdot8}\tfrac{1}{3\cdot4}
$$
$$
q'_{i-4} = 0, \quad q'_{i-2} = 0, \qquad\qquad q'_{i+2} = \tfrac{1}{4\cdot3}, \quad q'_{i+4} = \tfrac{1}{13\cdot8}\tfrac{1}{4}.
$$

In the expression for $\mathfrak{C}_i^2$ the coefficient of β^2 inside the bracket is

$$
\tfrac{1}{13\cdot8}\tfrac{1}{3\cdot4}[3(i-2)(i-3)(i-4)(i-5) + 3(i+3)(i+4)(i+5)(i+6)
$$
$$
+ 72(i-1)i(i+1)(i+2) + 8(i-2)(i-3)(i+3)(i+4)
$$
$$
- 24(i+1)(i+2)(i+3)(i+4) - 24(i-3)(i-2)(i-1)i]. \tag{42}
$$

Effecting the reduction and writing Σ for $\tfrac{1}{2}i(i+1)$, we find

$$
\mathfrak{C}_i^2 = \frac{i+2!}{i-2!}\{1 - \tfrac{1}{2}\beta(\Sigma-1) + \tfrac{1}{13\cdot8}\beta^2(19\Sigma^2 - 130\Sigma + 80)\} \quad . \quad . \quad (43)
$$

The coefficient of β^2 for E^2 may be got from (42) thus:—Multiply the first and second terms by 3, erase the third, fifth, and sixth terms, and multiply the fourth term by 4.

Effecting the reduction we find

$$
\mathsf{E}_i^2 = \frac{i+2!}{i-2!}\{1 + \tfrac{1}{2}\beta(\Sigma+3) + \tfrac{1}{13\cdot8}\beta^2(25\Sigma^2 + 186\Sigma + 368)\} \quad . \quad . \quad (44)
$$

Observe that there is no factor by which $\mathfrak{P}^2$ can be converted into P^2, so that this case cannot be verified like the general one.

When $s = 1$ we have

$$q_{s-4} = 0, \qquad q_{s-2} = 0, \qquad q_{s+2} = \tfrac{1}{16}[1 - \tfrac{1}{16}\beta i(i+1)], \qquad q_{s+4} = \tfrac{1}{768}$$

$$q'_{s-4} = 0, \qquad q'_{s-2} = 0, \qquad q'_{s+2} = \tfrac{3}{16}[1 + \tfrac{1}{16}\beta i(i+1)], \qquad q'_{s+4} = \tfrac{3}{768}.$$

The terms in βq_{s+2} and $\beta q'_{s+2}$ now contribute to the terms in β^2.

For $\mathfrak{C}_i^1$ the term in β inside the bracket is

$$\tfrac{1}{8\cdot 2}[(i-1)(i-2) + (i+2)(i+3)] = \tfrac{1}{8}[i(i+1) + 4].$$

The term in β^2, of which the first portion is carried over from the term in β, is

$$- \tfrac{1}{256}i(i+1)[(i-1)(i-2) + (i+2)(i+3)]$$
$$+ \tfrac{1}{128}\tfrac{1}{2\cdot 3}[(i-1)(i-2)(i-3)(i-4) + (i+2)(i+3)(i+4)(i+5)$$
$$+ 3(i-1)(i-2)(i+2)(i+3)].$$

This is equal to $-\tfrac{1}{768}[i^2(i+1)^2 - 56i(i+1) - 180].$

As we cannot now use the abridged notation with $\Sigma_i{}^1$, which is infinite, I write

$$j = i(i+1).$$

Thus $\qquad \mathfrak{C}_i^1 = -\genfrac{}{}{0pt}{}{i+1!}{i-1!}[1 + \tfrac{1}{8}\beta(j+4) - \tfrac{1}{768}\beta^2(j^2 - 56j - 180)] \Bigg\}$ . . (45).

For E_i^1 the coefficient of β is three times as great as before, and the coefficient of β^2 is

$$\tfrac{3}{128}i(i+1)[i(i+1)+4] + \tfrac{1}{128}\tfrac{1}{2\cdot 3}[5(i-1)(i-2)(i-3)(i-4)$$
$$+ 5(i+2)(i+3)(i+4)(i+5) + 27(i-1)(i-2)(i+2)(i+3)].$$

On effecting the reduction I find

$$\mathsf{E}_i^1 = -\genfrac{}{}{0pt}{}{i+1!}{i-1!}[1 + \tfrac{3}{8}\beta(j+4) + \tfrac{1}{768}\beta^2(55j^2 + 376j + 1044)]. \quad . (46).$$

When $s = 0$ we have only $\mathfrak{C}_i$ to determine.   Here

$$q_{s-4} = q_{s-2} = 0, \qquad q_{s+2} = \tfrac{1}{4}, \qquad q_{s+4} = \tfrac{1}{192}.$$

The term in β is $\tfrac{1}{4}[i(i-1) + (i+1)(i+2)] = \tfrac{1}{2}(j+1)$.

That in β^2 is

$$\tfrac{1}{192}[i(i-1)(i-2)(i-3) + (i+1)(i+2)(i+3)(i+4)$$
$$+ 8i(i-1)(i+1)(i+2)] = \tfrac{1}{64}(5j^2 + 14j + 12).$$

Therefore

$$\mathfrak{E}_i = 1 + \tfrac{1}{2}\beta(j+1) + \tfrac{1}{64}\beta^2(5j^2 + 14j + 12) \\ = 1 - \tfrac{1}{2}\beta(\Sigma - 1) + \tfrac{1}{64}\beta^2(5\Sigma^2 - 14\Sigma + 12) \Big\} \quad . \quad . \quad . \ (47).$$

since

$$\Sigma_i = -i(i+1) = -j.$$

Collecting results from (41), (43), (44), (45), 46), and (47),

$$(s > 2) \ \mathfrak{E}_i^s = (-)^s \frac{s+s!}{i-s!}\{1 - \tfrac{1}{2}\beta(\Sigma - 1) \\ + \tfrac{1}{32}\beta^2[-s^2(\Sigma^2 + 2\Sigma - 1) + 3(\Sigma^2 - 2\Sigma + 2) + 2T]\},$$

$$\mathfrak{E}_i^2 = \frac{s+2!}{i-2!}\{1 - \tfrac{1}{2}\beta(\Sigma - 1) + \tfrac{1}{320}\beta^2[19\Sigma^2 - 130\Sigma + 80]\},$$

$$\mathfrak{E}_i^1 = -\frac{s+1!}{s-1!}\{1 + \tfrac{1}{8}\beta(j+4) - \tfrac{1}{768}\beta^2(j^3 - 56j - 180]\},$$

$$\mathfrak{E}_i = 1 + \tfrac{1}{2}\beta(j+1) + \tfrac{1}{64}\beta^2(5j^2 + 14j + 12).$$

$$(s > 2) \ \mathsf{E}_i^s = (-)^s \frac{s+s!}{i-s!}\{1 + \tfrac{1}{2}\beta(\Sigma + 3) \\ + \tfrac{1}{32}\beta^2[s^2(3\Sigma^2 - 2\Sigma + 1) - (\Sigma^2 - 26\Sigma - 42) - 2T]\},$$

$$\mathsf{E}_i^2 = \frac{i+2!}{i-2!}\{1 + \tfrac{1}{2}\beta(\Sigma + 3) + \tfrac{1}{320}\beta^2[25\Sigma^2 + 186\Sigma + 368]\},$$

$$\mathsf{E}_i^1 = -\frac{i+1!}{i-1!}\{1 + \tfrac{3}{8}\beta(j+4) + \tfrac{1}{768}\beta^2(55j^2 + 376j + 1044)\},$$

where

$$\Sigma = \frac{i(i+1)}{s^2-1}, \qquad T = \frac{(i-1)i(i+1)(i+2)}{s^2-4}, \qquad j = i(i+1)$$

48).

PART II.

APPLICATION OF ELLIPSOIDAL HARMONIC ANALYSIS.

§ 11. *The Potential of an harmonic deformation of an Ellipsoid.*

A solid harmonic, or solution of LAPLACE'S equation, is the product of two P-functions of ν and of μ respectively, and of a cosine or sine function of ϕ. A surface harmonic is a P-function of μ multiplied by a cosine or sine function of ϕ.

We found

$$\mathfrak{P}^s(\nu) = \mathsf{P}^s(\nu) + \Sigma\beta''q_{s-2s}\mathsf{P}^{s-2s}(\nu) + \Sigma\beta''q_{s+2s}\mathsf{P}^{s+2s}(\nu),$$

where $\mathsf{P}^s(\nu) = \frac{(\nu^2-1)^{\frac12 t}}{2^t \cdot i!}\left(\frac{d}{d\nu}\right)^{i+t}(\nu^2-1)^i$; and a similar formula held for $\mathsf{P}^s(\nu)$.

Hitherto we have supposed $\mathsf{P}^s(\mu)$ to have exactly the same form as $\mathsf{P}^s(\nu)$. But since μ is less than unity this introduces an imaginary factor when t is odd, and

makes the succession of P's alternately positive and negative when t is even. As this is practically inconvenient I now define

$$P^t(\mu) = \frac{(1 - \mu^2)^{\frac{1}{2}t}}{2^i \, i \, !} \left(\frac{d}{d\mu}\right)^{i + t} (\mu^2 - 1)^i,$$

and then retaining the former meaning for the q coefficients, we give the following definition—

$$\mathfrak{P}^t(\mu) = P^t(\mu) + \Sigma \, (-)^n \beta^n q_{i - 2n} P^{i - 2n}(\mu) + \Sigma \, (-)^n \beta^n q_{i + 2n} P^{i + 2n}(\mu) \, ;$$

with a similar formula for $P^s(\mu)$.

Thus we need only remark that in the functions of μ the q's corresponding to odd powers of β enter with the opposite sign from that which holds in the functions of ν, and the whole of our preceding results are true with this definition of $P^t(\mu)$.

If ν_0 defines the ellipsoid to which the surface harmonic applies, we require the expression for the perpendicular p on the tangent plane at ν_0, μ, ϕ, and that for an element of area of the surface of the ellipsoid at the same point.

By the usual formula

$$\frac{k^2}{p^2} = \frac{x^2}{k^2 \left(\nu_0^2 - \frac{1 + \beta}{1 - \beta}\right)^2} + \frac{y^2}{k^2 (\nu_0^2 - 1)^2} + \frac{z^2}{k^4 \nu_0^3}$$

$$= -\frac{(1 - \beta)\left(\mu^2 - \frac{1 + \beta}{1 - \beta}\right)}{(1 + \beta)\left(\nu_0^2 - \frac{1 + \beta}{1 - \beta}\right)} \cos^2\phi - \frac{\mu^2 - 1}{\nu_0^2 - 1} \sin^2\phi + \frac{\mu^2 (1 - \beta \cos 2\phi)}{\nu_0^3 (1 + \beta)}$$

$$= \frac{(\nu_0^2 - \mu^2)\left(\nu_0^2 - \frac{1 - \beta \cos 2\phi}{1 - \beta}\right)}{\nu_0^2 (\nu_0^2 - 1)\left(\nu_0^2 - \frac{1 + \beta}{1 - \beta}\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

Let dn, dm, df be the three elements of the orthogonal arcs corresponding to variations of ν, μ, ϕ respectively.

Then by the formula at the end of § 1,

$$\left(\frac{dn}{k^2 \nu_0 d\nu_0}\right)^2 = \frac{(\nu_0^2 - \mu^2)\left(\nu_0^2 - \frac{1 - \beta \cos 2\phi}{1 - \beta}\right)}{k^2 \nu_0^2 (\nu_0^2 - 1)\left(\nu_0^2 - \frac{1 + \beta}{1 - \beta}\right)} = \frac{1}{p^2},$$

$$\left(\frac{dm}{k^2 d\mu}\right)^2 = \frac{(\nu_0^2 - \mu^2)\left(\frac{1 - \beta \cos 2\phi}{1 - \beta} - \mu^2\right)}{k^2 (1 - \mu^2)\left(\frac{1 + \beta}{1 - \beta} - \mu^2\right)}$$

$$\left(\frac{df}{\frac{k\mu \sin 2\phi}{1 - \beta} d\phi}\right)^2 = \frac{\left(\nu_0^2 - \frac{1 - \beta \cos 2\phi}{1 - \beta}\right)\left(\frac{1 - \beta \cos 2\phi}{1 - \beta} - \mu^2\right)}{\frac{\beta^2 \sin^2 2\phi}{(1 - \beta)^2} (1 - \beta \cos 2\phi)} \, .$$

$$\quad (50).$$

Therefore

$$\left(\frac{df}{k d\phi}\right)^2 = \frac{\left(\nu_0^2 - \frac{1 - \beta \cos 2\phi}{1 - \beta}\right)\left(\frac{1 - \beta \cos 2\phi}{1 - \beta} - \mu^2\right)(1 - \beta)}{1 - \beta \cos 2\phi} \quad . \quad . \quad . \quad (50),$$

and

$$\left(\frac{p\,dm\,df}{k^2 d\mu\,d\phi}\right)^2 = \nu_0{}^2(\nu_0{}^2 - 1)\left(\nu_0{}^2 - \frac{1+\beta}{1-\beta}\right)(1-\beta)\ \left(1 - \beta\cos 2\phi\right)(1-\mu^2)\frac{\left(\frac{1-\beta\cos 2\phi}{1-\beta} - \mu^2\right)^2}{\left(\frac{1+\beta}{1-\beta} - \mu^2\right)}, \Bigg\}\ (50).$$

$$\frac{d}{dn} = \frac{p}{k^2}\cdot\frac{d}{\nu_0 d\nu_0}.$$

Two functions, written in alternative form,

$$A\,\mathfrak{P}_i{}^s\left(\left\{{\nu\atop\nu_0}\right\}\right)\mathfrak{Q}_i{}^s\left(\left\{{\nu_0\atop\nu}\right\}\right)\mathfrak{P}_i{}^s(\mu)\,\mathfrak{C}_i{}^s(\phi),$$

are solutions of LAPLACE's equation, and together form a function V continuous at the surface of the ellipsoid $\nu = \nu_0$. Reading the upper line we have a function always finite inside the ellipsoid, and reading the lower line one always finite outside. Hence V is the potential of a layer of surface density on the ellipsoid ν_0, and by POISSON's equation that density is equal to $-\dfrac{1}{4\pi}\left[\dfrac{dV}{dn}\text{ (outside)} - \dfrac{dV}{dn}\text{ (inside)}\right]$.

Let the surface density, which it is our object to find, be

$$p\,\mathfrak{P}_i{}^s(\mu)\,\mathfrak{C}_i{}^s(\phi)\cdot\rho,$$

a surface harmonic multiplied by the perpendicular on to the tangent plane and by a quantity ρ.

$$\rho = -\frac{A}{4\pi k^2\nu_0}\left[\mathfrak{P}_i{}^s(\nu_0)\frac{d}{d\nu_0}\mathfrak{Q}_i{}^s(\nu_0) - \mathfrak{Q}_i{}^s(\nu_0)\frac{d}{d\nu_0}\mathfrak{P}_i{}^s(\nu_0)\right].$$

But $$\mathfrak{Q}_i{}^s(\nu_0) = \mathfrak{C}_i{}^s\,\mathfrak{P}_i{}^s(\nu_0)\int_{\nu_0}^{\infty}\frac{d\nu}{[\mathfrak{P}_i{}^s(\nu)]^2(\nu^2-1)^{\frac12}(\nu^2 - \frac{1+\beta}{1-\beta})^{\frac12}}.$$

Differentiating this logarithmically we find

$$\rho = \frac{A\mathfrak{C}_i{}^s}{4\pi k^2\nu_0(\nu_0{}^2-1)(\nu_0{}^2 - \frac{1+\beta}{1-\beta})},\text{ a constant.}$$

Hence surface density $p\,\mathfrak{P}_i{}^s(\mu)\mathfrak{C}_i{}^s(\phi)\cdot\rho$, where ρ is constant, gives rise to potential

$${\text{inside}\atop\text{outside}}\ \text{the ellipsoid}\quad \frac{4\pi k^2\rho\nu_0}{\mathfrak{C}_i{}^s}(\nu_0{}^2-1)^{\frac12}\left(\nu_0{}^2 - \frac{1+\beta}{1-\beta}\right)^{\frac12}\mathfrak{P}_i{}^s\left(\left\{{\nu\atop\nu_0}\right\}\right)\mathfrak{Q}_i{}^s\left(\left\{{\nu_0\atop\nu}\right\}\right)\mathfrak{P}_i{}^s(\mu)\,\mathfrak{C}_i{}^s(\phi)$$
$$\text{.   .   .   .   (51)}.$$

The same investigation holds good with $\mathfrak{S}_i{}^s(\phi)$, or with P, Q, C, S in place of the corresponding letters above.

3 T 2

. Imagine that the surface of a homogeneous ellipsoid of density ρ, defined by ν_0, receives a normal displacement δn, such that

$$\delta n = p \cdot \epsilon \mathfrak{P}_i^s(\mu)\, \mathfrak{C}_i^s(\phi).$$

Then the equivalent surface density is $p \cdot \epsilon \rho \mathfrak{P}_i^s(\mu)\, \mathfrak{C}_i^s(\phi)$, and we can at once write down the expressions for the internal and external potentials by means of (51).

If $x_0,\ y_0,\ z_0$ be the co-ordinates of a point on the surface, it is clear that the co-ordinates of the corresponding point on the deformed surface are

$$x = x_0\Big(1 + \frac{p^2\epsilon\mathfrak{P}_i^s(\mu)\,\mathfrak{C}_i^s(\phi)}{k^2\left(\nu_0^2 - \frac{1+\beta}{1-\beta}\right)}\Big),\quad y = y_0\Big(1 + \frac{p^2\epsilon\mathfrak{P}_i^s(\mu)\,\mathfrak{C}_i^s(\phi)}{k^2(\nu_0^2 - 1)}\Big),$$

$$z = z_0\Big(1 + \frac{p^2\epsilon\mathfrak{P}_i^s(\mu)\,\mathfrak{C}_i^s(\phi)}{k^3\nu_0^2}\Big).$$

Hence the equation to the deformed surface is

$$\frac{x^2}{k^2\left(\nu_0^2 - \frac{1+\beta}{1-\beta}\right)} + \frac{y^2}{k^2(\nu_0^2 - 1)} + \frac{z^2}{k^2\nu_0^2}\quad 1 + 2\epsilon\mathfrak{P}_i^s(\mu)\,\mathfrak{C}_i^s(\phi)\quad . \quad . \quad (52),$$

or since $\dfrac{x^2}{k^2\left(\nu^2 - \frac{1+\beta}{1-\beta}\right)} + \dfrac{y^2}{k^2(\nu^2 - 1)} + \dfrac{z^2}{k^2\nu^2} = 1$, it may be written

$$(\nu^2 - \nu_0^2)\frac{k^2}{p^2} = 2\epsilon\mathfrak{P}_i^s(\mu)\,\mathfrak{C}_i^s(\phi).$$

If we substitute for $\dfrac{k^2}{p^2}$ its value from (49), this may be written in the form

$$\frac{(\nu^2 - \nu_0^2)(\mu^2 - \nu_0^2)\left(\frac{1-\beta\cos^2\phi}{1-\beta} - \nu_0^2\right)}{\nu_0^2(\nu_0^2 - 1)\left(\nu_0^2 - \frac{1+\beta}{1-\beta}\right)} = 2\epsilon\mathfrak{P}_i^s(\mu)\,\mathfrak{C}_i^s(\phi)\quad . \quad . \quad . \quad (52).$$

This is the equation in elliptic co-ordinates to the deformed surface, but in actual computation the form involving rectangular co-ordinates might perhaps be more convenient.

§ 12. *The Potential of a homogeneous solid Ellipsoid.*

It is well known that the potential of a solid ellipsoid externally is equal to that of a "focaloid" shell of the same mass coincident with its external surface.

If ρ' be the density of the shell defined by ν_0 and $\nu_0 + \delta\nu$, we have

$$\tfrac{4}{3}\pi k^3\rho'\Big[\Big(\nu_0^2 + 2\nu_0\delta\nu - \frac{1+\beta}{1-\beta}\Big)^{\frac{1}{2}}(\nu_0^2 + 2\nu_0\delta\nu - 1)^{\frac{1}{2}}(\nu_0^2 + 2\nu_0\delta\nu)^{\frac{1}{2}}$$

$$- \Big(\nu_0^2 - \frac{1+\beta}{1-\beta}\Big)^{\frac{1}{2}}(\nu_0^2 - 1)^{\frac{1}{2}}\nu_0\Big] = \tfrac{4}{3}\pi k^3\rho\Big(\nu_0^2 - \frac{1+\beta}{1-\beta}\Big)^{\frac{1}{2}}(\nu_0^2 - 1)^{\frac{1}{2}}\nu_0.$$

Therefore
$$\rho' \nu_0 \delta\nu \left(\frac{1}{\nu_0{}^2 - \frac{1+\beta}{1-\beta}} + \frac{1}{\nu_0{}^2 - 1} + \frac{1}{\nu_0{}^2} \right) = \rho,$$

or
$$\rho' \nu_0 \delta\nu = \frac{\rho \nu_0{}^2 (\nu_0{}^2 - 1)\left(\nu_0{}^2 - \frac{1+\beta}{1-\beta}\right)}{3\nu_0{}^4 - \frac{4\nu_0{}^2}{1-\beta} + \frac{1+\beta}{1-\beta}}. \qquad \dots \qquad (53).$$

If δn be the thickness of the shell at the point where p is the perpendicular on the tangent plane,

$$\delta n = \nu_0 \delta\nu \cdot \frac{k^2}{p}.$$

If we multiply both sides of (53) by $\dfrac{k^2}{p}$, we see that the surface density of the focaloid shell is

$$p \cdot \frac{\rho \nu_0{}^2 (\nu_0{}^2 - 1)\left(\nu_0{}^2 - \frac{1+\beta}{1-\beta}\right)}{3\nu_0{}^4 - \frac{4\nu_0{}^2}{1-\beta} + \frac{1+\beta}{1-\beta}} \cdot \frac{k^2}{p^2}.$$

If therefore we can express $\dfrac{k^2}{p^2}$ in the form of surface harmonics, it will be easy to write down the external potential of the ellipsoid by means of the formula (51).

Before doing this I will, however, take one other step.

It is easy to see that

$$3\nu_0{}^4 - \frac{4\nu_0{}^3}{1-\beta} + \frac{1+\beta}{1-\beta} = 3\left(\nu_0{}^2 - \frac{2+B}{3(1-\beta)}\right)\left(\nu_0{}^2 - \frac{2-B}{3(1-\beta)}\right). \quad \dots \quad (54),$$

where for brevity $B = (1 + 3\beta^2)^{\frac{1}{2}}$.

Now on referring to § 7, (17) and (23), we see that

$$\mathfrak{P}_2(\nu) = P_2(\nu) + \frac{B-1}{6\beta} P_2{}^2(\nu), \qquad \mathfrak{P}_2{}^2(\nu) = -\frac{2(B-1)}{\beta} P_2(\nu) + P_2{}^2(\nu),$$

where
$$P_2(\nu) = \tfrac{3}{2}\nu^2 - \tfrac{1}{2}, \qquad P_2{}^2(\nu) = 3(\nu^2 - 1).$$

If then we put
$$\mathfrak{P}_2(\nu) = \alpha\nu^2 + \gamma,$$
$$\mathfrak{P}_2{}^2(\nu) = -\alpha'\nu^2 - \gamma', \qquad \text{or} \qquad \mathfrak{P}_2{}^2(\mu) = \alpha'\mu^2 + \gamma',$$

it is clear that

$$\alpha = \frac{B-1+3\beta}{2\beta}, \qquad \gamma = \frac{-B+1-\beta}{2\beta};$$

$$\alpha' = \frac{3(B-1-\beta)}{\beta}, \qquad \gamma' = \frac{-B+1+3\beta}{\beta};$$

and
$$\frac{\gamma}{\alpha} = \frac{B-2}{3(1-\beta)}, \qquad \frac{\gamma'}{\alpha'} = \frac{-B-2}{3(1-\beta)}.$$

It is obvious then that our expression (54) is equal to $-\dfrac{3}{aa'}\,\mathfrak{P}_2(\nu_0)\,\mathfrak{P}_2{}^2(\nu_0)$.   Then

since $-\dfrac{3}{aa'} = \dfrac{1+B}{3(1-\beta)}$, we have the surface density of the focaloid given by

$$p \cdot \frac{3(1+\beta)}{1+B} \cdot \frac{\rho\nu_0{}^2(\nu_0{}^2-1)\left(\nu_0{}^2-\frac{1+\beta}{1-\beta}\right)}{\mathfrak{P}_2(\nu_0)\,\mathfrak{P}_2{}^2(\nu_0)} \cdot \left(\frac{k}{p}\right)^2,$$

where

$$\frac{k^2}{p^2} = \frac{x_0{}^2}{k^2\left(\nu_0{}^2-\frac{1+\beta}{1-\beta}\right)^2} + \frac{y_0{}^2}{k^2(\nu_0{}^2-1)^2} + \frac{z_0{}^2}{k^2\nu_0{}^4}.$$

But since

$$\frac{x_0{}^2}{k^2\left(\nu_0{}^2-\frac{1+\beta}{1-\beta}\right)} + \frac{y_0{}^2}{k^2(\nu_0{}^2-1)} + \frac{z_0{}^2}{k^2\nu_0{}^2} = 1 \quad . \quad . \quad . \quad . \quad . \quad (55),$$

we have

$$\frac{k^2}{p^2} = \frac{1}{\nu_0{}^2} + \frac{\frac{1+\beta}{1-\beta}\cdot x_0{}^2}{k^2\nu_0{}^2\left(\nu_0{}^2-\frac{1+\beta}{1-\beta}\right)^2} + \frac{y_0{}^2}{k^2\nu_0{}^2(\nu_0{}^2-1)^2} \cdot \quad . \quad . \quad . \quad . \quad (56).$$

With the object of writing this function in surface harmonics, and besides to enable us to express a rotation potential in similar form, we have to reduce x^2, y^2, z^2 in the required manner.

I now drop the suffix zero, since we are not concerned with any particular ellipsoid.

Referring again to §7, (18) and (24), we have

$$\mathbb{C}_2(\phi) = 1 - \frac{B-1}{\beta}\cos 2\phi, \qquad \mathbb{C}_2{}^2(\phi) = \frac{B-1}{3\beta} + \cos 2\phi.$$

If then we put

$$\epsilon = \frac{2(1-B)}{\beta}, \qquad \zeta = \frac{B-1+\beta}{\beta};$$

$$\epsilon' = 2, \qquad \zeta' = \frac{B-1-3\beta}{3\beta};$$

we may write

$$\mathbb{C}_2(\phi) = \epsilon\cos^2\phi + \zeta, \qquad \mathbb{C}_2{}^2(\phi) = \epsilon'\cos^2\phi + \zeta'.$$

Let us assume, if possible,

$$k^2\left(\frac{1-\beta}{1+\beta}\right)\frac{-x^2}{\left(\nu^2-\frac{1+\beta}{1-\beta}\right)} = F\mathfrak{P}_2(\mu)\,\mathbb{C}_2(\phi) + G\mathfrak{P}_2{}^2(\mu)\,\mathbb{C}_2{}^2(\phi) + H,$$

or $\left(\mu^2-\dfrac{1+\beta}{1-\beta}\right)\cos^2\phi = F(\alpha\mu^2+\gamma)\,(\epsilon\cos^2\phi+\zeta)$

$$+ G(\alpha'\mu^2+\gamma')\,(\epsilon'\cos^2\phi+\zeta') + H.$$

From which it follows that

$$F\alpha\zeta + G\alpha'\zeta' = 0, \qquad F\gamma\zeta + G\gamma'\zeta' + H = 0,$$

$$F\alpha\epsilon + G\alpha'\epsilon' = 1, \qquad F\gamma\epsilon + G\gamma'\epsilon' = -\frac{1+\beta}{1-\beta}.$$

These equations give

$$F = \frac{1}{a\zeta} \cdot \frac{1}{\frac{\epsilon}{\zeta} - \frac{\epsilon'}{\zeta'}}, \quad G = -\frac{1}{a'\zeta'} \cdot \frac{1}{\frac{\epsilon}{\zeta} - \frac{\epsilon'}{\zeta'}}, \quad H = -\frac{\frac{\gamma}{a} - \frac{\gamma'}{a'}}{\frac{\epsilon}{\zeta} - \frac{\epsilon'}{\zeta'}},$$

and the condition

$$(1 - \beta)\left(\frac{\gamma\epsilon}{a\zeta} - \frac{\gamma'\epsilon'}{a'\zeta'}\right) + (1 + \beta)\left(\frac{\epsilon}{\zeta} - \frac{\epsilon'}{\zeta'}\right) = 0.$$

Now

$$\frac{\gamma\epsilon}{a\zeta} = \frac{1 + \beta - B}{1 - \beta}, \qquad \frac{\gamma'\epsilon'}{a'\zeta'} = \frac{1 + \beta + B}{1 - \beta},$$

$$\frac{\epsilon}{\zeta} = \frac{-1 - 3\beta + B}{1 + \beta}, \qquad \frac{\epsilon'}{\zeta'} = \frac{-1 - 3\beta - B}{1 + \beta}.$$

Since these values satisfy the condition amongst the coefficients, the assumed form for x^2 is justifiable.

I find then

$$F = \frac{1 + B}{2B} \cdot \frac{B - 2\beta}{3(1 - \beta)}, \qquad G = -\frac{1 + B}{4B} \cdot \frac{B + 2\beta}{3(1 - \beta)}, \qquad H = -\frac{1 + \beta}{3(1 - \beta)}.$$

Whence

$$\frac{3x^2}{k^2\left(\nu^2 - \frac{1+\beta}{1-\beta}\right)} = -\frac{1 + B}{2B} \cdot \frac{B - 2\beta}{1 + \beta} \, \mathfrak{P}_2(\mu) \, \mathbb{C}_2(\phi) + \frac{1 + B}{4B} \cdot \frac{B + 2\beta}{1 + \beta} \, \mathfrak{P}_2{}^2(\mu) \, \mathbb{C}_2{}^2(\phi) + 1. \quad (57)$$

This is the required expression for x^2 in surface harmonics.

Next assume

$$\frac{-y^2}{k^2(\nu^2 - 1)} = F_1 \mathfrak{P}_2(\mu) \, \mathbb{C}_2(\phi) + G_1 \mathfrak{P}_2{}^2(\mu) \, \mathbb{C}_2{}^2(\phi) + H_1.$$

If we put

$$\mathbb{C}_2(\phi) = \epsilon_1 \sin^2 \phi + \zeta_1, \qquad \mathbb{C}_2{}^2(\phi) = \epsilon_1{}' \sin^2 \phi + \zeta_1';$$

we have

$$\zeta_1 = \frac{-B + 1 + \beta}{\beta};$$

$$\qquad \qquad 3\beta$$

and

$$(\mu^2 - 1)\sin^2 \phi = F_1(a\mu^2 + \gamma)(\epsilon_1 \sin^2 \phi + \zeta_1) + G_1(a'\mu^2 + \zeta)(\epsilon_1{}' \sin^2 \phi + \zeta_1') + H_1.$$

Whence F_1, G_1, H_1 have the same forms as before, and the condition to be satisfied by the coefficients is

$$\frac{\gamma\epsilon_1}{a\zeta_1} - \frac{\gamma'\epsilon_1{}'}{a'\zeta_1'} + \frac{\epsilon_1}{\zeta_1} - \frac{\epsilon_1{}'}{\zeta_1'} = 0.$$

It will be found that the condition is satisfied, and that

$$F_1 = \frac{1 + B}{6B}, \qquad G_1 = \frac{1 + B}{12B}, \qquad H_1 = -\frac{1}{3}.$$

Hence

$$k^2 \frac{3y^2}{(\nu^2 - 1)} = -\frac{1 + B}{2B} \mathfrak{P}_2(\mu) \mathbb{C}_2(\phi) - \frac{1 + B}{4B} \mathfrak{P}_2{}^2(\mu) \mathbb{C}_2{}^2(\phi) + 1 \quad . \quad (58).$$

It follows from (55) that

$$\frac{3x^2}{k^2 \nu^2} = \frac{1 + B}{2B}\left(\frac{B - 2\beta}{1 + \beta} + 1\right) \mathfrak{P}_2(\mu) \mathbb{C}_2(\phi) - \frac{1 + B}{4B}\left(\frac{B + 2\beta}{1 + \beta} - 1\right) \mathfrak{P}_2{}^2(\mu) \mathbb{C}_2{}^2(\phi) + 1$$
$$. \quad . \quad . \quad . \quad (59).$$

Whence

$$\frac{3}{k^2}(y^2 + z^2) = \frac{1 + B}{2B}\left(1 + \nu^2 \frac{B - 2\beta}{1 + \beta}\right) \mathfrak{P}_2(\mu) \mathbb{C}_2(\phi)$$
$$+ \frac{1 + B}{4B}\left(1 - \nu^2 \frac{B + 2\beta}{1 + \beta}\right) \mathfrak{P}_2{}^2(\mu) \mathbb{C}_2{}^2(\phi) + 2\nu^2 - 1 \quad . \quad (60).$$

This is needed to express the rotation potential $\frac{1}{2}\omega^2(y^2 + z^2)$. If we add $\frac{3x^2}{k^2}$ to this

$$\frac{3}{k^2}(x^2 + y^2 + z^2) = \frac{1 + B}{2B} \cdot \frac{B + 1}{1 - \beta} - \frac{3\beta}{\beta} \mathfrak{P}_2(\mu) \mathbb{C}_2(\phi)$$
$$- \frac{1 + B}{4B} \cdot \frac{B - 1}{1 - \beta} + \frac{3\beta}{\beta} \mathfrak{P}_2{}^2(\mu) \mathbb{C}_2{}^2(\phi) + 3\nu^2 - 1 - \frac{2}{1 - \beta} \cdot \quad . \quad (61).$$

This expression will be needed hereafter.

Returning now to the formation of the expression for k^2/p^3, I find

$$\frac{3k^2}{p^3} = \frac{1}{\nu^3(\nu^2 - 1)\left(\nu^2 - \frac{1 + \beta}{1 - \beta}\right)}\left[-\frac{1 + B}{2B} \cdot \frac{\nu^2(B + 1 - 3\beta) - (B + 1 - \beta)}{1 - \beta} \mathfrak{P}_2(\mu) \mathbb{C}_2(\phi)\right.$$
$$+ \frac{1 + B}{4B} \cdot \frac{\nu^2(B - 1 + 3\beta) - (B - 1 + \beta)}{1 - \beta} \mathfrak{P}_2{}^2(\mu) \mathbb{C}_2{}^2(\phi)$$
$$\left. + 3\nu^4 - \frac{4\nu^2}{1 - \beta} + \frac{1 + \beta}{1 - \beta}\right].$$

On considering the forms of the functions $\mathfrak{P}_2(\nu)$, $\mathfrak{P}_2{}^2(\nu)$, it is found that this result may be written thus :

$$\frac{k^2}{p^3} = \frac{\mathfrak{P}_2(\nu) \mathfrak{P}_2{}^2(\nu)}{3\nu^3(\nu^2 - 1)\left(\nu^2 - \frac{1 + \beta}{1 - \beta}\right)} \cdot \frac{1 + B}{3(1 - \beta)}\left[1 + B \frac{\mathfrak{P}_2(\mu) \mathbb{C}_2(\phi)}{2B \mathfrak{P}_2(\nu)} + \frac{3\beta}{2B} \frac{\mathfrak{P}_2{}^2(\mu) \mathbb{C}_2{}^2(\phi)}{\mathfrak{P}_2{}^2(\nu)} + 1\right].$$

Therefore. writing $\mathfrak{P}_0(\mu) \mathbb{C}_0(\phi)$ for unity, the surface density of the focaloid shell, for which $\nu = \nu_0$, is

$$p\rho\left[\frac{1 + B}{6B} \frac{\mathfrak{P}_2(\mu) \mathbb{C}_2(\phi)}{\mathfrak{P}_2(\nu_0)} + \frac{\beta}{2B} \frac{\mathfrak{P}_2{}^2(\mu) \mathbb{C}_2{}^2(\phi)}{\mathfrak{P}_2{}^2(\nu_0)} + \frac{1}{3}\mathfrak{P}_0(\mu) \mathbb{C}_0(\phi)\right]. \quad . \quad . \quad (62)$$

By means of (51), we now at once write down the external potential of the ellipsoid. It is

$$V = \frac{M_0}{k}\left[\frac{1+B}{2B}\frac{\mathfrak{C}_2(\nu)}{\mathfrak{C}_2}\mathfrak{P}_2(\mu)\mathfrak{C}_2(\phi) + \frac{3\beta}{2B}\frac{\mathfrak{C}_2{}^2(\nu)}{\mathfrak{C}_2{}^2}\mathfrak{P}_2{}^2(\mu)\mathfrak{C}_2{}^2(\phi) + \frac{\mathfrak{C}_0(\nu)}{\mathfrak{C}_0}\right] \quad . \,(63).$$

In this expression M_0 denotes the mass of the ellipsoid, and the $\mathfrak{C}$'s are merely coefficients determined approximately in § 10.

In order to find the potential internally, let

$$r^2 = x^2 + y^2 + z^2;$$

and, as suggested by the form of (61), let

$$\frac{r_0{}^2}{k^2} = \frac{1+B}{6B} \cdot \frac{B+1}{1-\beta}\frac{1-3\beta}{\mathfrak{P}_2(\nu_0)}\mathfrak{P}_2(\mu)\mathfrak{C}_2(\phi)$$
$$-\frac{1+B}{12B} \cdot \frac{B-1+3\beta}{1-\beta}\frac{\mathfrak{P}_2{}^2(\nu)}{\mathfrak{P}_2{}^2(\nu_0)}\mathfrak{P}_2{}^2(\mu)\mathfrak{C}_2{}^2(\phi) + \nu_0{}^2 - \frac{2}{3(1-\beta)}.$$

Then $r_0{}^2$ is a solution of LAPLACE's equation throughout the interior of the ellipsoid, and at the surface, where $\nu = \nu_0$, it is equal to $x^2 + y^2 + z^2$.

Now consider the function

$$V = -\tfrac{2}{3}\pi\rho\,(r^2 - r_0{}^2) + \frac{M_0}{k}\left[\frac{1+B}{2B}\frac{\mathfrak{C}_2(\nu_0)}{\mathfrak{C}_2\mathfrak{P}_2(\nu_0)}\mathfrak{P}_2(\nu)\mathfrak{P}_2(\mu)\mathfrak{C}_2(\phi)\right.$$
$$\left.+\frac{3\beta}{2B}\frac{\mathfrak{C}_2{}^2(\nu_0)}{\mathfrak{C}_2{}^2\mathfrak{P}_2{}^2(\nu_0)}\mathfrak{P}_2{}^3(\nu)\mathfrak{P}_2{}^2(\mu)\mathfrak{C}_2{}^2(\phi) + \frac{\mathfrak{C}_0(\nu_0)}{\mathfrak{C}_0}\right]. \quad . \,(64).$$

The whole of it, excepting the term in r^2, is a solution of LAPLACE's equation for space inside the ellipsoid, and the term in r^2 gives $\nabla^2 V = -4\pi\rho$. Also at the surface, where $\nu = \nu_0$, this expression agrees with (63). Hence we have found the potential of the ellipsoid internally.

The potential at an internal point does not lend itself to expression in elliptic co-ordinates, but it may be given another form which is perhaps more convenient.

In our present notation the well-known formula is

$$V = \tfrac{3}{2}\frac{M_0}{k}\int_{\nu_0}^{\infty}\left(1 - \frac{x^2}{k^2(\nu^2 - \frac{1+\beta}{1-\beta})} - \frac{y^2}{k^2(\nu^2-1)} - \frac{z^2}{k^2\nu^2}\right)\frac{d\nu}{(\nu^2 - \frac{1+\beta}{1-\beta})^{\frac{1}{2}}(\nu^2-1)^{\frac{1}{2}}\nu} .$$

Since $\mathfrak{P}_0(\nu) = 1$, $\mathfrak{P}_1{}^1(\nu) = \left(\nu^2 - \frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}$, $\mathfrak{P}_1{}^1(\nu) = (\nu^2-1)^{\frac{1}{2}}$, $\mathfrak{P}_1(\nu) = \nu$, the integrals may be expressed in terms of the Q-functions, and we have (omitting the divisors $\mathfrak{C}$ and $\mathbf{E}$ for brevity)

$$V = \tfrac{3}{2}\frac{M_0}{k}\left(\frac{\mathfrak{C}_0(\nu_0)}{\mathfrak{P}_0(\nu_0)} - \frac{x^2}{k^2}\frac{Q_1{}^1(\nu_0)}{P_1{}^1(\nu_0)} - \frac{y^2}{k^2}\frac{\mathfrak{C}_1{}^1(\nu_0)}{\mathfrak{P}_1{}^1(\nu_0)} - \frac{z^2}{k^2}\frac{\mathfrak{C}_1(\nu_0)}{\mathfrak{P}_1(\nu_0)}\right) \quad . \,. \,(65).$$

In this we may substitute the expressions for x^2, y^2, z^2 found above.

It may be worth noting that

$$\frac{Q_1{}^1(\nu)}{P_1{}^1(\nu)} + \frac{\mathfrak{C}_1{}^1(\nu)}{\mathfrak{P}_1{}^1(\nu)} + \frac{\mathfrak{C}_1(\nu)}{\mathfrak{P}_1(\nu)} = \frac{1}{\nu(\nu^2-1)^{\frac{1}{2}}(\nu^2 - \frac{1+\beta}{1-\beta})^{\frac{1}{2}}}.$$

Also $P_1{}^1(\nu) Q_1{}^1(\nu) + \mathfrak{P}_1{}^1(\nu)\mathfrak{Q}_1{}^1(\nu) + \mathfrak{P}_1(\nu)\mathfrak{Q}_1(\nu) + \mathfrak{P}_0(\nu)\mathfrak{Q}_0(\nu) = 0.$

This last follows from the fact that if a, b, c are the axes of the ellipsoid, and if Ψ denotes the function $\int_0^\iota \frac{du}{ABC}$ (proportional to our $\mathfrak{Q}_0(\nu_0)$), Ψ is a homogeneous function of degree -1 in a, b, c, and therefore

$$a\frac{d\Psi}{da} + b\frac{d\Psi}{db} + c\frac{d\Psi}{dc} = -\Psi.$$

§ 13. *Preparation for the Integration of the square of a surface harmonic over the Ellipsoid.*

If it is intended to express any function in harmonics, it is necessary to know the integrals over the surface of the ellipsoid of the squares of surface harmonics multiplied by the perpendicular on the tangent plane.

The surface harmonic has one of the eight forms

$$V_i{}^s = \lfloor \mathfrak{P}_i{}^s(\mu) \quad \text{or} \quad P_i{}^s(\mu)\rfloor \times \begin{bmatrix} \mathfrak{C}_i{}^s(\phi) & \text{or} & C_i{}^s(\phi) \\ \mathfrak{S}_i{}^s(\phi) & & S_i{}^s(\phi) \end{bmatrix},$$

and the P-functions are expressible in terms of the P's where

$$P_i{}^s(\mu) = \frac{(1-\mu^2)^{\frac{1}{2}s}}{2^i \cdot i!}\left(\frac{d}{d\mu}\right)^{i+s}(\mu^2-1)^i.$$

I shall in this portion of the investigation frequently write $\mu = \sin\theta$, and shall omit the μ or θ or ϕ in the P-, C-, S-functions. Also I may very generally omit the subscript i, as elsewhere.

If $d\sigma$ denotes the element of surface of the ellipsoid, and

$$M = k^3\nu(\nu^2-1)^{\frac{1}{2}}\left(\nu^2 - \frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}},$$

so that $\frac{4}{3}\pi M$ is the volume of the ellipsoid, we have, by (50) of § 11,

$$\frac{pd\sigma}{d\theta d\phi} = \frac{M(1-\beta)^{\frac{1}{2}}}{(1-\beta\cos2\phi)^{\frac{1}{2}}} \cdot \frac{(\cos^2\theta + \frac{\beta}{1-\beta} - \frac{\beta\cos2\phi}{1-\beta})}{(\frac{1+\beta}{1-\beta} - \sin^2\theta)^{\frac{1}{2}}},$$

Then $\int p(V_i{}^s)^2 d\sigma = M(1-\beta)^{\frac{1}{2}}\iint \frac{\cos^2\theta + \frac{\beta}{1-\beta} - \frac{\beta\cos2\phi}{1-\beta}}{(1-\beta\cos2\phi)^{\frac{1}{2}}(\frac{1+\beta}{1-\beta} - \sin^2\theta)^{\frac{1}{2}}}(V_i{}^s)^2 d\theta d\phi,$

where the limits of θ are $\frac{1}{2}\pi$ to $-\frac{1}{2}\pi$, and of ϕ are 2π to 0.

It will be legitimate to develop $p\,d\sigma$ in powers of $\sec^2\theta$ up to any given power, provided $(V_i^s)^2$ involves as a factor such a power of $\cos^2\theta$ that the whole function to be integrated does not become infinite at the poles where $\theta = \pm \frac{1}{2}\pi$.

I shall at present limit the developments to the square of β.

We know that P^s is of the same form as $\mathfrak{P}^s$, but with the additional factor

$$\left(\frac{\frac{1+\beta}{1-\beta} - \sin^2\theta}{\cos^2\theta}\right)^{\frac{1}{2}}.$$

Suppose then that

$$\Pi_0 + \beta\Pi_1 + \beta^2\Pi_2 = (\mathfrak{P}^s)^2 \quad\text{or}\quad \frac{\cos^2\theta}{\frac{1+\beta}{1-\beta} - \sin^2\theta}(\mathsf{P}^s)^2;$$

and let

$$\gamma = 1 - \cos 2\phi,$$

then we put

$$F_1 = \frac{\cos^2\theta + \gamma(\beta + \beta^2)}{(\frac{1+\beta}{1-\beta} - \sin^2\theta)^{\frac{1}{2}}}(\Pi_0 + \beta\Pi_1 + \beta^2\Pi_2),$$

and

$$F_2 = \frac{\frac{1+\beta}{1-\beta} - \sin^2\theta}{\cos^2\theta}F_1.$$

Now suppose that K^2, a function independent of θ, denotes one of the four

$$\frac{(\mathfrak{C}^s)^2 \text{ or } (\mathfrak{S}^s)^2 \text{ or } (\mathsf{C}^s)^2 \text{ or } (\mathsf{S}^s)^2}{(1 - \beta\cos 2\phi)^{\frac{1}{2}}}.$$

Then in the cases involving $\mathfrak{P}$-functions and P-functions respectively, we have in alternative form—

$$\text{for } \begin{Bmatrix}\mathfrak{P}^s \\ \mathsf{P}^s\end{Bmatrix}, \quad \int p\,(V_i^s)^2 d\sigma = M\,(1 - \beta)^{\frac{1}{2}} \iint K^2 \begin{Bmatrix}F_1 \\ F_2\end{Bmatrix} d\theta\,d\phi.$$

If it be supposed that the development in powers of $\sec^2\theta$ is justifiable

$$F_1 = \cos\theta\left[1 + \frac{\gamma(\beta + \beta^2)}{\cos^2\theta}\right]\left[1 - \frac{\beta + \beta^2}{\cos^2\theta} + \frac{3}{2}\frac{\beta^2}{\cos^4\theta}\right][\Pi_0 + \beta\Pi_1 + \beta^2\Pi_2],$$

$$= \Pi_0\cos\theta + \beta\left[\frac{\Pi_0(\gamma - 1)}{\cos\theta} + \Pi_1\cos\theta\right] + \beta^2\left[\frac{\Pi_0(\gamma - 1)}{\cos^3\theta} + \frac{\Pi_0(\frac{3}{2} - \gamma)}{\cos^3\theta}\right.$$
$$\left. + \frac{\Pi_1(\gamma - 1)}{\cos\theta} + \Pi_2\cos\theta\right].$$

And F_2 has a similar form, save that $\gamma + 1$ replaces $\gamma - 1$, and $\gamma - \frac{1}{2}$ replaces $\frac{3}{2} - \gamma$.

It is clear that unless Π_0 is divisible by $\cos^3\theta$ and Π_1 by $\cos\theta$, $\int F_1 d\theta$ and $\int F_2 d\theta$ will have infinite elements at the poles, and the development is not legitimate.

Since $\mathsf{P}^s = \frac{\cos^s\theta}{2^s \cdot s!}\frac{d^{i+s}}{d\mu^{i+s}}(\mu^2 - 1)^i$, it follows that the power of $\cos\theta$ by which P^s is divisible increases as s increases.

Let us consider the case of $s = 2$.

Then

$$\Pi_0 + \beta\Pi_1 + \beta^2\Pi_2 = \lceil P^2 - \beta q_0 P - \beta q_4 P^4 + \beta^2 q_0 P^6 \rceil^2$$
$$= (P^2)^2 - 2\beta (q_0 PP^2 + q_4 P^2 P^4)$$
$$+ \beta^2 [2q_0 P^2 P^6 + (q_0 P)^2 + (q_4 P^4)^2 + 2q_0 q_4 PP^4]$$

(or the same with accented q's for the other case).

From this it is clear that Π_0 is divisible by $\cos^4\theta$ and Π_1 by $\cos^2\theta$, and the method of development is legitimate when $s = 2$, but it is not so when $s = 0$ and $s = 1$.

The investigation then separates into the general case, and the cases $s = 0$, $s = 1$.

§ 14. *Integration in the general case.*

We have

$$\mathfrak{P} = P^s - \beta q_{s-2} P^{s-2} - \beta q_{s+2} P^{s+2} + \beta^2 q_{s-4} P^{s-4} + \beta^2 q_{s+4} P^{s+4},$$

and

$$(\mathfrak{P})^2 = (P^s)^2 - 2\beta (q_{s-2} P^s P^{s-2} + q_{s+2} P^s P^{s+2}) + 2\beta^2 (q_{s-4} P^s P^{s-4} + q_{s+4} P^s P^{s+4})$$
$$+ \beta^2 [(q_{s-2} P^{s-2})^2 + (q_{s+2} P^{s+2})^2 + 2q_{s-2} q_{s+2} P^{s-2} P^{s+2}].$$

Also $\left[P^s \left(\frac{\cos^2\theta}{1 - \beta} - \sin^2\theta \right)^{\frac{1}{2}} \right]^2$ has the same form with accented q's, so that it will be merely necessary to accent the q's to obtain the second case.

We have then

$$\Pi_0 = (P^s)^2, \quad \Pi_1 = -2 (q_{s-2} P^s P^{s-2} + q_{s+2} P^s P^{s+2}),$$
$$\Pi_2 = 2 (q_{s-4} P^s P^{s-4} + q_{s+4} P^s P^{s+4}) + (q_{s-2} P^{s-2})^2 + (q_{s+2} P^{s+2})^2$$
$$+ 2q_{s-2} q_{s+2} P^{s-2} P^{s+2}.$$

Then since $\cos\theta d\theta = d\mu$,

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} F_1 d\theta = \int_{-1}^{+1} (P^s)^2 d\mu + \beta \int_{-1}^{+1} \frac{(\gamma-1)(P^s)^2}{1-\mu^2} d\mu - 2\beta \int_{-1}^{+1} (q_{s-2} P^s P^{s-2} + q_{s+2} P^s P^{s+2}) d\mu$$
$$+ \beta^2 \int_{-1}^{+1} \frac{(\gamma-1)(P^s)^2}{1-\mu^2} d\mu + \beta^2 \int_{-1}^{+1} \frac{(\frac{1}{4}-\gamma)(P^s)^2}{(1-\mu^2)^2} d\mu$$
$$- 2\beta^2 \int_{-1}^{+1} (\gamma-1)(q_{s-2} \frac{P^s P^{s-2}}{1-\mu^2} + q_{s+2} \frac{P^s P^{s+2}}{1-\mu^2}) d\mu$$
$$+ \beta^2 \int_{-1}^{+1} [2q_{s-4} P^s P^{s-4}$$
$$+ 2q_{s+4} P^s P^{s+4} + (q_{s-2} P^{s-2})^2 + (q_{s+2} P^{s+2})^2 + 2q_{s-2} q_{s+2} P^{s-2} P^{s+2}] d\mu$$
$$\qquad\qquad\qquad\qquad\qquad . \quad . \quad . \quad . \quad (66).$$

And $\int F_2 d\theta$ has the same form, but with accented q's, and with $\gamma + 1$ replacing $\gamma - 1$, and $\gamma - \frac{1}{2}$ replacing $\frac{3}{4} - \gamma$.

It is now necessary to evaluate the several definite integrals involved in this expression.

It is well known that

$$\int_{-1}^{+1} (P_i{}^s(\mu))^2 d\mu = \frac{2}{2i+1} \cdot \frac{i+s!}{i-s!}.$$

It is easy to see that it is possible to express $P_i{}^{s+2k}$ in the form

$$P_i{}^{s+2k} = AP_i{}^s + BP_{i-2}^s + CP_{i-4}^s + \ldots,$$

where A, B, C . . . do not involve μ.

The value of A may be found by considering only the highest power of μ on each side of the identity.

Now

$$P_i{}^{s+2k} = \frac{(1-\mu^2)^{\frac{1}{2}s+k}}{2^i \cdot i!} \left(\frac{d}{d\mu}\right)^{i+s+2k} (\mu^2-1)^i,$$

$$= (-)^{\frac{1}{2}s+k} \frac{2i!}{2^i \cdot i! \; i-s-2k!} \mu^i + \ldots;$$

and

$$P_i{}^s = (-)^{\frac{1}{2}s} \frac{2i!}{2^i \cdot i! \; i-s!} \mu^i + \ldots.$$

Therefore

$$A = (-)^k \frac{i-s!}{i-s-2k!}.$$

Then, since the integral of the product of two P's of different orders vanishes, we have

$$\int_{-1}^{+1} P_i{}^s P_i{}^{s+2k} d\mu = (-)^k \frac{i-s!}{i-s-2k!} \int_{-1}^{+1} (P_i{}^s)^2 d\mu = (-)^k \frac{2}{2i+1} \cdot \frac{i+s!}{i-s-2k!}.$$

We will next consider $\int_{-1}^{1} \frac{P_i{}^s P_i{}^{s+2k}}{1-\mu^2} d\mu$, where k is not zero.

The differential equation gives

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{dP_i{}^{s+2k}}{d\mu}\right] + i(i+1)P_i{}^{s+2k} - \frac{(s+2k)^2}{1-\mu^2}P_i{}^{s+2k} = 0,$$

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{dP_i{}^s}{d\mu}\right] + i(i+1)P_i{}^s - \frac{s^2}{1-\mu^2}P_i{}^s = 0.$$

Multiply the first of these by $P_i{}^s$ and the second by $P_i{}^{s+2k}$ and subtract, and we have

$$\frac{4k(s+k)P_i{}^s P_i{}^{s+2k}}{1-\mu^2} = P_i{}^{s+2k}\frac{d}{d\mu}\left[(1-\mu^2)\frac{dP_i{}^s}{d\mu}\right] - P_i{}^s\frac{d}{d\mu}\left[(1-\mu^2)\frac{dP_i{}^{s+2k}}{d\mu}\right].$$

Therefore[*]

$$4k(s+k)\int_{-1}^{+1}\frac{P^s P^{s+2k}}{1-\mu^2}\,d\mu = (1-\mu^2)\left[P^{s+2k}\frac{dP^s}{d\mu} - P^s\frac{dP^{s+2k}}{d\mu}\right], \text{ between limits } \pm 1,$$

$$= 0.$$

Again since by (11)

$$\frac{P^{s+2k}}{1-\mu^2} = AP^{s+2k+2} + BP^{s+2k} + CP^{s+2k-2},$$

it follows that

$$\int_{-1}^{+1}\frac{P^s P^{s+2k}}{(1-\mu^2)^2}\,d\mu = 0, \text{ unless } k = 0 \text{ or } 1.$$

It remains to find the integrals of $\dfrac{(P^s)^2}{1-\mu^2}$, $\dfrac{(P^s)^2}{(1-\mu^2)^2}$, and $\dfrac{P^s P^{s+2}}{(1-\mu^2)^2}$.

We have seen in (11) (transformed to accord with our present definition of P^s) that

$$\frac{P^s}{1-\mu^2} = \frac{1}{4s(s+1)}P^{s+2} + \frac{1}{2}\left[\frac{i(i+1)}{s^2-1}+1\right]P^s + \frac{\{i,s\}\{i,s-1\}}{4s(s-1)}P^{s-2}.$$

Hence

$$\int\frac{(P^s)^2}{1-\mu^2}\,d\mu = \frac{1}{4s(s+1)}\int P^s P^{s+2}\,d\mu + \frac{1}{2}\left[\frac{i(i+1)}{s^2-1}+1\right]\int(P^s)^2\,d\mu$$
$$+ \frac{\{i,s\}\{i,s-1\}}{4s(s-1)}\int P^s P^{s-2}\,d\mu,$$

$$\int\frac{(P^s)^2}{(1-\mu^2)^2}\,d\mu = \frac{1}{4s(s+1)}\int\frac{P^s P^{s+2}}{1-\mu^2}\,d\mu + \frac{1}{2}\left[\frac{i(i+1)}{s^2-1}+1\right]\int\frac{(P^s)^2}{1-\mu^2}\,d\mu$$
$$+ \frac{\{i,s\}\{i,s-1\}}{4s(s-1)}\int\frac{P^s P^{s-2}}{1-\mu^2}\,d\mu,$$

$$\int\frac{P^s P^{s+2}}{(1-\mu^2)^2}\,d\mu = \frac{1}{4s(s+1)}\int\frac{(P^{s+2})^2}{1-\mu^2}\,d\mu + \frac{1}{2}\left[\frac{i(s+1)}{s^2-1}+1\right]\int\frac{P^s P^{s+2}}{1-\mu^2}\,d\mu$$
$$+ \frac{\{i,s\}\{i,s-1\}}{4s(s-1)}\int\frac{P^{s-2}P^{s+2}}{1-\mu^2}\,d\mu.$$

The first of these involves integrals already determined; on introducing them on the right and reducing we find the result to be $\dfrac{1}{s}\cdot\dfrac{i+s!}{i-s!}$.

The first and last terms of the second integral vanish, and the integral is clearly $\dfrac{1}{2}\left[\dfrac{i(i+1)}{s^2-1}+1\right]\cdot\dfrac{1}{s}\cdot\dfrac{i+s!}{i-s!}$.

The second and third terms of the third integral vanish, and the whole is clearly $\dfrac{1}{4s(s+1)(s+2)}\dfrac{i+s+2!}{i-s-2!}$.

Collecting results we have

$$
\left.
\begin{aligned}
\int_{-1}^{+1} (\mathbf{P}^s)^2\, d\mu &= \frac{2}{2i+1}\cdot\frac{i+s\,!}{i-s\,!}\\[4pt]
\int_{-1}^{+1} \mathbf{P}^s\mathbf{P}^{s+2k}\, d\mu &= (-)^k\frac{2}{2i+1}\cdot\frac{i+s\,!}{i-s-2k\,!}\\[4pt]
\int_{-1}^{+1} \frac{\mathbf{P}^s\mathbf{P}^{s+2k}}{1-\mu^2}\, d\mu &= 0.\\[4pt]
\int_{-1}^{+1} \frac{(\mathbf{P}^s)^2}{1-\mu^2}\, d\mu &= \frac{1}{s}\cdot\frac{i+s\,!}{i-s\,!}\\[4pt]
\int_{-1}^{+1} \frac{(\mathbf{P}^s)^2}{(1-\mu^2)^2}\, d\mu &= \tfrac{1}{2}\Big[\frac{i(i+1)}{s^2-1}+1\Big]\cdot\frac{1}{s}\cdot\frac{i+s\,!}{i-s\,!}\\[4pt]
\int_{-1}^{+1} \frac{\mathbf{P}^s\mathbf{P}^{s+2}}{(1-\mu^2)^2}\, d\mu &= \frac{1}{4s(s+1)(s+2)}\cdot\frac{i+s+2\,!}{i-s-2\,!}\\[4pt]
\int_{-1}^{+1} \frac{\mathbf{P}^s\mathbf{P}^{s-2}}{(1-\mu^2)^2}\, d\mu &= \frac{1}{4s(s-1)(s-2)}\cdot\frac{i+s\,!}{i-s\,!}
\end{aligned}
\right\} \quad\dots\dots\ (67)
$$

Then by means of (66) and (67)

$$
\begin{aligned}
\int F_1 d\theta ={}& \frac{2}{2i+1}\frac{i+s\,!}{i-s\,!}+\beta(\gamma-1)\cdot\frac{1}{s}\cdot\frac{i+s\,!}{i-s\,!}\\[4pt]
&+2\beta\Big(q_{s-2}\frac{2}{2i+1}\cdot\frac{i+s-2\,!}{i-s\,!}+q_{s+2}\cdot\frac{2}{2i+1}\cdot\frac{i+s\,!}{i-s-2\,!}\Big)\\[4pt]
&+\beta^2(\gamma-1)\cdot\frac{1}{s}\cdot\frac{i+s\,!}{i-s\,!}+\beta^2(\tfrac{3}{2}-\gamma)\tfrac{1}{2}\Big[\frac{i(i+1)}{s^2-1}+1\Big]\cdot\frac{1}{s}\cdot\frac{i+s\,!}{i-s\,!}\\[4pt]
&+\frac{2\beta^2}{2i+1}\Big[2q_{s-4}\frac{i+s-4\,!}{i-s\,!}+2q_{s+4}\frac{i+s\,!}{i-s-4\,!}+(q_{s-2})^2\frac{i+s-2\,!}{i-s+2\,!}\\[4pt]
&\qquad\qquad+(q_{s+2})^2\frac{i+s+2\,!}{i-s-2\,!}+q_{s-2}q_{s+2}\frac{i+s-2\,!}{i-s-2\,!}\Big].
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\int F_1 d\theta ={}& \frac{2}{2i+1}\cdot\frac{i+s\,!}{i-s\,!}\Big[1+\frac{2\beta q_{s-2}}{(i+s)(i+s-1)}+2\beta q_{s+2}(i-s)(i-s-1)\\[4pt]
&\qquad+\frac{2\beta^2 q_{s-4}}{(i+s)(i+s-1)(i+s-2)(i+s-3)}\\[4pt]
&\qquad+2\beta^2 q_{s+4}(i-s)(i-s-1)(i-s-2)(i-s-3)\\[4pt]
&\qquad+\frac{\beta^2(q_{s-2})^2}{(i-s+1)(i-s+2)(i+s)(i+s-1)}\\[4pt]
&\qquad+\beta^2(q_{s+2})^2(i+s+1)(i+s+2)(i-s)(i-s-1)\\[4pt]
&\qquad+2\beta^2 q_{s-2}q_{s+2}\frac{(i-s)(i-s-1)}{(i+s)(i+s-1)}\Big]\\[4pt]
&+\beta\cdot\frac{1}{s}\frac{i+s\,!}{i-s\,!}\Big\{(1+\beta)(\gamma-1)+\beta(\tfrac{3}{2}-\gamma)\cdot\tfrac{1}{2}\Big[\frac{i(i+1)}{s^2-1}+1\Big]\Big\}.\quad.\ (68).
\end{aligned}
$$

Also $\int F_2 d\theta$ has a similar form with accented q's, $\gamma + 1$ for $\gamma - 1$, and $\gamma - \frac{1}{2}$ for $\frac{3}{2} - \gamma$.

When we substitute for γ its value $1 - \cos 2\phi$, and write as before

the last term in $\int F_1 d\theta$ becomes

$$+ \beta \cdot \frac{1}{s} \cdot \frac{i + s!}{i - s!} \{\tfrac{1}{4}\beta(\Sigma + 1) - \cos 2\phi [1 - \tfrac{1}{2}\beta(\Sigma - 1)]\} \quad . \quad . \quad (68).$$

Also the last term in $\int F_2 d\theta$ becomes

$$+ \beta \cdot \frac{1}{s} \cdot \frac{i + s!}{i - s!} \{2[1 + \tfrac{1}{8}\beta(\Sigma + 9)] - \cos 2\phi [1 + \tfrac{1}{8}\beta(\Sigma + 3)]\}. \quad . (68).$$

But it will appear later that we only need the parts of these terms which involve $\cos 2\phi$ developed as far as the first power of β; hence in both cases we may write the latter term inside { } simply as $- \cos 2\phi$.

Our general formulæ for the q coefficients apply for all values of s down to $s = 3$, inclusive, although the result for $s = 3$ needs proper interpretation. Hence the present result applies down to $s = 3$, inclusive.

I have just re-defined Σ, and I remind the reader that

$$\Upsilon = \frac{(i - 1)i(i + 1)(i + 2)}{s^2 - 4}.$$

Then if in (68) we introduce for the q's their values, we find that the coefficient of the term in β is

$$\frac{2q_{s-2}}{(i + s)(i + s - 1)} + q_{s+2}(i - s)(i - s - 1) = - \tfrac{1}{2}(\Sigma - 1)$$

The coefficient of the term in β^2 is

$$\tfrac{1}{8}s \left\{ \frac{(i - s + 1)(i - s + 2)(i - s + 3)(i - s + 4)}{(s - 1)(s - 2)} + \frac{(i - s)(i - s - 1)(i - s - 2)(i - s - 3)}{(s + 1)(s + 2)} \right.$$
$$+ \frac{(i + s)(i + s - 1)(i - s + 1)(i - s + 2)}{(s - 1)^2} + \frac{(i + s + 1)(i + s + 2)(i - s)(i - s - 1)}{(s + 1)^2}$$
$$\left. - 2\frac{(i - s + 1)(i - s + 2)(i - s)(i - s - 1)}{(s^2 - 1)} \right\}.$$

If this be reduced by a process similar to that employed in § 10, we find

$$\int F_1 d\theta = \frac{2}{2i+1} \frac{i+s!}{i-s!} \{1 - \tfrac{1}{2}\beta(\Sigma - 1) + \tfrac{1}{12}\beta^2 [3\Sigma^2 - 6\Sigma + 6 - s^2(\Sigma^2 + 2\Sigma - 1) + 2\mathrm{T}]\}$$
$$+ \beta \cdot \frac{1}{s} \cdot \frac{i+s!}{i-s!} \{\tfrac{1}{4}\beta(\Sigma+1) - \cos 2\phi\} \quad . (69).$$

We know that $\mathbf{P}^s$ is derivable from $\mathfrak{P}^s$ by multiplication by $1/C_i^s$, and we have found in (33), § 9,

$$\frac{1}{(C_i^s)^2} = 1 + \beta(\Sigma + 1) + \tfrac{1}{8}\beta^2 [3\Sigma^2 + 8\Sigma + 5 + s^2\Sigma^2 - \mathrm{T}].$$

Hence multiplying (69) by $\dfrac{1}{(C_i^s)^2}$ we have

$$\int F_2 d\theta = \frac{2}{2i+1} \frac{i+s!}{i-s!} \{1 + \tfrac{1}{2}\beta(\Sigma + 3) + \tfrac{1}{12}\beta^2 [-\Sigma^2 + 26\Sigma + 42$$
$$+ s^2(3\Sigma^2 - 2\Sigma + 1) - 2\mathrm{T}]\} + \beta \cdot \frac{1}{s} \cdot \frac{i+s!}{i-s!} \{\tfrac{1}{4}\beta(\Sigma + 1) - \cos 2\phi\} \quad . (70).$$

I have also obtained this result by direct development. It may be thought surprising that the last term is now the same in both formulæ, notwithstanding the difference in the earlier stages, but if the reader will go through the analysis he will see how this has been brought about. The formulæ (69) and (70) also hold true when $s = 3$ (as I have verified), notwithstanding the fact that $\mathbf{P}^3$ is not to be derived from $\mathfrak{P}^3$ by a factor.

The next step is the integration with respect to ϕ.
We have

$$\begin{Bmatrix} \mathfrak{C}_i^s \\ \mathfrak{S}_i^s \end{Bmatrix} = \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} s\phi + \beta \left[p_{s-2} \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} (s-2)\phi + p_{s+2} \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} (s+2)\phi \right]$$
$$+ \beta^2 \left[p_{s-4} \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} (s-4)\phi + p_{s+4} \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} (s+4)\phi \right].$$

Therefore

$$\begin{Bmatrix} (\mathfrak{C}_i^s)^2 \\ (\mathfrak{S}_i^s)^2 \end{Bmatrix} = \tfrac{1}{2} \pm \tfrac{1}{2} \cos 2s\phi + \beta [(p_{s-2} + p_{s+2}) \cos 2\phi \pm p_{s-2} \cos 2(s-1)\phi$$
$$\pm p_{s+2} \cos 2(s+1)\phi] + \beta^2 [\tfrac{1}{2}(p_{s-2})^2 + \tfrac{1}{2}(p_{s+2})^2 + (p_{s-4} + p_{s+4}$$
$$+ p_{s-2} p_{s+2}) \cos 4\phi \pm p_{s-2} p_{s+2} \cos 2s\phi \pm (p_{s-4} + \tfrac{1}{2}(p_{s-2})^2) \cos 2(s-2)\phi$$
$$\pm (p_{s+4} + \tfrac{1}{2}(p_{s+2})^2) \cos 2(s+2)\phi].$$

Also $\dfrac{(\mathfrak{C}_i^s)^2 \text{ or } (\mathfrak{S}_i^s)^2}{1 - \beta \cos 2\phi}$ have the same forms with accented p's.

Accordingly, with unaccented p's, we have to multiply this expression by

$(1 - \beta \cos 2\phi)^{-\frac{1}{2}}$, and with accented p's we multiply by $(1 - \beta \cos 2\phi)^{\frac{1}{2}}$, and we shall then have the functions denoted above by K^2.

The function K^2 has to be multiplied by a function of the form $A + B \beta \cos 2\phi$, and integrated from $\phi = 2\pi$ to 0. It follows that the only terms in K^2 which will not vanish are those independent of ϕ and those in $\cos 2\phi$; moreover, the latter terms are only required as far as the first power of β.

Now
$$(1 - \beta \cos 2\phi)^{-1} = 1 + \tfrac{1}{2}\beta \cos 2\phi + \tfrac{3}{16}\beta^2 (1 + \cos 4\phi),$$
$$(1 - \beta \cos 2\phi)^1 = 1 - \tfrac{1}{2}\beta \cos 2\phi - \tfrac{1}{16}\beta^2 (1 + \cos 4\phi).$$

Then omitting terms which will vanish on integration

$$\frac{(\mathbf{S}_i')^2}{(1 - \beta \cos 2\phi)^1} = \tfrac{1}{2} \{1 + \beta^2 [(p_{i-2})^2 + (p_{i+2})^2 + \tfrac{1}{2}p_{i-2} + \tfrac{1}{2}p_{i+2} + \tfrac{3}{16}]\}$$
$$+ \beta(p_{i-2} + p_{i+2} + \tfrac{1}{4}) \cos 2\phi,$$

$$\frac{(\mathbf{C}_i' \text{ or } \mathbf{S}_i')^2}{(1 - \beta \cos 2\phi)^1} = \tfrac{1}{2} \{1 + \beta^2 [(p'_{i-2})^2 + (p'_{i+2})^2 - \tfrac{1}{2}p'_{i-2} - \tfrac{1}{2}p'_{i+2} - \tfrac{1}{16}]\}$$
$$+ \beta(p'_{i-2} + p'_{i+2} - \tfrac{1}{4}) \cos 2\phi.$$

However, the latter formula is not needed except for verification, because it will be derivable from the former by multiplication by $\dfrac{1}{(D_i')^2}$.

Now if we substitute for the p's their values as given in (27), § 8, we find

$$\frac{(\mathbf{C}_i' \text{ or } \mathbf{S}_i')^2}{(1 - \beta \cos 2\phi)^1} = \tfrac{1}{2}\{1 + \tfrac{1}{32}\beta^2[\Sigma^2 + 4\Sigma + 6 + s^2(\Sigma^2 - 2\Sigma + 1)]\} + \tfrac{1}{4}\beta(\Sigma + 1)\cos 2\phi.$$

And multiplying by $\dfrac{1}{(D_i')^2}$ or $1 - \tfrac{1}{8}\beta^2(\Sigma + 3)$, or developing directly

$$\frac{(\mathbf{C}_i' \text{ or } \mathbf{S}_i')^2}{(1 - \beta \cos 2\phi)^1} = \tfrac{1}{2}\{1 + \tfrac{1}{32}\beta^2[\Sigma^2 - 6 + s^2(\Sigma^2 - 2\Sigma + 1)]\} + \tfrac{1}{4}\beta(\Sigma + 1)\cos 2\phi.$$

These represent the K^2 of our integrals.

Then

$$\int p\left(\mathbf{P}_i \left\{ \begin{matrix} \mathbf{C}_i' \\ \mathbf{S}_i' \end{matrix} \right\}^2 \right) d\sigma = M (1 - \beta)^{\frac{1}{2}} \iint F_i \frac{(\mathbf{C}_i' \text{ or } \mathbf{S}_i')^2}{(1 - \beta \cos 2\phi)^1} \, d\theta \, d\phi,$$

$$= \frac{2\pi M (1 - \beta)^{\frac{1}{2}}}{2i + 1} \cdot \frac{i + s!}{i - s!} \{1 - \tfrac{1}{2}\beta(\Sigma - 1)$$
$$+ \tfrac{1}{32}\beta^2[3\Sigma^2 - 6\Sigma + 6 - s^2(\Sigma^2 + 2\Sigma - 1) + 2\Upsilon]\}$$
$$\times \{1 + \tfrac{1}{32}\beta^2[\Sigma^2 + 4\Sigma + 6 + s^2(\Sigma^2 - 2\Sigma + 1)]\}$$
$$+ \tfrac{1}{4}\pi M (1 - \beta)^{\frac{1}{2}}\left[\beta^2 \cdot \tfrac{1}{s} \cdot \frac{i + s!}{i - s!}(\Sigma + 1) - \beta^2 \cdot \tfrac{1}{s} \cdot \frac{i + s!}{i - s!}(\Sigma + 1)\right].$$

$$= \frac{2\pi M}{2i+1}\frac{i+s!}{i-s!}(1-\beta)^{\frac{1}{2}}\{1-\tfrac{1}{2}\beta(\Sigma-1)$$
$$+\tfrac{1}{16}\beta^2[2\Sigma^2-\Sigma+6-s^2(2\Sigma-1)+\mathbf{T}]\}$$
$$= \frac{2\pi M}{2i+1}\frac{i+s!}{i-s!}\{1-\tfrac{1}{2}\beta\Sigma+\tfrac{1}{16}\beta^2[2\Sigma^2+3\Sigma-s^2(2\Sigma-1)+\mathbf{T}]\}$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad . \quad . \quad . \quad . \quad (71).$$

The following results may be obtained either by direct development, or by multiplication by either or both the factors $\dfrac{1}{(\mathbf{C}_i^s)^2}$ and $\dfrac{1}{(\mathbf{D}_i^s)^2}$. The former converts $\mathfrak{P}$ into $\mathbf{P}$, the latter $\mathfrak{C}$ or $\mathfrak{S}$ into $\mathbf{C}$ or $\mathbf{S}$.

$$\int \rho\left(\mathfrak{P}_i^s\left\{\begin{matrix}\mathbf{C}_i^s\\ \mathbf{S}_i^s\end{matrix}\right\}\right)^2 d\sigma = \frac{2\pi M}{2i+1}\frac{i+s!}{i-s!}\{1-\tfrac{1}{2}\beta\Sigma$$
$$+\tfrac{1}{16}\beta^2[2\Sigma^2+\Sigma-6-s^2(2\Sigma-1)+\mathbf{T}]\},$$

$$\int \rho\left(\mathbf{P}_i^s\left\{\begin{matrix}\mathfrak{C}_i^s\\ \mathfrak{S}_i^s\end{matrix}\right\}\right)^2 d\sigma = \quad . \quad . \quad . \quad \{1+\tfrac{1}{2}\beta(\Sigma+2)$$
$$+\tfrac{1}{16}\beta^2[11\Sigma+10+s^2(2\Sigma^2-2\Sigma+1)-\mathbf{T}]\},$$

$$\int \rho\left(\mathbf{P}_i^s\left\{\begin{matrix}\mathbf{C}_i^s\\ \mathbf{S}_i^s\end{matrix}\right\}\right)^2 d\sigma = \quad . \quad . \quad . \quad \{1+\tfrac{1}{2}\beta(\Sigma+2)$$
$$+\tfrac{1}{16}\beta^2[9\Sigma+4+s^2(2\Sigma^2-2\Sigma+1)-\mathbf{T}]\}$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (71).$$

§ 15. *Integration in the case of $s = 2$.*

Although the development in powers of $\sec^2\theta$ is still legitimate in this case, yet the formulæ found in the last section fail because $\mathbf{T}$ contains s^2-4 in the denominator. Moreover since $\mathfrak{P}^2$ is not convertible into $\mathbf{P}^2$ by a factor each case must be considered separately.

We now have $q_{i-4}=0$, $q'_{i-4}=0$, and therefore from (68)

$$\int F_1 d\theta = \frac{2}{2i+1}\frac{i+2!}{i-2!}\left\{1+2\beta\left[\frac{q_0}{(i+1)(i+2)}+q_1(i-2)(i-3)\right]\right.$$
$$+\beta^2\left[2q_6(i-2)(i-3)(i-4)(i-5)+\frac{(q_0)^2}{(i-1)i(i+1)(i+2)}\right.$$
$$\left.+(q_4)^2(i+3)(i+4)(i-2)(i-3)+2q_0q_4\frac{(i-2)(i-3)}{(i+1)(i+2)}\right]\right\}$$
$$+\tfrac{1}{2}\beta\frac{i+2!}{i-2!}\{\tfrac{1}{2}\beta(\Sigma+1)-\cos 2\phi\}.$$

$\int F_2 d\theta$ is equal to the same with accented q's, and the last term equal to

$$\tfrac{1}{2}\beta\frac{i+2!}{i-2!}\{2+\tfrac{1}{2}\beta(\Sigma+9)-\cos 2\phi\}.$$

We now have

$$q_0 = -\tfrac{1}{6}\{i,1\}\{i,2\}, \quad q_4 = \tfrac{1}{144}, \quad q_6 = \tfrac{1}{1728}, \text{ associated with a cosine function,}$$
$$q'_0 = \quad 0 \quad , \quad q'_4 = \tfrac{1}{144}, \quad q'_6 = \tfrac{1}{1728}, \text{ associated with a sine function.}$$

It is well to note that these values are given by the general formula, because this consideration shows that much of the previous reductions is still applicable.

Effecting the reductions I find

$$
\int F_1 d\theta = \frac{2}{2i+1} \cdot \frac{i+2\,!}{i-2\,!} \{1 - \tfrac{1}{2}\beta\,(\Sigma - 1) + \tfrac{1}{84}\beta^2 \left[\tfrac{14}{4}\Sigma^2 - \tfrac{65}{4}\Sigma + 20 \right.
$$
$$
\left. + 2\,(2i+1)(3\Sigma - 1)\right]\} + \tfrac{1}{2}\beta\frac{i+2\,!}{i-2\,!}\{\tfrac{1}{2}\beta\,(\Sigma + 1) - \cos 2\phi\},
$$
$$
= \frac{2}{2i+1}\frac{i+2\,!}{i-2\,!}\{1 - \tfrac{1}{2}\beta\,(\Sigma - 1) + \tfrac{1}{840}\beta^2\,(19\Sigma^2 - 130\Sigma + 80)\}
$$
$$
+ \tfrac{1}{2}\beta\frac{i+2\,!}{i-2\,!}\{\tfrac{1}{2}\beta\,(5\Sigma + 1) - \cos 2\phi\}.
$$

This integral will be associated with $\mathbb{C}_i^2$ and $\mathbb{O}_i^2$, and in the present case $\Sigma = \tfrac{1}{3}\,i\,(i+1)$.

In the same way

$$
\int F_2 d\theta = \frac{2}{2i+1}\frac{i+2\,!}{i-2\,!}\{1 + \tfrac{1}{2}\beta\,[\Sigma + 3 - (2i+1)] + \tfrac{1}{84}\beta^2 \left[\tfrac{24}{4}\Sigma^2 + \tfrac{9}{4}\Sigma + 92 \right.
$$
$$
\left. - 6(2i+1)(\Sigma + 5)]\} + \tfrac{1}{2}\beta\frac{i+2\,!}{i-2\,!}\{2 + \tfrac{1}{2}\beta(\Sigma + 9) - \cos 2\phi\},
$$
$$
= \frac{2}{2i+1}\frac{i+2\,!}{i-2\,!}\{1 + \tfrac{1}{2}\beta\,(\Sigma + 3) + \tfrac{1}{840}\beta^2\,(25\Sigma^2 + 186\Sigma + 368)\}
$$
$$
+ \tfrac{1}{2}\beta\frac{i+2\,!}{i-2\,!}\{-\tfrac{1}{2}\beta(\Sigma - 3) - \cos 2\phi\}.
$$

This will be associated with $\mathbb{S}_i^2$ and $\mathbb{S}_i^2$.

Now turning to the cosine and sine functions, we find that they must be treated apart, but the integral involving $\mathbb{O}_i^2$ may be derived from that in $\mathbb{C}_i^2$ by the factor $\left[\frac{1}{D_i^2(\cos)}\right]^2$; and similarly $\mathbb{S}_i^2$ from $\mathbb{S}_i^2$ by the factor $\left[\frac{1}{D_i^2(\sin)}\right]^2$. These factors were evaluated in (36), § 9.

We now have $p_{i-1} = 0$, $p'_{i-1} = 0$; also for the sine function $p_{i-1} = p_0 = 0$. Then

$$
(\mathbb{C}_i^2)^2 = \tfrac{1}{2} + \tfrac{1}{2}\cos 4\phi + \beta\,[(2p_0 + p_4)\cos 2\phi + p_4\cos 6\phi]
$$
$$
+ \beta^2 [(p_0)^2 + \tfrac{1}{2}(p_4)^2 + (p_0 + p_0 p_4)\cos 4\phi + p_0 p_4\cos 4\phi
$$
$$
+ (p_0 + \tfrac{1}{2}(p_4)^2)\cos 8\phi],
$$
$$
(\mathbb{S}_i^2)^2 = \tfrac{1}{2} - \tfrac{1}{2}\cos 4\phi + \beta\,[p_4\cos 2\phi - p_4\cos 6\phi]
$$
$$
+ \beta^2 [\tfrac{1}{2}(p_4)^2 + p_9\cos 4\phi - (p_6 + \tfrac{1}{8}(p_4)^2)\cos 8\phi].
$$

Then as far as material

$$(1 - \beta \cos 2\phi)^{\frac{1}{2}} \frac{(\mathfrak{C}_i f)^2}{} = \tfrac{1}{8}\{1 + \beta^2[(p_i)^2 + 2(p_0)^2 + p_0 + \tfrac{1}{2}p_i + \tfrac{9}{8}p_i]\}$$
$$+ \beta(p_i + 2p_0 + \tfrac{3}{8})\cos 2\phi,$$

$$(1 - \beta \cos 2\phi)^{\frac{1}{2}} \frac{(\mathfrak{S}_i f)^2}{} = \tfrac{1}{2}\{1 + \beta^2[(p_i)^2 + \tfrac{1}{2}p_i + \tfrac{3}{8}s_i]\} + \beta(p_i + \tfrac{1}{8})\cos 2\phi.$$

Now $\qquad p_0 = \tfrac{1}{8}\{i, 2\}, \quad p_4 = -\tfrac{1}{24}\{i, 3\} \quad$ and

$$\frac{(\mathfrak{C}_i f)^2}{(1 - \beta \cos 2\phi)^{\frac{1}{2}}} = \tfrac{1}{8}\{1 + \tfrac{1}{64}\beta^2(19\Sigma^2 - 8\Sigma + 22)\} + \tfrac{1}{8}\beta(5\Sigma + 1)\cos 2\phi,$$

$$\frac{(\mathfrak{S}_i f)^2}{(1 - \beta \cos 2\phi)^{\frac{1}{2}}} = \tfrac{1}{2}\{1 + \tfrac{1}{64}\beta^2(\Sigma^2 - 8\Sigma + 18)\} - \tfrac{1}{8}\beta(\Sigma - 3)\cos 2\phi.$$

We now multiply these by $\int F_1 d\theta$ and $\int F_2 d\theta$ respectively, and the last terms disappear as before. I remark that the disappearance of the terms which do not involve the factor $1/(2i + 1)$ affords an excellent test of the correctness of the laborious reductions throughout all this part of the work.

Then we have

$$\int \rho(\mathfrak{P}_i{}^s \mathfrak{C}_i{}^s)^2 d\sigma = \frac{2\pi M}{2i+1} \frac{i+2!}{i-2!}(1-\beta)[1 - \tfrac{1}{2}\beta(\Sigma - 1) + \tfrac{1}{240}\beta^2(19\Sigma^2 - 180\Sigma + 80)]$$
$$\times [1 + \tfrac{1}{64}\beta^2(19\Sigma^2 - 8\Sigma + 22)]$$
$$= \frac{2\pi M}{2i+1} \frac{i+2!}{i-2!}\{1 - \tfrac{1}{2}\beta\Sigma + \tfrac{1}{240}\beta^2(95\Sigma^2 - 98\Sigma + 72)\} \quad . \quad . \quad . \quad (72).$$

If we multiply this by $\left[\frac{1}{D_i{}^2}(\cos)\right]^2$ or $1 - \tfrac{1}{16}\beta^2(5\Sigma + 7)$, we obtain the result when $\mathbf{C}_i{}^s$ replaces $\mathfrak{C}_i{}^s$; the only change is that the last term inside $\{\ \}$ now becomes $+ \tfrac{1}{240}\beta^2(95\Sigma^2 - 178\Sigma - 40)$.

Again

$$\int \rho(\mathbf{P}_i{}^s \mathfrak{S}_i{}^s)^2 d\sigma = \frac{2\pi M}{2i+1} \frac{i+2!}{i-2!}(1-\beta)[1 + \tfrac{1}{2}\beta(\Sigma + 3) + \tfrac{1}{240}\beta^2(25\Sigma^2 + 186\Sigma + 368)]$$
$$\times [1 + \tfrac{1}{64}\beta^2(\Sigma^2 - 8\Sigma + 18)]$$
$$= \frac{2\pi M}{2i+1} \frac{i+2!}{i-2!}[1 + \tfrac{1}{2}\beta(\Sigma + 2) + \tfrac{1}{240}\beta^2(29\Sigma^2 + 90\Sigma + 216)]. \quad . \quad . \quad . \quad (72).$$

If we multiply this by $\left[\frac{1}{D_i{}^2}(\sin)\right]^2$ or $1 + \tfrac{1}{16}\beta^2(\Sigma + 5)$, obtain the result when $\mathbf{S}_i{}^s$ replaces $\mathfrak{S}_i{}^s$; the only change is that the last term inside $\{\ \}$ now becomes $+ \tfrac{1}{240}\beta^2(29\Sigma^2 + 106\Sigma + 136)$.

This terminates the integrals, which can be completely determined by this method of developing in powers of $\sec^2 \theta$.

§ 16. *Portion of the Integration in the case of $s = 1$.*

The preceding method may be used for finding the four integrals

$$\int p \left\{ \begin{matrix} (\mathfrak{S}_i^1)^2 \\ (\mathfrak{S}_i^1)^2 \end{matrix} \right\} [(\mathfrak{P}_i^1)^2 - (P_i^1)^2] d\sigma$$

and

$$\int p \left\{ \begin{matrix} (\mathfrak{C}_i^1)^2 \\ (\mathfrak{C}_i^1)^2 \end{matrix} \right\} \left[(P_i^1)^2 - \left(P_i^1 \sqrt{\frac{1+\beta}{1-\beta} - \sin^2\theta}}{\cos^2\theta} \right)^2 \right] d\theta.$$

There will then remain four integrals of the type $\int p (\mathfrak{S}_i^1)^2 (P_i^1)^2 d\sigma$ to evaluate.

The first pair of our integrals are clearly to be treated by putting $\Pi_0 = 0$, $q_{i-2} = q_{-1} = 0$, $q_{i-4} = q_{-3} = 0$, and then determining $\int F_1 d\theta$. The condition for the second pair only differs in the accentuation of the q's which vanish, and in the use of $\int F_2 d\theta$.

The vanishing of Π_0 makes

$$F_1 = \beta \Pi_1 \cos\theta + \beta^2 \left(\frac{\Pi_1(\gamma-1)}{\cos\theta} + \Pi_2 \cos\theta \right),$$

$$F_2 = \beta \Pi_1 \cos\theta + \beta^2 \left(\frac{\Pi_3(\gamma+1)}{\cos\theta} + \Pi_2 \cos\theta \right).$$

In the first of these

$$\Pi_1 = -2q_3 P^1 P^5, \quad \Pi_2 = 2q_5 P^1 P^5 + (q_3)^2 (P^3)' :$$

and in the second the form is the same with accented q's.

Also since $\int \frac{P^1 P^3}{1-\mu^2} d\mu = 0$, we have $\int \frac{\Pi_1 d\theta}{\cos\theta} = 0$.

Hence

$$\int F_1 d\theta = -2\beta q_3 \int P^1 P^5 d\mu + 2\beta^2 \int [(q_5 P^1 P^5 + \tfrac{1}{2}(q_3)^2 (P^3)^2] d\mu,$$

$$= \frac{2}{2i+1} \frac{i+1}{i-1} [2\beta q_3(i-1)(i-2) + 2\beta^2 q_5(i-1)(i-2)(i-3)(i-4) + \beta^2 (q_3)^2 (i+2)(i+3)(i-1)(i-2)] ;$$

and $\int F_2 d\theta$ is the same with accented q's.

It is only necessary to pursue the cases $\int p (\mathfrak{S}_i^1)^2 F_1 d\sigma$ and $\int p (\mathfrak{C}_i^1)^2 F_2 d\sigma$, since the other pair of integrals may be determined by means of multiplication by the appropriate factors, determined in § 9.

Now for $\mathfrak{P}_1{}^1 \mathfrak{S}_1{}^1$, associated with F_1,

$$q_3 = \tfrac{2}{15}[1 - \tfrac{1}{16}\beta i(i+1)], \quad q_5 = \tfrac{1}{768};$$

and for $\mathfrak{P}_1{}^1 \mathfrak{C}_1{}^1$, associated with F_2,

$$q_3' = \tfrac{2}{15}[1 + \tfrac{1}{16}\beta i(i+1)], \quad q_5' = \tfrac{1}{768}.$$

Therefore

$$\int F_1 d\theta = \frac{2}{2i+1}\frac{i+1}{i-1}\bigg\{ \tfrac{1}{5}\beta(i-1)(i-2) + \tfrac{1}{128}\beta^2[\tfrac{1}{3}(i-1)(i-2)(i-3)(i-4)$$
$$+ \tfrac{2}{3}(i+2)(i+3)(i-1)(i-2) - (i-1)(i-2)i(i+1)] \bigg\},$$

$$\int F_2 d\theta = \frac{2}{2i+1}\frac{i+1}{i-1}\bigg\{ \tfrac{2}{5}\beta(i-1)(i-2) + \tfrac{1}{128}\beta^2[\tfrac{1}{3}(i-1)(i-2)(i-3)(i-4)$$
$$+ \tfrac{2}{3}(i+2)(i+3)(i-1)(i-2) + 3(i-1)(i-2)i(i+1)] \bigg\}.$$

In the present case we cannot use Σ as an abridgement, since it is infinite; I therefore now write

$$j = i(i+1).$$

Effecting the reductions we have

$$\int F_1 d\theta = \frac{2}{2i+1}\frac{i+1}{i-1}\bigg\{ \tfrac{1}{5}\beta(j - 2(2i+1) + 4) + \tfrac{1}{128}\beta^2[-\tfrac{1}{3}j^2 + \tfrac{9}{3}j + 30 - 16(2i+1)] \bigg\},$$

$$\int F_2 d\theta = \frac{2}{2i+1}\frac{i+1}{i-1}\bigg\{ \tfrac{2}{5}\beta(j - 2(2i+1) + 4) + \tfrac{1}{128}\beta^2[\tfrac{5}{6}j^2 + \tfrac{143}{3}j + 174 - 16(2i+1)(j+5)] \bigg\}.$$

The former of these is associated with $\mathfrak{S}$, the latter with $\mathfrak{C}$.

In the cosine and sine functions we have

$$p_{-2} = p_{-1} = 0, \quad p_{-4} = p_{-3} = 0, \quad \text{and}$$

$$(\mathfrak{S}_1{}^1)^2 = \tfrac{1}{2} - \tfrac{1}{2}\cos 2\phi + \beta(p_3 \cos 2\phi - p_3 \cos 4\phi)$$
$$+ \beta^2[\tfrac{1}{2}(p_3)^2 + p_5 \cos 4\phi - (p_5 + \tfrac{1}{2}(p_3)^2)\cos 6\phi],$$

$$(\mathfrak{C}_1{}^1)^2 = \tfrac{1}{2} + \tfrac{1}{2}\cos 2\phi + \beta(p_3 \cos 2\phi + p_3 \cos 4\phi)$$
$$+ \beta^2[\tfrac{1}{2}(p_3)^2 + p_5 \cos 4\phi + (p_5 + \tfrac{1}{2}(p_3)^2)\cos 6\phi].$$

As far as material, we then have

$$\frac{(\mathfrak{S}_1{}^1)^2}{(1 - \beta\cos 2\phi)^2} = \tfrac{1}{2}\{1 - \tfrac{1}{4}\beta + \beta^2[(p_3)^2 + \tfrac{1}{2}p_3 + \tfrac{3}{16}]\}$$
$$+ (-\tfrac{1}{2} + \tfrac{1}{4}\beta + \beta p_3)\cos 2\phi,$$

$$= \tfrac{1}{2}\{1 - \tfrac{1}{4}\beta + \beta^2[(p_3)^2 + \tfrac{1}{2}p_3 + \tfrac{3}{16}]\}\{1 - (1 - 2\beta p_3 - \tfrac{1}{2}\beta)\cos 2\phi\};$$

$$\frac{(\mathfrak{C}_i^j)^2}{(1-\beta\cos 2\phi)^4} = \tfrac{1}{2}\{1 + \tfrac{1}{4}\beta + \beta^2[(p_3)^2 + \tfrac{1}{2}p_3 + \tfrac{3}{16}]\} + (\tfrac{1}{4} + \tfrac{1}{8}\beta + \beta p_3)\cos 2\phi,$$

$$= \tfrac{1}{2}\{1 + \tfrac{1}{4}\beta + \beta^2[(p_3)^2 + \tfrac{1}{2}p_3 + \tfrac{3}{16}]\}\{1 + (1 + 2\beta p_3 + \tfrac{1}{4}\beta)\cos 2\phi\}.$$

Now both for sines and cosines, to the order necessary for our present purpose.
$p_3 = -\tfrac{1}{16}\{i, 2\}$.   Therefore, introducing j for $i(i+1)$,

$$\left.\begin{aligned}
\frac{(\mathfrak{S}_i^j)^2}{(1-\beta\cos 2\phi)^4} &= \tfrac{1}{2}\{1 - \tfrac{1}{4}\beta + \tfrac{1}{384}\beta^2(j^2-12j+68)\}\{1 - [1 + \tfrac{1}{8}\beta j - \tfrac{1}{2}\beta]\cos 2\phi\} \\
\frac{(\mathfrak{C}_i^j)^2}{(1-\beta\cos 2\phi)^4} &= \tfrac{1}{2}\{1 + \tfrac{1}{4}\beta + \tfrac{1}{384}\beta^2(j^2-12j+68)\}\{1 + [1 - \tfrac{1}{8}\beta j + \tfrac{1}{2}\beta]\cos 2\phi\}
\end{aligned}\right\} \quad . \quad (73).$$

Observe that $\int F_1 d\theta$ and $\int F_2 d\theta$ do not involve $\cos 2\phi$, and are of the first order in β.
Hence, as far as material for the *present* portion of the work.

$$\frac{(\mathfrak{S}_i^j)^2}{(1-\beta\cos 2\phi)^4} = \tfrac{1}{2}(1 - \tfrac{1}{4}\beta), \qquad \frac{(\mathfrak{C}_i^j)^2}{(1-\beta\cos 2\phi)^4} = \tfrac{1}{2}(1 + \tfrac{1}{4}\beta) \quad . \quad . \quad . \quad (73).$$

Also, to the first order, from (37),

$$\left[\frac{1}{D_i^j}(\sin)\right]^2 = 1 + \tfrac{1}{2}\beta, \qquad \left[\frac{1}{D_i^j}(\cos)\right]^2 = 1 - \tfrac{1}{2}\beta.$$

Therefore as far as necessary

$$\frac{(\mathbf{S}_i^j)^2}{(1-\beta\cos 2\phi)^4} = \tfrac{1}{2}(1 + \tfrac{1}{4}\beta) \qquad \frac{(\mathbf{C}_i^j)^2}{(1-\beta\cos 2\phi)^4} = \tfrac{1}{2}(1 - \tfrac{1}{4}\beta).$$

Hence

$$\begin{aligned}
\int p(\mathbf{S}_i^j)^2[(\mathfrak{P}_i^j)^2 - (\mathbf{P}_i^j)^2]d\sigma &= \frac{2\pi M}{2i+1}\frac{i+1!}{i-1!}(1-\beta)^4(1-\tfrac{1}{4}\beta)\{\tfrac{1}{8}\beta(j-2(2i+1)+4) \\
&\qquad + \tfrac{1}{384}\beta^2[-\tfrac{1}{2}j^2 + \tfrac{34}{8}j + 30 - 16(2i+1)]\}, \\
&= \frac{2\pi M}{2i+1}\frac{i+1!}{i-1!}\{\tfrac{1}{8}\beta(j+4) - \tfrac{1}{768}\beta^2(j^2+16j+108)\} \\
&\qquad + \pi M\frac{i+1!}{i-1!}(-\tfrac{1}{2}\beta + \tfrac{1}{8}\beta^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (74).
\end{aligned}$$

For $\mathbf{S}_i^j$ we have only to replace the factor $1 - \tfrac{1}{4}\beta$ by $1 + \tfrac{1}{4}\beta$, and find

$$\begin{aligned}
\int p(\mathbf{S}_i^j)^2[(\mathfrak{P}_i^j)^2 - (\mathbf{P}_i^j)^2]d\sigma &= \frac{2\pi M}{2i+1}\frac{i+1!}{i-1!}\{\tfrac{1}{8}\beta(j+4) - \tfrac{1}{768}\beta^2(j^2-32j-84)\} \\
&\qquad + \pi M\frac{i+1!}{i-1!}(-\tfrac{1}{2}\beta - \tfrac{1}{8}\beta^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (74).
\end{aligned}$$

Again, omitting intermediate steps,

$$\int p\,(\mathfrak{C}_i{}^1)^2\left[(P_i{}^1)^2-\left(P_i{}^1\sqrt{\tfrac{1+\beta}{1-\beta}-\sin^2\theta}\over\cos^2\theta\right)^2\right]d\sigma$$

$$=\frac{2\pi M}{2i+1}\frac{i+1!}{i-1!}\{\tfrac23\beta\,(j+4)+\tfrac{1}{70}\pi\beta^2(55j^2+304j+756)\}$$

$$-\pi M\frac{i+1!}{i-1!}[\tfrac23\beta+\tfrac16\beta^2(2j+7)]\qquad\qquad(74).$$

$$\int p(\mathbf{O}_i{}^1)^2\left[(P_i{}^1)^2-\left(P_i{}^1\sqrt{\tfrac{1+\beta}{1-\beta}-\sin^2\theta}\over\cos^2\theta\right)^2\right]d\sigma$$

$$=\frac{2\pi M}{2i+1}\frac{i+1!}{i-1!}\{\tfrac23\beta\,(j+4)+\tfrac{1}{105}\beta^2(55j^2+160j+180)\}$$

$$-\pi M\frac{i+1!}{i-1!}[\tfrac23\beta+\tfrac16\beta^2\,(2j+1)]\quad.\qquad(74)$$

§ 17   *Portion of the Integration in the case of s=0.*

We are to find $\int p\left\{\tfrac{(\mathfrak{C}_i)^2}{(\mathbf{O}_i)^2}\right\}[(\mathbf{P}_i)^2-(P_i)^2]d\sigma$, leaving two integrals of the type $\int p(\mathfrak{C}_i)^2\,(P_i)^2d\sigma$ to be determined subsequently.

It is only necessary to consider $\mathfrak{C}_i$, since the other case is determinable from it by multiplication to $\frac{1}{(D_i)^2}$, as found in (38) of § 9.

Following the procedure of the case where $s=1$, we have

$$\int F_i d\theta=-2\beta q_2\int PP'd\mu+2\beta^2\int[q_4PP'+\tfrac13(q_2)^2(P')^2]\,d\mu,$$

$$=\frac{2}{2i+1}[2\beta q_2 i\,(i-1)+2\beta^2 q_4 i\,(i-1)\,(i-2)\,(i-3)+\beta^2\,(q_2)^2(i+1)\,(i+2)\,i(i-1)].$$

Then since $q_2=\tfrac14$, $q_4=\tfrac{1}{128}$,

$$\int F_i d\theta=\frac{2}{2i+1}\{\tfrac13\beta[j+1-(2i+1)]+\tfrac{1}{64}\beta^2[5j^2+14j+12-4\,(2i+1)\,(j+3)]\}.$$

Now $\qquad(\mathfrak{C}_i)^2=1+2\beta p_2\cos2\phi+\beta^2[(2p_2+\tfrac12(p_2)^2)\cos4\phi+\tfrac12\,(p_2)^2],$

and as far as material

$$\frac{(\mathfrak{C}_i)^2}{(1-\beta\cos2\phi)^2}=1+\beta^2[\tfrac12p_2+\tfrac12(p_2)^2+\tfrac{1}{16}]+\beta\,(\tfrac12+2p_2)\cos2\phi,$$

$$=\{1+\tfrac{1}{16}\beta^2(j^2-4j+6)\}\{1-\tfrac12\beta\,(j-1)\cos2\phi\},\quad.\quad.\quad(75),$$

since $\qquad p_2=-\tfrac14 i\,(i+1)=-\tfrac14 j.$

At present we only require this to the first power of β, and since $\int \mathbf{F}_i d\theta$ does not contain $\cos 2\phi$, the expression (75) as far as at present needed is simply unity.

Again, by (38) of § 9,

$$\frac{1}{(\mathbf{D}_i)^2} = 1 + \tfrac{1}{8}\beta^2 (j-3),$$

therefore by multiplication

$$\frac{(\mathbf{C}_i)^2}{(1-\beta\cos 2\phi)^2} = \{1 + \tfrac{1}{16}\beta^2(j^2-6)\}\,\{1 - \tfrac{1}{2}\beta(j-1)\cos 2\phi\} \quad . \quad . \quad . \quad (76).$$

This is also unity to the order at present needed.

Hence

$$\int p \left\{\frac{(\mathbf{C}_i)^2}{(\mathbf{O}_i)^2}[(\mathbf{P}_i)^2 - (\mathbf{P}_i)^2]d\sigma = \frac{4\pi}{2i+1}\,\mathbf{M}\,(1-\beta)^{\frac{1}{4}}\{\tfrac{1}{2}\beta[j+1-(2i+1)]\right.$$
$$+ \tfrac{1}{64}\beta^2[5j^2+14j+12-4(2i+1)(j+3)]\}$$
$$= \frac{4\pi\mathbf{M}}{2i+1}\{\tfrac{1}{2}\beta(j+1) + \tfrac{1}{64}\beta^2(5j^2-2j-4)\}$$
$$- \pi\mathbf{M}\{2\beta + \tfrac{1}{4}\beta^2(j-1)\} \quad . \quad . \quad (77).$$

§ 18. *Preparation for the Integrations when $s=1$ and 0.*

We have now to evaluate the three integrals

$$\mathbf{L} = \int p\,(\mathbf{S}_i{}^1\,\mathbf{P}_i{}^1)^2\,d\sigma,$$
$$\mathbf{M} = \int p\left(\mathbf{C}_i\,\frac{\mathbf{P}_i{}^1}{\cos\theta}\right)^2\left(\frac{1+\beta}{1-\beta} - \sin^2\theta\right)d\sigma, \qquad . \quad (78),$$
$$\mathbf{N} = \int p\,(\mathbf{C}_i\,\mathbf{P}_i)^2\,d\sigma$$

and from these to determine three others when $\mathbf{S}$, $\mathbf{O}$ replace $\mathbf{S}$, $\mathbf{C}$.

We have

$$\frac{p\,d\sigma}{d\theta\,d\phi} = \left\{\frac{\mathbf{M}(1-\beta)^{\frac{1}{4}}\left(\frac{1-\beta}{1+\beta}\right)^{\frac{1}{4}}}{(1-\frac{1-\beta}{1+\beta}\sin^2\theta)^{\frac{1}{4}}}\right\}\left\{\frac{\cos^2\theta + \frac{\beta(1-\cos 2\phi)}{1-\beta}}{(1-\beta\cos 2\phi)^2}\right\},$$

$$\frac{p\left(\frac{1+\beta}{1-\beta}-\sin^2\theta\right)d\sigma}{d\theta\,d\phi} = \left\{\mathbf{M}\,(1-\beta)^{\frac{1}{4}}\left(\frac{1+\beta}{1-\beta}\right)^{\frac{1}{4}}\left(1 - \frac{1-\beta}{1+\beta}\sin^2\theta\right)^{\frac{1}{4}}\right\}\left\{\frac{\cos^2\theta + \frac{\beta(1-\cos 2\phi)}{1-\beta}}{(1-\beta\cos 2\phi)^2}\right\}.$$

It is the second factor which alone involves ϕ, and as I shall now first integrate with respect to ϕ, the first factor may be dropped for the moment, and the second factor multiplied by the squares of the cosine or sine functions. Since the integration is from $\phi = 2\pi$ to 0, those terms which vanish on integration may be dropped.

For brevity write

$$j = i(i+1),$$
$$j_0 = \tfrac{1}{16}(j^2 - 4j + 6),$$
$$j_1 = \tfrac{1}{216}(j^3 - 12j + 68).$$

Then we have seen in (73) and (75) that

$$\frac{(\mathfrak{S}_i{}^1)^2}{(1-\beta\cos 2\phi)^4} = \tfrac{1}{2}(1 - \tfrac{1}{4}\beta + \beta^2 j_1)\{1 - [1 + \tfrac{1}{8}\beta(j-4)]\cos 2\phi\},$$

$$\frac{(\mathfrak{C}_i{}^1)^2}{(1-\beta\cos 2\phi)^4} = \tfrac{1}{2}(1 + \tfrac{1}{4}\beta + \beta^2 j_1)\{1 + [1 - \tfrac{1}{8}\beta(j-4)]\cos 2\phi\},$$

$$\frac{(\mathfrak{C}_i)^2}{(1-\beta\cos 2\phi)^3} = (1 + \beta^2 j_0)\{1 - \tfrac{1}{2}\beta(j-1)\cos 2\phi\}.$$

Therefore

$$\int_0^{2\pi} \frac{\cos^2\theta + \frac{\beta(1-\cos 2\phi)}{1-\beta}}{(1-\beta\cos 2\phi)^4} \cdot (\mathfrak{S}_i{}^1)^2\, d\phi = \pi(1 - \tfrac{1}{4}\beta + \beta^2 j_1)\{\cos^2\theta + \tfrac{3}{2}\beta + \tfrac{1}{16}\beta^2(j+20)\},$$

$$\int_0^{2\pi} \quad\cdots\quad (\mathfrak{C}_i{}^1)^2\, d\phi = \pi(1 + \tfrac{1}{4}\beta + \beta^2 j_1)\{\cos^2\theta + \tfrac{1}{2}\beta + \tfrac{1}{16}\beta^2(j+4)\}, \qquad (78).$$

$$\int_0^{2\pi} \quad\cdots\quad (\mathfrak{C}_i)^2\, d\phi = 2\pi(1 + \beta^2 j_0)\{\cos^2\theta + \beta + \tfrac{1}{4}\beta^2(j+3)\}$$

Now pick out the parts of $pd\sigma$ and of these integrals (78) which are independent of θ, and write

$$\mathrm{F} = \pi \mathrm{M}(1-\beta)^{\frac{1}{2}}\left(\frac{1-\beta}{1+\beta}\right)^{\frac{1}{2}}(1 - \tfrac{1}{4}\beta + \beta^2 j_1)$$
$$= \pi \mathrm{M}\{1 - \tfrac{7}{4}\beta + \tfrac{1}{216}\beta^2(j^2 - 12j + 388)\},$$

$$\mathrm{G} = \pi \mathrm{M}(1-\beta)^{\frac{1}{2}}\left(\frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}(1 + \tfrac{1}{4}\beta + \beta^2 j_1)$$
$$= \pi \mathrm{M}\{1 + \tfrac{3}{4}\beta + \tfrac{1}{216}\beta^2(j^2 - 12j + 68)\}, \qquad (79).$$

$$\mathrm{H} = 2\pi \mathrm{M}(1-\beta)^{\frac{1}{2}}\left(\frac{1-\beta}{1+\beta}\right)^{\frac{1}{2}}(1 + \beta^2 j_0)$$
$$= 2\pi \mathrm{M}\{1 - \tfrac{3}{2}\beta + \tfrac{1}{16}\beta^2(j^2 - 4j + 34)\}$$

Also write

$$f = \tfrac{2}{3}(1 + \tfrac{1}{24}\beta j + \tfrac{1}{8}\beta),$$
$$g = \tfrac{1}{3}\{1 + \tfrac{1}{8}\beta j + \tfrac{1}{8}\beta\},$$
$$h = 1 + \tfrac{1}{4}\beta j + \tfrac{3}{8}\beta, \qquad\qquad\qquad (80).$$
$$\kappa'^2 = 1 - \kappa^2 = \frac{1-\beta}{1+\beta}, \text{ so that } \kappa^2 = 2\beta + 2\beta^2 + \cdots$$

Lastly, in accordance with the usual notation for elliptic integrals, write

$$\Delta^2 = 1 - \frac{1-\beta}{1+\beta}\sin^2\theta = 1 - \kappa'^2\sin^2\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (80).$$

Then we have

$$\left.\begin{array}{l}
L = F\displaystyle\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\cos^2\theta + \beta f}{\Delta}(P_i{}^1)^2 d\theta \\[2ex]
M = G\displaystyle\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} (\cos^2\theta + \beta g).\Delta.(P_i{}^1)^2 d\theta \\[2ex]
N = H\displaystyle\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\cos^2\theta + \beta h}{\Delta}(P_i{}^1)^2 d\theta
\end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (81).$$

The next step is to express the squares of the P's in a series of powers of $\cos^2\theta$.

When $P_i{}^s(\mu) = \frac{(1-\mu^2)^{\frac{1}{2}s}}{2^i.i!}\left(\frac{d}{d\mu}\right)^{i+s}(\mu^2-1)^i$, it is known that

$$P_i(\mu\mu' + (1-\mu^2)^{\frac{1}{2}}(1-\mu'^2)^{\frac{1}{2}}\cos\phi) = P_i(\mu)P_i(\mu') + 2\sum_{s=1}^{s=i}\frac{i-s!}{i+s!}P_i{}^s(\mu)P_i{}^s(\mu')\cos s\phi.$$

By putting $\mu = \mu'$ we see that $2\frac{i-s!}{i+s!}(P_i{}^s(\mu))^2$ is the coefficient of $\cos s\phi$ in the expansion of $P_i(1 - (1-\mu^2)2\sin^2\frac{1}{2}\phi)$. By TAYLOR's theorem this last is equal to

$$\sum_{r=0}^{r=i}(-)^r\frac{(1-\mu^2)^r}{r!}(2\sin^2\tfrac{1}{2}\phi)^2\left(\frac{d^r}{d\mu^r}P_i(\mu),\ \mu=1\right).$$

Now

$$\left(\frac{d}{d\mu}\right)^r P_i = \frac{1}{2^i.i!}\left(\frac{d}{d\mu}\right)^{i+r}(\mu^2-1)^i = \frac{1}{2^r.r!}\frac{i+r!}{i-r!}[1 + \text{terms involving powers of } \mu^2-1],$$

$$= \frac{1}{2^r.r!}\frac{i+r!}{i-r!}, \text{ when } \mu=1.$$

Also

$$\sin^{2r}\tfrac{1}{2}\phi = \left(\frac{e^{\frac{1}{2}\phi\sqrt{-1}} - e^{-\frac{1}{2}\phi\sqrt{-1}}}{2\sqrt{-1}}\right)^{2r}$$

$$= \sum_{-r}^{+r}(-)^{r-t}\frac{1}{2^{2r}}\cdot\frac{2r!}{2r-t!\,t!}\smile$$

On putting $r-t=s$, we see that the coefficient of $\cos s\phi$ in $\sin^{2r}\tfrac{1}{2}\phi$ is

$$\frac{(-)^r}{2^{2r-1}}\cdot\frac{2r!}{r-s!\,r+s!}.$$

Hence we have[*]

$$2\frac{i-s!}{s+s!}(P_i{}^s(\mu))^2 = 2\sum_{r=s}^{r=i}(-)^{r+s}\frac{2r!}{2^{2r}r-s!(r!)^2r+s!}\frac{s+r!}{i-r!}\cos^{2r}\theta \quad . \quad . \quad (82).$$

Now suppose

$$(P_i{}^1)^2 = \sum_i^i \gamma_{2r-2}\cos^{2r}\theta,$$

$$(P_i)^2 = \sum_0^i \alpha_{2r}\cos^{2r}\theta.$$

Then clearly   (82).

$$\gamma_{2r-2} = (-)^{r+}\frac{2r!}{2^{2r}r-1!(r!)^2r+1!}\cdot\frac{}{i-r!}\cdot\frac{}{i-1!},$$

$$\alpha_{2r} = (-)^r\frac{2r!}{2^{2r}(r!)^2}\frac{i+r!}{i-r!}$$

Therefore

$$\frac{L}{F} = \sum_{r=1}^{r=i}\gamma_{2r-2}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\frac{\cos^{2r+2}\theta+\beta f\cos^{2r}\theta}{\Delta}d\theta,$$

$$\frac{M}{G} = \sum_{r=1}^{r=i}\gamma_{2r-2}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}(\cos^{2r}\theta+\beta g\cos^{2r-2}\theta)\Delta d\theta, \quad\Bigg\} \quad . \quad . \quad . \quad . \quad (83).$$

$$\frac{N}{G} = \sum_{r=0}^{r=i}\alpha_{2r}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\frac{\cos^{2r+2}\theta+\beta h\cos^{2r}\theta}{\Delta}d\theta$$

The evaluation of these integrals depends on two integrals only, namely, $\int\dfrac{\cos^{2n}\theta}{\Delta}d\theta$ and $\int\cos^{2n}\theta\cdot\Delta d\theta$, and these will be considered in the next section.

§ 19. *Evaluation of the Integrals* $\int\dfrac{\cos^{2n}\theta}{\Delta}d\theta$ *and* $\int\cos^{2n}\theta\Delta d\theta$.

I will denote these integrals D and E respectively, and I propose to find their values in series proceeding by powers of κ^2.

The usual notation is adopted where $\Pi(x)$ is such a function that it is equal to $x\Pi(x-1)$; accordingly when x is a positive integer $\Pi(x) = x!$.

Since κ'^2 is less than unity

$$\frac{1}{\Delta} = \sum_0^\infty\frac{\frac{1}{2}\cdot\frac{3}{2}\cdots\frac{2r-1}{2}}{r!}\kappa'^{2r}\sin^{2r}\theta;$$

and since

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\cos^{2n}\theta\sin^{2r}\theta d\theta = \pi\frac{\frac{1}{2}\cdot\frac{3}{2}\cdots\frac{2r-1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdots\frac{2n-1}{2}}{n+r!},$$

$$D = \int\frac{\cos^{2n}\theta}{\Delta}d\theta = \pi\frac{\frac{1}{2}\cdot\frac{3}{2}\cdots\frac{2n-1}{2}}{n!}\sum_0^\infty\frac{(\frac{1}{2}\cdot\frac{3}{2}\cdots\frac{2r-1}{2})^2\kappa'^2}{(n+1)(n+2)\ldots(n+r)r!},$$

---

[*] Mr. HOBSON kindly gave me this proof when I had shown him the series which I believed to hold true.

or, with the usual notation for hypergeometric series,

$$D = \pi \frac{2n!}{2^{2n}(n!)^2} F(\tfrac{1}{2}, \tfrac{1}{2}, n+1, \kappa'^2).$$

This series is of no service, since it proceeds by powers of κ'^2, which in our case is nearly unity. It is required then to transform the series into one proceeding by powers of κ^2.

It is known that, if $\kappa^2 + \kappa'^2 = 1$,

$$F(a, b, c, \kappa'^2) = \frac{\Pi(c-a-b-1)\,\Pi(c-1)}{\Pi(c-a-1)\,\Pi(c-b-1)} F(a, b, 1+a+b-c, \kappa^2)$$

$$+ \kappa^{2(c-a-b)} \frac{\Pi(a+b-c-1)\,\Pi(c-1)}{\Pi(a-1)\,\Pi(b-1)} F(c-a, c-b, c-a-b+1, \kappa^2).^*$$

If we apply this theorem with $a = b = \tfrac{1}{2}$, $c = n+1$, the first F becomes $F(\tfrac{1}{2}, \tfrac{1}{2}, 1-n, \kappa^2)$, whose nth and all subsequent terms involve zero factors in the denominators. Also the coefficient of the second F involves $\Pi(-n-1)$, which has an infinite factor. Hence the formula leads to an indeterminate result. Let us therefore put $c = n+1+\epsilon$, and proceed to the limit when $\epsilon = 0$.

We have then

$$D = \text{Limit } \pi \frac{2n!}{2^{2n}(n!)^2} \left\{ \frac{\Pi(n-1+\epsilon)\,\Pi(n+\epsilon)}{[\Pi(n-\tfrac{1}{2}+\epsilon)]^2} F(\tfrac{1}{2}, \tfrac{1}{2}, 1-n-\epsilon, \kappa^2) \right.$$

$$\left. + \kappa^{2n+\epsilon} \frac{\Pi(-n-1-\epsilon)\,\Pi(n+\epsilon)}{[\Pi(-\tfrac{1}{2})]^2} F(n+\tfrac{1}{2}+\epsilon, n+\tfrac{1}{2}+\epsilon, n+1+\epsilon, \kappa^2) \right\}.$$

Now $\Pi(\epsilon) = 1 + \epsilon \Pi'(0)$, $\Pi(-\tfrac{1}{2}+\epsilon) = \Pi(-\tfrac{1}{2}) \left(1 + \epsilon \dfrac{\Pi'(-\tfrac{1}{2})}{\Pi(-\tfrac{1}{2})}\right)$.

Therefore, when ϵ is very small,

$$\frac{\Pi(n-1+\epsilon)}{\Pi(n-1)} = 1 + \epsilon \left(-\frac{1}{n-1} + \frac{1}{n-2} + \ldots + \tfrac{1}{2} + 1\right) + \epsilon \Pi'(0)$$

$$= 1 + \epsilon \left(\Pi'(0) + \sum_1^n \frac{1}{i} - \frac{1}{n}\right)$$

$$\frac{\Pi(n+\epsilon)}{\Pi(n)} = 1 + \epsilon \left(\Pi'(0) + \sum_1^n \frac{1}{i}\right)$$

$$\frac{\Pi(n-\tfrac{1}{2}+\epsilon)}{\Pi(n-\tfrac{1}{2})} = 1 + 2\epsilon \left(\frac{1}{2n-1} + \frac{1}{2n-3} + \ldots + \tfrac{1}{3} + 1\right) + \epsilon \frac{\Pi'(-\tfrac{1}{2})}{\Pi(-\tfrac{1}{2})}$$

$$= 1 + \epsilon \left(\frac{\Pi'(-\tfrac{1}{2})}{\Pi(-\tfrac{1}{2})} + 2\sum_1^n \frac{1}{2i-1}\right)$$

---

* I have to thank Mr. HOBSON for giving me this formula, and for showing me the procedure whereby it can be made effective.

Hence for the coefficient of the first series we have

$$\frac{\Pi(n-1+\epsilon)\,\Pi(n+\epsilon)}{[\Pi(n-\tfrac{1}{2}+\epsilon)]^2} = \frac{\Pi(n-1)\,\Pi(n)}{\Pi(n-\tfrac{1}{2})}\left\{1+\epsilon\left(2\Pi'(0)+2\sum_1^n\frac{1}{i}-\frac{1}{n}-2\frac{\Pi(-\tfrac{1}{2})}{\Pi(-\tfrac{1}{2})}-4\sum_1^n\frac{1}{2i-1}\right)\right\}$$

But

$$\Pi\left(-\tfrac{1}{2}\right)=\pi^{\frac{1}{2}} \qquad\qquad \Pi'(0)=\frac{\Pi'(-\tfrac{1}{2})}{\Pi(-\tfrac{1}{2})}=\log_\epsilon 4 \;*$$

$$[\Pi(n-\tfrac{1}{2})]^2=\pi\left(\frac{2n\,!}{2^{2n}\,n\,!}\right)^2 \qquad \Pi(n)\,\Pi(n-1)=n\,!\,n-1\,!$$

Therefore

$$\pi\,\frac{2n\,!}{2^{2n}(n\,!)^2}\,\frac{\Pi(n-1+\epsilon)\,\Pi(n+\epsilon)}{[\Pi(n-\tfrac{1}{2}+\epsilon)]^2}=2^{2n}\,\frac{n\,!\,n-1\,!}{2n\,!}\left\{1+\epsilon\left(2\log 4-\frac{1}{n}+2\sum_1^n\frac{1}{i}-4\sum_1^n\frac{1}{2i-1}\right)\right\}$$

This is true from $n=\infty$ to 1, but in the case of $n=0$ we have

$$\Pi(-1+\epsilon)=\frac{1}{\epsilon}\Pi(\epsilon)=\frac{1}{\epsilon}+\Pi'(0)$$

so that in that case

$$\pi\,\frac{\Pi(-1+\epsilon)\,\Pi(\epsilon)}{[\Pi(-\tfrac{1}{2}+\epsilon)]^2}=\frac{1}{\epsilon}+2\log 4$$

Now consider the coefficient of the second series

We have

$$\kappa^{2n+2\epsilon}=\kappa^{2n}(1+2\epsilon\log_\epsilon\kappa),$$

and since

$$\Pi(-x)\,\Pi(x-1)=\frac{\pi}{\sin\pi x}$$

$$\Pi(-n-1-\epsilon)\,\Pi(n+\epsilon)=\frac{\pi}{\sin(n+1+\epsilon)\pi}=\frac{(-)^{n+1}}{\epsilon}\pi \qquad [\Pi(-\tfrac{1}{2})]=\pi$$

Therefore the coefficient of the second series is $(-)^{n+1}\dfrac{2n}{2^{2n}(n\,!)^2}\,\dfrac{\kappa^{2n}}{\epsilon}\,(1+2\epsilon\log\kappa)$, and

$$D=\frac{2^{2n}\,n\,!\,n-1\,!}{2n\,!}\left[1+\epsilon\left(2\log 4-\frac{1}{n}+2\sum_1^n\frac{1}{i}-4\sum_1^n\frac{1}{2i-1}\right)\right]F(\tfrac{1}{2}\;\tfrac{1}{2}\;1-n-\epsilon,\kappa^2)$$

$$+(-)^{n+1}\frac{2n\,!}{2^{2n}(n\,!)^2}\,\kappa^{2n}\,\frac{1}{\epsilon}\,(1+2\epsilon\log\kappa)\,F(n+\tfrac{1}{2}+\epsilon,\,n+\tfrac{1}{2}+\epsilon,\,n+1+\epsilon\;\kappa)$$

The case of $n=0$ is an exception, for the coefficient of the first F has the part inside $[\;\;]$ replaced by $\dfrac{1}{\epsilon}(1+2\epsilon\log 4)$

---

* Proved by differentiating the known formula $\Pi(x-1)\,\Pi(x-\tfrac{1}{2})=\Pi(2x-1)\cdot\dfrac{(4\pi)^{\frac{1}{2}}}{4^x}$, and putting $x=\tfrac{1}{2}$.

It remains to consider these two F series.

$$F\left(\tfrac{1}{2}, \tfrac{1}{2}, 1-n-\epsilon, \kappa^2\right) = \sum_0^\infty \frac{[1.3\ldots(2r-1)]^2 \kappa^{2r}}{2^{2r}(1-n-\epsilon)(2-n-\epsilon)\ldots(r-n-\epsilon).r!}$$

$$= \sum_0^{n-1}(-)^r \frac{[1.3\ldots(2r-1)]^2 \kappa^{2r}}{2^{2r}(n-1+\epsilon)(n-2+\epsilon)\ldots(n-r+\epsilon)r!}$$

$$+(-)^n \sum_n^\infty \frac{[1.3\ldots(2r-1)]^2 \kappa^{2r}}{2^{2r}(n-1+\epsilon)\ldots(1+\epsilon)\epsilon(1-\epsilon)\ldots(r-n-\epsilon).r!}$$

When $r < n$

$$\frac{1}{(n-1+\epsilon)(n-2+\epsilon)\ldots(n-r+\epsilon)} = \frac{1}{(n-1)(n-2)\ldots(n-r)}\left(1-\epsilon\sum_{n-1}^{n-1}\frac{1}{t}\right)$$

When $r > n$, put $r = n+s$, and

$$\frac{1}{(n-1+\epsilon)(n-2+\epsilon)\ldots(1+\epsilon)\epsilon(1-\epsilon)\ldots(s-\epsilon)} = \frac{1}{n-1!\,s!}\cdot\frac{1}{\epsilon}\left[1-\epsilon\sum_1^n\frac{1}{t}+\epsilon\sum_1^s\frac{1}{r}+\frac{\epsilon}{n}\right].$$

Also when $r = n+s$

$$\kappa^{2r}\frac{[1.3\ldots(2r-1)]^2}{2^{2r}.r!} = \frac{(2n!)^2}{2^{4n}(n!)^2}\kappa^{2n}\cdot\frac{[(2n+1)(2n+3)\ldots(2n+2s-1)]^2}{2^{2s}(n+1)(n+2)\ldots(n+\kappa)}\kappa^{2s}.$$

Thus

$$F\left(\tfrac{1}{2}, \tfrac{1}{2}, 1-n-\epsilon, \kappa^2\right) = \sum_{r=0}^{r=n-1}(-)^r \frac{[1.3\ldots(2r-1)]^2}{2^{2r}(n-1)\ldots(n-r)r!}\left(1-\epsilon\sum_{n-1}^{n-1}\frac{1}{t}\right)\kappa^{2r}$$

$$+(-)^n\frac{(2n!)^2}{2^{4n}(n!)^4}n\kappa^{2n}\sum_{s=0}^{s=\infty}\frac{[(2n+1)\ldots(2n+2s-1)]^2}{2^{2s}(n+1)\ldots(n+s)s!}\cdot\frac{1}{\epsilon}\left(1-\epsilon\sum_1^n\frac{1}{t}+\frac{\epsilon}{n}+\epsilon\sum_1^s\frac{1}{t}\right)\kappa^{2s}.$$

It follows that we may write the first term of D as follows:—

$$(-)^n\frac{2n!}{2^{2n}(n!)^2}\frac{\kappa^{2n}}{\epsilon}\,F\left(n+\tfrac{1}{2}, n+\tfrac{1}{2}, n+1, \kappa^2\right) + \frac{2^{2n}n!\,n-1!}{2n!}\sum_0^{n-1}(-)^r\frac{[1.3\ldots(2r-1)]^2}{2^{2r}(n-1)\ldots(n-r).r!}\kappa^{2r}$$

$$+(-)^n\frac{2n!}{2^{2n}(n!)^2}\kappa^{2n}\sum_0^\infty\frac{[(2n+1)\ldots(2n+2s-1)]^2}{2^{2s}(n+1)\ldots(n+s).s!}\left[\sum_1^n\frac{1}{t}+\sum_1^s\frac{1}{t}-4\sum_1^n\frac{1}{2t-1}+2\log 4\right]\kappa^{2s}.$$

The first of these terms becomes infinite when $\epsilon = 0$.

Turning to the second F we have

$$F\left(n+\tfrac{1}{2}+\epsilon, n+\tfrac{1}{2}+\epsilon, n+1+\epsilon, \kappa^2\right) = \sum_0^\infty\frac{[2n+1+2\epsilon)(2n+3+2\epsilon)\ldots(2n+2s-1+2\epsilon)]^2}{2^{2s}(n+1+\epsilon)(n+2+\epsilon)\ldots(n+s+\epsilon).s!}\kappa^{2r}$$

$$= \sum_0^\infty\frac{[(2n+1)(2n+3)\ldots(2n+2s-1)]^2}{2^{2s}(n+1)(n+2)\ldots(n+s).s!}\left[1+4\epsilon\sum_{n+1}^{n+s}\frac{1}{2t-1}-\epsilon\sum_{n+}^{n+s}\frac{1}{t}\right]\kappa^{2r}.$$

Thus the second term of D is

$$(-)^{n+1}\frac{2n!}{2^{2n}(n!)^2}\frac{\kappa^{2n}}{\epsilon}F\left(n+\tfrac{1}{2},n+\tfrac{1}{2},n+1,\kappa^2\right)$$

$$+(-)^{n+1}\frac{2n!}{2^{2n}(n!)^2}\kappa^{2n}\sum_0^{s}\frac{[(2n+1)..(2n+2s-1)]^2}{2^{2s}(n+1)\ (n+s)s!}\left[4\sum_{n+1}^{n+s}\frac{1}{2l-1}-\sum_{1}^{s+1}\frac{1}{l}+2\log\kappa\right]\kappa^{2s}.$$

The first term of this becomes infinite when $\epsilon=0$, but it is equal and opposite to the infinite term in the first part of D, and they annihilate one another.

Hence

$$D=\frac{2^{2n}n!\,n-1!}{2n!}\sum_0^{l-1}\frac{(-)^l[1\ \,]..(2l-1)]^l}{2^{2l}(n-1)..(n-l)l!}\kappa^{2l}$$

$$+(-)^n\frac{2n!}{2^{2n}(n!)^2}\kappa^2\sum_0^{s}\frac{[(2n+1)..(2n+2s-1)]^2}{2^{2s}(n+1)..(n+s)s!}\left[2\log\frac{4}{\kappa}+\sum_1^{s}\frac{1}{l}+\sum_1^{s}\frac{1}{l}-4\sum_1^{s}\frac{1}{2l-1}\right]\kappa^{2s}$$

On examining the case of $n=0$ we find that this formula also embraces it

provided we interpret $\overset{0}{\underset{1}{\sum}}$ as meaning zero

The coefficient in the last term admits of some simplification, for

$$\sum_1^{s}\frac{1}{l}+\sum_1^{s}\frac{1}{l}-4\sum_1^{n+s}\frac{1}{2l-1}=-\left[2\sum_1^{s+s}\frac{1}{l(2l-1)}+\sum_{l-1}^{l}\frac{1}{l+s}\right]$$

We thus conclude that D or

$$\int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi}\frac{\cos^{2n}\theta}{\Delta}d\theta=\frac{2^{2n}n!\,n-1!}{2n!}\left[1-\frac{1^2}{2^2(n-1)1!}\kappa^2+\frac{1^2\ 3^2}{2^4(n-1)(n-2)2!}\kappa^4-\dots\text{to }n\text{ terms}\right]$$

$$+(-)^{s}\frac{2n!}{2^{2n}(n!)^2}\kappa^2\left[\qquad\left(2\log\frac{4}{\kappa}-\sum_1^{s}\frac{1}{l}-2\sum_1^{n}\frac{1}{l(2l-1)}\right)\right.$$

$$+\frac{(2n+1)^2}{2^2(n+1)1!}\left(2\log\frac{4}{\kappa}-\sum_2^{s+1}\frac{1}{l}-2\sum_1^{n+1}\frac{1}{l(2l-1)}\right)\kappa^2$$

$$\left.+\frac{(2n+1)^2(2n+3)^2}{2^4(n+1)(n+2)2!}\left(2\log\frac{4}{\kappa}-\sum_3^{n+2}\frac{1}{l}-2\sum_1^{n+2}\frac{1}{l(2l-1)}\right)\kappa^4+\dots\right]\quad.\ (84).$$

The second integral E may be found as follows :—

$$E_n=\int\cos^{2n}\theta\ \Delta d\theta=\int\cos^{2n}\theta\left[\kappa^2+(1-\kappa^2)\cos^2\theta\right]\frac{d\theta}{\Delta},$$

$$=\kappa^2 D_n+(1-\kappa^2)\ D_{n+1}\quad.\quad.\quad.\quad.\quad.\quad(85).$$

From this I find E or

$$\int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} \cos^{2n}\theta \, . \, \Delta d\theta = \frac{2^{2n+1}(n!)^2}{2n+1!} + \frac{2^{2n-1}n!\,n-1!}{2n+1!} \sum_{0}^{\frac{1}{2}n-} (-)^r \frac{1 \cdot 3^2 \ldots (2r-1)^2 (2r+1)}{2^{2r}(n-1)(n-2)\ldots(n-r)\, . \, r+1!} \, \kappa^{2r+1}$$

$$+ \frac{(-)^n \, 2n!}{2^{2n+1}n!\,n+1!} \, \kappa^{2n} \sum_{1}^{\infty} \frac{(2n+1)(2n+3)^2 \ldots (2n+2s-3)^2(2n+2s-1)}{2^{2s-2}(n+2)(n+3)\ldots(n+s)\, . \, s-1!} \, \kappa^{2s}$$

$$\times \left[2 \log \frac{4}{\kappa} + \sum_{1}^{n+s!} \frac{1}{i} + \sum_{1}^{s-1!} \frac{1}{i} - 4 \sum_{1}^{n+s-1!} \frac{1}{2i-1} - \frac{2}{2n+2s-1} \right] \kappa^{2s} .$$

This is applicable also to the case of $n = 0$, provided that $\sum_{0}$ is interpreted as zero.

In the particular case in hand I find, however, that it is shorter not to use this general formula, but to carry out the transformation (85) in the particular cases where the result is needed.

§ 20. *Reduction of preceding integrals; disappearance of logarithmic terms.*

In the application of the integrals of the last section, we are to put $\kappa^2 = 1 - \dfrac{1-\beta}{1+\beta}$, and only to develop as far as β^2.

Then to the proposed order $\kappa^2 = 2\beta(1 - \beta)$, $\kappa^4 = 4\beta^2$.

Also

$$2 \log \frac{4}{\kappa} = \log \frac{8}{\beta} + \log (1+\beta) = \log \frac{8}{\beta} + \beta - \tfrac{1}{2}\beta^2 .$$

It will now facilitate future developments to adopt an abridged notation. I write then

$$f(n) = \frac{2^{2n}\,n!\,n-1!}{2n!} .$$

and observe that $f(n+1) = \dfrac{2n}{2n+1} f(n)$, and $f(1) = 2$, $f(2) = \tfrac{4}{3}$.

Since κ^2 is of the first order in β, only the first series in the D integral (84) enters when n is greater than 2. In that case

$$D = f(n) \left[1 - \frac{\beta - \beta^2}{2(n-1)} + \frac{9\beta^2}{8(n-1)(n-2)} \right],$$

$$= f(n) \left[1 - \frac{\beta}{2(n-1)} + \frac{(4n+1)\beta^2}{8(n-1)(n-2)} \right] . \quad \ldots \ldots \quad (86).$$

This result may be obtained very shortly without reference to the general formula; for when n is greater than 2

$$D = \left(\frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\cos^{2n}\theta \, d\theta}{(\cos^2\theta + 2\beta + 2\beta^2)^{\frac{1}{2}}}$$

$$\left(\frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \cos^{2n-1}\theta \left[1 - \frac{\beta + \beta^2}{\cos^2\theta} + \frac{3\beta^2}{2\cos^4\theta} \right] d\theta.$$

The integral of an odd power of $\cos\theta$ is easily determined, and it will be found that the result (86) is obtained. It is, however, clear that if n is not greater than 2 the development in powers of $\sec^2\theta$ is not legitimate.

When n is not greater than 2 the formula (84) of the last section is necessary, and we find

$$\int\frac{\cos^4\theta}{\Delta}d\theta = \frac{3}{2}\beta^2\log\frac{8}{\beta} + f(2)\left[1 - \frac{1}{2}\beta - \frac{61}{16}\beta^2\right];$$

$$\int\frac{\cos^2\theta}{\Delta}d\theta = -(\beta + \frac{5}{4}\beta^2)\log\frac{8}{\beta} + f(1)\left[1 + \frac{3}{2}\beta + \frac{19}{16}\beta^2\right]; \qquad .(87).$$

$$\int\frac{d\theta}{\Delta} = (1 + \frac{1}{2}\beta + \frac{1}{16}\beta^2)\log\frac{8}{\beta} - \frac{5}{16}\beta^2$$

We have now to find the second integral E, and this may be done more easily than by reference to the general formula of the last section.

We have

$$E = \int\cos^{2n}\theta\,\Delta d\theta = (1 - \kappa^2)\int\frac{\cos^{2n+2}\theta}{\Delta}d\theta + \kappa^2\int\frac{\cos^{2n}\theta}{\Delta}d\theta,$$

$$= (1 - 2\beta + 2\beta^2)\int\frac{\cos^{2n+2}\theta}{\Delta}d\theta + 2\beta(1 - \beta)\int\frac{\cos^{2n}\theta}{\Delta}d\theta.$$

It will be observed that even when n is 2 the general formula (86) gives the D integral as far as the first power of β. Hence in finding E we may use that general formula except when $n = 0, 1$.

Then since $f(n) = \frac{2n+1}{2n}f(n+1)$, when n is greater than 1,

$$E = f(n+1)\left[(1 - 2\beta + 2\beta^2)\left(1 - \frac{\beta}{2n} + \frac{(4n+5)\beta^2}{8n(n-1)}\right) + \frac{2n+1}{n}\beta(1-\beta)\left(1 - \frac{\beta}{2(n-1)}\right)\right],$$

$$= f(n+1)\left[1 + \frac{\beta}{2n} - \frac{(4n-1)}{8n(n-1)}\beta^2\right]. \qquad . \quad . \quad . \quad . \quad . \quad (88).$$

But when $n = 1$,

$$E = (1 - 2\beta + 2\beta^2).\frac{3}{8}\beta^2\log\frac{8}{\beta} - 2\beta(1-\beta).\beta(1 + \frac{3}{4}\beta)\log\frac{8}{\beta}$$

$$\qquad + f(2)\left[(1 - 2\beta + 2\beta^2)(1 - \frac{1}{2}\beta - \frac{61}{16}\beta^2) + 3\beta(1-\beta)(1 + \frac{3}{2}\beta)\right].$$

$$= -\frac{1}{2}\beta^2\log\frac{8}{\beta} + f(2)\left[1 + \frac{1}{2}\beta + \frac{1}{16}\beta^2\right]. \qquad . \quad . \quad . \quad . \quad (89).$$

And when $n = 0$,

$$E = -(1 - 2\beta + 2\beta^2)\beta(1 + \frac{5}{4}\beta)\log\frac{8}{\beta} + 2\beta(1-\beta)(1 + \frac{1}{2}\beta)\log\frac{8}{\beta}$$

$$\qquad + f(1)\left[(1 - 2\beta + 2\beta^2)(1 + \frac{3}{2}\beta + \frac{19}{16}\beta^2) - 2\beta(1-\beta).\frac{5}{16}\beta^2\right].$$

$$= \beta(1 - \frac{1}{2}\beta)\log\frac{8}{\beta} + f(1)\left[1 - \frac{1}{2}\beta + \frac{5}{16}\beta^2\right] \qquad . \quad . \quad . \quad . \quad . \quad (90).$$

I now wish to show that, in the use to which these integrals are to be put, the logarithmic terms disappear.

The following is a table of these integrals collected from (87), (89), (90), in as far only as they involve logarithms :—

$$\int \frac{d\theta}{\Delta} = (1 + \tfrac{1}{2}\beta + \tfrac{1}{16}\beta^2) \log \frac{8}{\beta}, \qquad \int \Delta d\theta = \beta(1 - \tfrac{1}{4}\beta) \log \frac{8}{\beta},$$

$$\int \frac{\cos^2 \theta}{\Delta} d\theta = -\beta (1 + \tfrac{5}{4}\beta) \log \frac{8}{\beta}, \quad \int \cos^2 \theta \Delta\, d\theta = -\tfrac{1}{2}\beta^2 \log \frac{8}{\beta}.$$

$$\int \frac{\cos^4 \theta}{\Delta} d\theta = \tfrac{3}{2}\beta^2 \log \frac{8}{\beta}.$$

Then the formulæ (83) for L, M, N, in so far only as is at present material, are

$$\frac{\mathrm{L}}{\mathrm{F}} = \int \left[\frac{\gamma_0}{\Delta} (\cos^4 \theta + \beta f \cos^2 \theta) + \frac{\gamma_2}{\Delta} \beta f \cos^4 \theta \right] d\theta.$$

$$\frac{\mathrm{M}}{\mathrm{G}} = \int \left[\gamma_0 \Delta \ (\cos^2 \theta + \beta g) + \gamma_2 \ \Delta \beta g \cos^2 \theta \right] d\theta.$$

$$\frac{\mathrm{N}}{\mathrm{H}} = \int \left[\frac{\alpha_0}{\Delta} (\cos^2 \theta + \beta h) + \frac{\alpha_2}{\Delta} (\cos^4 \theta + \beta h \cos^2 \theta) + \frac{\alpha_4}{\Delta} \beta h \ \cos^4 \theta \right] d\theta.$$

On using the integrals and only retaining squares of β, we find

$$\frac{\mathrm{L}}{\mathrm{F}} = \beta^2 \gamma_0 \ (\tfrac{3}{2} - f) \log \frac{8}{\beta}.$$

$$\frac{\mathrm{M}}{\mathrm{G}} = \beta^2 \gamma_0 \ (g - \tfrac{1}{2}) \log \frac{8}{\beta}.$$

$$\frac{\mathrm{N}}{\mathrm{H}} = \{\beta\alpha_0 \ [-(1 + \tfrac{5}{4}\beta) + h \ (1 + \tfrac{1}{2}\beta)] + \beta^2 \alpha_2(\tfrac{3}{2} - h)\} \log \frac{8}{\beta}.$$

But by definition of f and g and of the α's in (82), to the order zero of small quantities,

$$f = \tfrac{3}{2}, \ g = \tfrac{1}{2}, \ h = 1 + \tfrac{1}{4}\beta j + \tfrac{3}{4}\beta^2, \ \alpha_0 = 1, \ \alpha_2 = -\tfrac{1}{4}i(i+1) = -\tfrac{1}{2}j.$$

Thus the logarithmic terms entirely disappear, and henceforth may be dropped.

Thus, *as far as material*, we have the following table of integrals :—

$$\int \frac{d\theta}{\Delta} = -\tfrac{5}{16}\beta^2, \qquad \int \frac{\cos^2\theta}{\Delta}\,d\theta = f(1)[1 + \tfrac{3}{8}\beta + \tfrac{1}{1}\tfrac{9}{6}\beta^2].$$

$$\int \frac{\cos^4\theta}{\Delta}\,d\theta = f(2)[1 - \tfrac{1}{8}\beta - \tfrac{6}{1}\tfrac{1}{6}\beta^2].$$

$$\int \frac{\cos^{2n}\theta}{\Delta}\,d\theta = f(n)\left[1 - \frac{\beta}{2(n-1)} + \frac{(4n+1)\beta^2}{8(n-1)(n-2)}\right],\ n>2.$$

$$\int \Delta d\theta = f(1)[1 - \tfrac{1}{2}\beta + \tfrac{3}{16}\beta^2].$$

$$\int \cos^2\theta\Delta d\theta = f(2)[1 + \tfrac{1}{2}\beta + \tfrac{1}{1}\tfrac{1}{6}\beta^2].$$

$$\int \cos^{2n}\theta\Delta d\theta = f(n+1)\left[1 + \frac{\beta}{2n} - \frac{(4n-1)\beta^2}{8n(n-1)}\right],\ n>1$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\cdot\ \ \cdot\ \ \cdot\ \ \cdot\ \ \cdot\ \ \cdot\ \ (91).$$

Before using these for the determination of L, M, N, it is well to obtain one other result.

We have seen in (82) that

$$(P^s)^2 = \frac{i+s\,!}{i-s\,!}\sum_s(-)^{r+s}\frac{2r\,!}{2^{2r}r+s\,!(r\,!)^2r-s\,!}\cos^{2r}\theta.$$

Therefore $\qquad \displaystyle\int_{-1}^{+1}(P^s)^2 d\mu = \frac{i+s\,!}{i-s\,!}\sum_s(-)^{r+s}f(r+1)\frac{2r\,!}{2^{2r}r+s\,!(r\,!)^2r-s\,!}.$

But this integral is equal to $\dfrac{2}{2i+1}\dfrac{i+s\,!}{i-s\,!};$ therefore

$$\sum_s(-)^{r+s}f(r+1)\frac{2r\,!}{2^{2r}r+s\,!(r\,!)^2r-s\,!} = \frac{2}{2i+1}.$$

Putting $s=1$ and 0, and comparing with the values of α_{2r}, γ_{2r-2} in (82), we have

$$\sum_0 \alpha_{2r}f(r+1) = \frac{2}{2i+1}.$$

$$\sum_1 \gamma_{2r-2}f(r+1) = \frac{2}{2i+1}\cdot\frac{i+1\,!}{i-1\,!},\quad \Big\}\qquad\cdots\cdots\quad (92).$$

§ 21. *Integrals of the squares of harmonics when $s=1$ and $s=0$.*

In (83) we have

$$\frac{L}{F} = \sum_1^i\int_{-\frac12\pi}^{\frac12\pi}\frac{\gamma_{2r-2}}{\Delta}(\cos^{2r+2}\theta + \beta f\cos^{2r}\theta)\,d\theta.$$

Therefore, noting that $f(r) = \dfrac{2r+1}{2r}f(r+1)$, and using the integrals (91),

$$\frac{L}{F} = \sum_2^i \gamma_{2r-2}f(r+1)\left[1 - \frac{\beta}{2r} + \frac{(4r+5)\beta^2}{8r(r-1)} + \frac{2r+1}{2r}\beta\left(1 - \frac{\beta}{2(r-1)}\right)f\right]$$
$$+ \gamma_0 f(2)[1 - \tfrac{1}{2}\beta - \tfrac{6}{1}\tfrac{1}{6}\beta^2 + \tfrac{3}{2}\beta(1+\tfrac{3}{8}\beta)f].$$

Substituting for f (which the reader must not confuse with the functional f in use here) its value (80), the term of order zero is $\overset{i}{\underset{2}{\Sigma}}\gamma_{2r-2}f(r+1)$. By (92) this is equal

The term of the first order in β is

$$\beta\overset{i}{\underset{2}{\Sigma}}\gamma_{2r-2}f(r+1)\left(-\frac{1}{2r}+\frac{3(2r+1)}{4r}\right)+\beta\gamma_0 f(2)\left(-\frac{1}{2}+\frac{9}{4}\right).$$

which may be reduced to the form

$$\beta\overset{i}{\underset{1}{\Sigma}}\gamma_{2r-2}f(r+1)+\tfrac{1}{2}\beta\overset{i}{\underset{1}{\Sigma}}\gamma_{2r-2}f(r),\text{ and is equal to }\frac{2\beta}{2i+1}\frac{i+1\,!}{i-1\,!}+\tfrac{1}{2}\beta\overset{i}{\underset{1}{\Sigma}}\gamma_{2r-2}f(r).$$

The term of the second order in β is

$$\beta^2\overset{i}{\underset{2}{\Sigma}}\gamma_{2r-2}f(r+1)\left[\frac{4r+5}{8r(r-1)}+\frac{3(2r+1)}{4r}\left(-\frac{1}{2(r-1)}+\tfrac{3}{4}j+\tfrac{5}{8}\right)\right]$$
$$+\beta^2\gamma_0 f(2)\left[-\tfrac{61}{16}+\tfrac{9}{4}\left(\tfrac{5}{8}+\tfrac{1}{4\cdot4}j+\tfrac{5}{8}\right)\right].$$

This may be reduced to the form

$$\beta^2\left[\tfrac{1}{2}\overset{i}{\underset{1}{\Sigma}}\gamma_{2r-2}f(r+1)+\tfrac{3}{16}(j+12)\overset{i}{\underset{1}{\Sigma}}\gamma_{2r-2}f(r)-\tfrac{1}{4}\gamma_0\right],$$

of which the first term is $\tfrac{1}{2}\beta^2\cdot\dfrac{2}{2i+1}\dfrac{i+1\,!}{i-1\,!}$.

Therefore

$$\frac{\text{L}}{\text{F}}=(1+\beta+\tfrac{1}{2}\beta^2)\frac{2}{2i+1}\frac{i+1\,!}{i-1\,!}+\tfrac{1}{2}\beta[1+\tfrac{3}{8}\beta(j+12)]\overset{i}{\underset{1}{\Sigma}}\gamma_{2r-2}f(r)-\tfrac{1}{4}\beta^2\gamma_0.$$

Now

$$\gamma_{2r-2}f(r)=(-)^{r+1}\frac{i+1\,!}{i-1\,!}\cdot\frac{1}{r+1\,!\,r\,!}\frac{i+r\,!}{i-r\,!},$$

and

$$\overset{i}{\underset{1}{\Sigma}}\frac{(-)^{r+1}}{r+1\,!\,r\,!}\frac{i+r\,!}{i-r\,!}=\frac{(i+1)\,i}{1\,!\,2}-\frac{(i+1)(i+2)\,i(i-1)}{2\,!\,2.3}+\cdots$$
$$=1-F(i+1,-i,2,1).$$

It is known that

$$F(a,\,b,\,c,\,1)=\frac{\Pi(c-1)\Pi(c-a-b-1)^{*}}{\Pi(c-a-1)\Pi(c-b-1)}\qquad\cdots\cdots\quad(93)$$

Then since $\Pi(-i)$ contains an infinite factor

$$F(i+1,-i,2,1)=\frac{\Pi(1)\,\Pi(0)}{\Pi(-i)\Pi(i+1)}=0,$$

therefore

$$\overset{i}{\underset{1}{\Sigma}}\gamma_{2r-2}f(r)=\frac{i+1\,!}{i-1\,!}.\qquad\cdots\cdots\cdots\cdots\quad(93).$$

* I have again to thank Mr. Hobson for this formula, which is due to Gauss.

Also
$$\gamma_0 = \tfrac{1}{4}j^2(i+1)^2 = \tfrac{1}{4}j\,\frac{i+1\,!}{i-1\,!}.$$

Hence
$$\frac{L}{F} = \frac{2}{2i+1}\,\frac{i+1\,!}{i-1\,!}(1+\beta+\tfrac{1}{8}\beta^2) + \tfrac{1}{2}\beta\,\frac{i+1\,!}{i-1\,!}(1+\tfrac{3}{8}\beta).$$

Introducing the value of F as defined in (79), we have L or

$$\int p(\mathfrak{S}_i{}^1 P_i{}^1)^2 d\sigma = \frac{2\pi M}{2i+1}\cdot\frac{i+1\,!}{i-1\,!}\left[1-\tfrac{3}{2}\beta+\tfrac{1}{16}\beta^2(j^2-12j+68)\right]$$
$$+\pi M\,\frac{i+1\,!}{i-1\,!}[\tfrac{1}{2}\beta-\tfrac{1}{8}\beta^2].$$

We have in (74) obtained $\int p(\mathfrak{S}_i{}^1)^2[(\mathfrak{P}_i{}^1)^2-(P_i{}^1)^2]d\sigma$, and if it be added to our last result we see that the term which does not involve the factor $1/(2i+1)$ is annihilated, and

$$\int p(\mathfrak{P}_i{}^1\mathfrak{S}_i{}^1)^2 d\sigma = \frac{2\pi M}{2i+1}\,\frac{i+1\,!}{i-1\,!}\left[1+\tfrac{1}{2}\beta(j-2)+\tfrac{1}{16}\beta^2(j^2-26j+48)\right]\quad . \quad (94).$$

Now from (37) the square of the factor for converting $\mathfrak{S}_i{}^1$ into $\mathsf{S}_i{}^1$ is

$$\left[\frac{1}{\mathfrak{p}_i{}^1}(\sin)\right]^2 = 1+\tfrac{1}{2}\beta+\tfrac{1}{16}\beta^2(j-8).$$

Therefore

$$\int p(\mathfrak{P}_i{}^1\mathsf{S}_i{}^1)^2 d\sigma = \frac{2\pi M}{2i+1}\cdot\frac{i+1\,!}{i-1\,!}\left[1+\tfrac{1}{2}\beta(j+2)+\tfrac{1}{16}\beta^2(j^2+10j-96)\right].\quad . \quad (94).$$

These are two of the required integrals.

* * *

Next we have from (83)

$$\frac{M}{G} = \overset{i}{\underset{1}{\Sigma}}\int\gamma_{2i-2}\left(\cos^{2r}\theta+\beta g\cos^{2r-2}\theta\right)\Delta\,d\theta.$$

Noting as before that $f(r) = \dfrac{2r+1}{2r}f(r+1)$, and using the integrals (91),

$$\frac{M}{G} = \overset{i}{\underset{1}{\Sigma}}\gamma_{2r-2}f(r+1)\left[1+\frac{\beta}{2r}-\frac{(4r-1)\beta^2}{8(r-1)r}+\frac{2r+1}{2r}\,\beta g\left(1+\frac{\beta}{2(r-1)}\right)\right]$$
$$+\gamma_0 f(2)\left[1+\tfrac{1}{2}\beta+\tfrac{11}{16}\beta^2+\tfrac{3}{2}\beta g\left(1-\tfrac{1}{2}\beta\right)\right].$$

Substituting for g its value from (80), I find the term of order zero to be $\overset{i}{\underset{1}{\Sigma}}\gamma_{2r-2}f(r+1)$, or $\dfrac{2}{2i+1}\cdot\dfrac{i+1\,!}{i-1\,!}$.

The term of the first order is $\beta\overset{i}{\underset{2}{\Sigma}}\gamma_{2r-2}f(r+1)\left[\dfrac{1}{2r}+\dfrac{2r+1}{4r}\right]+\beta\gamma_0 f(2)\left(\tfrac{3}{4}+\tfrac{1}{2}\right)$.

This may be reduced to the form $-\beta \sum_{1}^{i}\gamma_{2r-2} f(r+1) + \frac{2}{3}\beta \sum_{1}^{i}\gamma_{2r-2} f(r)$; which by (92) and (93) becomes $-\frac{2\beta}{2i+1}\frac{i+1!}{i+1!} + \frac{2}{3}\beta\frac{i+1!}{i-1!}$.

The term of the second order is

$$\beta^2 \sum_{2}^{i}\gamma_{2r-4} f(r+1)\left[-\frac{(4r-1)}{8r(r-1)} + \frac{2r+1}{4r}\left(\frac{1}{2(r-1)} + \frac{1}{8}j + \frac{1}{2}\right)\right] + \beta^2\gamma_0 f(2)\left(\frac{1}{16} + \frac{3}{16}j\right).$$

This is reducible to

$$\tfrac{1}{2}\beta^2 \sum_{1}^{i}\gamma_{2r-2} f(r+1) + \tfrac{1}{16}\beta^2 (j-4)\sum_{1}^{i}\gamma_{2r-2} f(r) + \tfrac{3}{4} \beta^2\gamma_0 \; ; \text{ which becomes}$$

$$\tfrac{1}{2}\beta^2 \frac{2}{2i+1}\cdot\frac{i+1!}{i-1!} + \tfrac{1}{16} \beta^2 (j-4)\frac{i+1!}{i-1!} + \tfrac{3}{16} \beta^2 \frac{i+1!}{i-1!}j.$$

Therefore

$$\frac{M}{G} = \frac{2}{2i+1}\frac{i+1!}{i-1!}(1-\beta+\tfrac{1}{2}\beta^2) + \frac{i+1!}{i-1!}\left[\tfrac{2}{3}\beta + \tfrac{1}{8}\beta^2 (j-1)\right].$$

Introducing for G its value (79), we find M or

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} p\left(\mathbb{C}_i^1 \frac{P_i^1}{\cos\theta}\right)^2\left(\frac{1+\beta}{1-\beta}-\sin^2\theta\right)d\sigma = \frac{2\pi M}{2i+1}\frac{i+1!}{i-1!}\left[1-\tfrac{1}{2}\beta+\tfrac{1}{16}\beta^2(j^2-12j+4)\right]$$
$$+ \pi M\frac{i+1!}{i-1!}\left[\tfrac{2}{3}\beta+\tfrac{1}{8}\beta^2(2j+7)\right].$$

But in (74) we have $\int p(\mathbb{C}_i^1)^2\left[(\mathbf{P}_i^1)^2-\left(P_i^1\sqrt{\frac{\frac{1+\beta}{1-\beta}-\sin^2\theta}{\cos^2\theta}}\right)^2\right]d\sigma$. If this be added to the result just found the term which has not $1/(2i+1)$ as a factor is annihilated, and

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} p(\mathbf{P}_i^1\mathbb{C}_i^1)^2 d\sigma = \frac{2\pi M}{2i+1}\frac{i+1!}{i-1!}\left[1+\tfrac{1}{3}\beta(3j+10)+\tfrac{1}{3\cdot 4\cdot 7}\beta^2(29j^2+134j+384)\right]. \quad (95)$$

Now from (37) the square of the factor for converting $\mathbb{C}_i^1$ into $\mathbf{O}_i^1$ is

$$\left[\frac{1}{D_i^1}(\cos)\right]^2 = 1-\tfrac{1}{2}\beta+\tfrac{1}{32}\beta^2(j-8).$$

Therefore

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} p(\mathbf{P}_i^1\mathbf{O}_i^1)^2 d\sigma = \frac{2\pi M}{2i+1}\frac{i+1!}{i-1!}\left[1+\tfrac{3}{8}\beta(j+2)+\tfrac{1}{3\cdot 8\cdot 4}\beta^2(29j^2+74j+48)\right]. \quad . \quad (95)$$

These last two complete the solution for $s=1$.

Next we have from (83)

$$\frac{N}{H} = \overset{i}{\underset{0}{\Sigma}} \int \frac{a_{2r}}{\Delta}(\cos^{2r+2}\theta + \beta h \cos^{2r}\theta)\, d\theta.$$

Proceeding as before;

$$\frac{N}{H} = \overset{i}{\underset{2}{\Sigma}} \alpha_{2r} f(r+1)\left[1 - \frac{\beta}{2r} + \frac{4r+5}{8r(r-1)}\beta^2 + \frac{2r+1}{2r}\beta h\left(1 - \frac{\beta}{2(r-1)}\right)\right]$$

$$+ \alpha_2 f(2)\left[1 - \tfrac{1}{4}\beta - \tfrac{6}{16}\beta^2 + \tfrac{3}{8}\beta h\left(1 + \tfrac{3}{8}\beta\right)\right]$$

$$+ \alpha_0 f(1)\left[1 + \tfrac{3}{4}\beta + \tfrac{1}{16}\beta^2\right] + \alpha_0 \beta h\left(-\tfrac{5}{16}\beta^2\right).$$

Substituting for h its value (80), we find that the term of order zero is $\overset{i}{\underset{0}{\Sigma}}\alpha_{2r}f(r+1)$,

and by (92) this is equal to $\dfrac{2}{2i+1}$.

The term of the first order is

$$\beta \overset{i}{\underset{2}{\Sigma}} \alpha_{2r} f(r+1)\left[-\frac{1}{2r} + \frac{2r+1}{2r}\right] + \tfrac{1}{2}\beta\alpha_2 + 3\beta\alpha_0,$$

which may be written in the form

$$\beta\overset{i}{\underset{0}{\Sigma}}\alpha_{2r}f(r+1) + \beta\alpha_0, \text{ and is equal to } \frac{2\beta}{2i+1} + \beta.$$

The term of the second order is

$$\beta^2 \overset{i}{\underset{2}{\Sigma}}\alpha_{2r}f(r+1)\left[\frac{4r+5}{8r(r-1)} - \frac{2r+1}{4r(r-1)} + \frac{2r+1}{8r}(j+3)\right] + \beta^2\alpha_2(\tfrac{1}{2}j - \tfrac{7}{12}) + \tfrac{1}{16}\beta^2\alpha_0,$$

which is equal to

$$\tfrac{3}{8}\beta^2\overset{i}{\underset{2}{\Sigma}}\alpha_{2r}\frac{f(r+1)}{r(r-1)} + \tfrac{1}{8}\beta^2(j+3)\overset{i}{\underset{2}{\Sigma}}\alpha_{2r}f(r) + \beta^2\alpha_2(\tfrac{1}{2}j - \tfrac{7}{12}) + \tfrac{1}{16}\beta^2\alpha_0,$$

Now

$$\tfrac{3}{8}\frac{f(r+1)}{r(r-1)} = \tfrac{1}{4}f(r+1) - \tfrac{3}{4}f(r) + \tfrac{1}{4}\frac{r}{r-1}f(r).$$

Hence the term may be written

$$\tfrac{1}{8}\beta^2\overset{i}{\underset{2}{\Sigma}}\alpha_{2r}f(r+1) + \tfrac{1}{4}\beta^2\overset{i-1}{\underset{1}{\Sigma}}\alpha_{2r+2}\frac{r+1}{r}f(r+1) + \tfrac{1}{4}\beta^2 j\overset{i}{\underset{2}{\Sigma}}\alpha_{2r}f(r) + \beta^2\alpha_2(\tfrac{1}{2}j - \tfrac{7}{12}) + \tfrac{1}{16}\beta^2\alpha_0,$$

But

$$\alpha_{2r}f(r) = (-)^r\frac{1}{(r!)^2}\frac{i+r!}{i-r!}.$$

And

$$\frac{r+1}{r}\alpha_{2r+2}f(r+1) = -(-)^r\frac{1}{(r!)^2(r+1)r}\left[i(i+1) - r(r+1)\right]\frac{i+r!}{i-r!}$$

$$= \left[-\frac{(-)^r}{(r!)^2 r} \cdot \frac{j}{(r+1)^2} + \frac{(-)^r}{(r!)^2(r+1)}\right]\frac{i+r!}{i-r!}.$$

In the preceding formula the sum of this last function had limits $i-1$ to 1, but as we now see that it vanishes when $r=i$, the upper limit may be changed to i.

It follows that the terms of the second order are

$$\tfrac{1}{2}\beta^2\sum_0^i a_{2r}f(r+1)-\tfrac{1}{2}\beta^2 a_2 f(2)-\tfrac{1}{2}\beta^2 a_0 f(1).$$

$$+\tfrac{1}{8}\beta^2\sum_1^i\left[-\frac{(-)^r}{(r!)^2r}\cdot\frac{j}{(r+1)^2}+\frac{(-)^r}{(r!)^2(r+1)}\right]\frac{i+r!}{i-r!}+\tfrac{1}{4}\beta^3 j\sum_1^i\frac{(-)^r}{r(r!)^2}\frac{i+r!}{i-r!}$$

$$-\tfrac{1}{4}\beta^3 j a_2 f(1)+\beta^2 a_2(\tfrac{1}{2}j-\tfrac{7}{12})+\tfrac{1}{12}\beta^2 a_0.$$

The term in a_2 in this expression will be found to be $-\tfrac{1}{4}a_2$. That in α_0 will be found to be $\tfrac{1}{12}a_0$. Then since $a_2=-\tfrac{1}{2}j$, $a_0=1$, these terms are together $\tfrac{1}{8}\beta^2(5j+11)$. The whole may then be written

$$\tfrac{1}{2}\beta^2\sum_0^i a_{2r}f(r+1)+\tfrac{1}{8}\beta^2(j+1)\sum_1^i\frac{(-)^r}{r!\,r+1!}\frac{i+r!}{i-r!}+\tfrac{1}{8}\beta^3 j\sum_1^i\frac{(-)^r}{(r+1)^2}\frac{i+r!}{i-r!}+\tfrac{1}{8}\beta^2(5j+11).$$

Now

$$\sum_1^i\frac{(-)^r}{r!\,r+1!}\frac{i+r!}{i-r!}=-\frac{(i+1)i}{1!\,2}+\frac{(i+1)(i+2)i(i-1)}{2!\,2.3}-\ldots=F(i+1,-i,2,1)-1=-1$$

$$\sum_1^i\frac{(-)^r}{(r+1)^2}\frac{i+r!}{i-r!}=-\frac{i(i+1)}{2!\,2!}+\frac{(i+1)(i+2)i(i-1)}{3!\,3!}-\ldots$$

$$=-\frac{1}{i(i+1)}\left[\frac{i(i+1)(-i-1)(-i)}{2!\,1.2}+\frac{i(i+1)(i+2)(-i-1)(-i)(-i+1)}{3!\,1.2.3}+\ldots\right]$$

$$=-\frac{1}{i(i+1)}\left\{F(i,-i-1,1,1)-1+i(i+1)\right\}$$

$$=-\frac{1}{j}F(i,-i-1,1,1)+\frac{1}{j}-1$$

$$=\frac{1}{j}-1.$$

The last result follows from the fact that in accordance with (93) the sum of the hypergeometric series has an infinite factor in the denominator, and vanishes.

Then since by (92) $\sum_0^i a_{2r}f(r+1)=\dfrac{2}{2i+1}$, the terms of the second order are found to be

$$\tfrac{1}{2}\beta^2\cdot\frac{2}{2i+1}+\tfrac{1}{8}\beta^2 j+\tfrac{11}{8}\beta^2.$$

Hence, collecting terms,

$$\frac{N}{H}=(1+\beta+\tfrac{1}{2}\beta^2)\frac{2}{2i+1}+\beta+\tfrac{1}{8}\beta^2(j+11).$$

Substituting for H its value (79), we have N or

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}p\,(\mathfrak{C},\mathrm{P}_i)^2\,d\sigma=\frac{4\pi\mathrm{M}}{2i+1}\left[1-\tfrac{1}{2}\beta+\tfrac{1}{32}\beta^2(j^2-4j+2)\right]+2\pi\mathrm{M}\left[\beta+\tfrac{1}{8}\beta^2(j-1)\right].$$

But we have already found in (77) the value of $\int p\,(\mathfrak{C}_i)^2\,[(\mathfrak{P}_i)^2-(\mathrm{P}_i)^2]\,d\sigma$, and on adding it to the last result the term independent of $1/(2i+1)$ disappears, and we have

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} p\,(\mathfrak{P}_i\,\mathfrak{C}_i)^2\,d\sigma = \frac{4\pi\mathrm{M}}{2i+1}\left[1+\tfrac{1}{2}\beta j+\tfrac{1}{64}\beta^2\,(7j^2-10j)\right] \quad . \quad . \quad . \quad (96).$$

The square of the factor whereby $\mathfrak{C}_i$ is converted into $\mathbf{C}_i$, was found in (38), namely,

$$\frac{1}{(\mathrm{D}_i)^2}=1+\tfrac{1}{8}\,\beta^2\,(j-3).$$

Hence $\displaystyle\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} p\,(\mathfrak{P}_i\,\mathbf{C}_i)^2\,d\sigma = \frac{4\pi\mathrm{M}}{2i+1}\left[1+\tfrac{1}{2}\beta j+\tfrac{1}{64}\beta^2\,(7j^2-2j-24)\right] \quad . \quad . \quad (96).$

These are the last of the required integrals.

§ 22. *Table of Integrals of squares of harmonics.**

In this section the results obtained in (71), (72), (94), (95), and (96) collected.

---

* After having completed the evaluation of all these integrals, I found that they may be evaluated very shortly by means of the factors $\mathfrak{C}$ and $\mathbf{E}$ of (48), § 10.

I find that for *all* values of s (writing the eight forms in a single formula),

$$\int p \left\{\begin{matrix}(\mathfrak{P}_i{}^s)^2 \\ (\mathrm{P}_i{}^s)^2\end{matrix}\right. \times \left.\begin{matrix}(\mathfrak{C}_i{}^s)^2 \text{ or } (\mathbf{C}_i{}^s)^2 \\ (\mathfrak{B}_i{}^s)^2 \text{ or } (\mathbf{8}_i{}^s)^2\end{matrix}\right\} d\sigma = \frac{4\pi\mathrm{M}}{2i+1}(1-\beta)^{\frac{1}{2}} \cdot \left\{\begin{matrix}\mathfrak{C}_i{}^s \\ \mathbf{E}_i{}^s\end{matrix}\right. \times \left\{\text{const. part of } \frac{(\mathfrak{C}_i{}^s \text{ or } \mathbf{C}_i{}^s \text{ or } \mathfrak{B}_i{}^s \text{ or } \mathbf{8}_i{}^s)^2}{(1-\beta\cos 2\phi)^{\frac{1}{2}}}\right\}.$$

I leave the reader to verify that this is so.

Unfortunately I have hitherto been unable to prove the truth of this except by the laborious method in the text. I do not therefore know whether the result remains true for higher degrees of approximation, although I suspect it does so. If it should be true, it would be very easy to compute the integrals when higher powers of β are included.

It may be worth mentioning that the variables are separable in the integrals. Thus, when $\mathfrak{P}_i{}^s\,\mathfrak{C}_i{}^s$ denotes any one of the eight forms,

$$\frac{1}{\mathrm{M}(1-\beta)^{\frac{1}{2}}}\int p\,[\mathfrak{P}_i{}^s\,\mathfrak{C}_i{}^s]^2\,d\sigma = \frac{1}{\sqrt{1+\beta}}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\frac{(\mathfrak{P}_i{}^s)^2\,d\theta}{\Delta}\int_0^{2\pi}(1-\beta\cos 2\phi)^{\frac{1}{2}}\,(\mathfrak{C}_i{}^s)^2\,d\phi$$

$$- \sqrt{\frac{1-\beta}{1+\beta}}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\frac{\sin^2\theta}{\Delta}\,(\mathfrak{P}_i{}^s)^2}{\Delta}\,d\theta\int_0^{2\pi}\frac{(\mathfrak{C}_i{}^s)^2}{(1-\beta\cos 2\phi)^{\frac{1}{2}}}\,d\phi.$$

The ϕ integrals present no difficulty, but with regard to the others we are met by the impossibility of expanding in powers of $\sec^2\theta$ for the lower orders. It would be a great step in the right direction, if it could be proved that all the terms which do not involve the factor $\dfrac{1}{2i+1}$ necessarily vanish.

It may be well to remind the reader that $M = k^3 \nu (\nu^2 - 1)^{\frac{1}{2}} \left(\nu^2 - \frac{1+\beta}{1-\beta} \right)^{\frac{1}{2}}$,

$$\Sigma = \frac{i(i+1)}{s^2 - 1},$$

$$T = \frac{(i-1)i(i+1)(i+2)}{s^2 - 4},$$

$$j = i(i+1).$$

First when $s > 2$.

Types $\begin{Bmatrix} \text{EEC} \\ \text{OOS} \end{Bmatrix}$ $\int p \left(\mathfrak{P}_i^s(\mu) \right) \begin{Bmatrix} \mathfrak{C}_i^s(\phi) \\ \mathfrak{S}_i^s \end{Bmatrix}^2 d\sigma$

$$= \frac{2\pi M}{2i+1} \frac{i+s!}{i-s!} \{ 1 - \tfrac{1}{2}\beta\Sigma + \tfrac{1}{16}\beta^2 [2\Sigma^2 + 3\Sigma - s^2(2\Sigma - 1) + T] \}.$$

Types $\begin{Bmatrix} \text{OEC} \\ \text{EOS} \end{Bmatrix}$ $\int p \left(\mathfrak{P}_i^s(\mu) \right) \begin{Bmatrix} \mathfrak{C}_i^s(\phi) \\ \mathfrak{S}_i^s \end{Bmatrix}^2 d\sigma$

$$= \frac{2\pi M}{2i+1} \frac{i+s!}{i-s!} \{ 1 - \tfrac{1}{2}\beta\Sigma + \tfrac{1}{16}\beta^2 [2\Sigma^2 + \Sigma - 6 - s^2(2\Sigma - 1) + T] \}.$$

Types $\begin{Bmatrix} \text{OOC} \\ \text{EES} \end{Bmatrix}$ $\int p \left(P_i^s(\mu) \right) \begin{Bmatrix} \mathfrak{C}_i^s(\phi) \\ \mathfrak{S}_i^s \end{Bmatrix}^2 d\sigma$

$$= \frac{2\pi M}{2i+1} \frac{i+s!}{i-s!} \{ 1 + \tfrac{1}{2}\beta(\Sigma + 2) + \tfrac{1}{16}\beta^2 [11\Sigma + 10 + s^2(2\Sigma^2 - 2\Sigma + 1) - T] \}.$$

Types $\begin{Bmatrix} \text{EOC} \\ \text{OES} \end{Bmatrix}$ $\int p \left(P_i^s(\mu) \right) \begin{Bmatrix} \mathfrak{C}_i^s(\phi) \\ \mathfrak{S}_i^s \end{Bmatrix}^2 d\sigma$

$$= \frac{2\pi M}{2i+1} \frac{i+s!}{i-s!} \{ 1 + \tfrac{1}{2}\beta(\Sigma + 2) + \tfrac{1}{16}\beta^2 [9\Sigma + 4 + s^2(2\Sigma^2 - 2\Sigma + 1) - T] \}.$$

Secondly, when $s = 2$, $\Sigma = \tfrac{1}{3}i(i+1)$.

Type EEC $\quad \int p \left(\mathfrak{P}_i^2(\mu) \mathfrak{C}_i^2(\phi) \right)^2 d\sigma = \frac{2\pi M}{2i+1} \frac{i+2!}{i-2!} \{ 1 - \tfrac{1}{2}\beta\Sigma + \tfrac{1}{116}\beta^2 (95\Sigma^2 - 98\Sigma + 72) \}.$

Type OEC $\quad \int p \left(\mathfrak{P}_i^2(\mu) \mathbf{C}_i^2(\phi) \right)^2 d\sigma = \frac{2\pi M}{2i+1} \frac{i+2!}{i-2!} \{ 1 - \tfrac{1}{2}\beta\Sigma + \tfrac{1}{116}\beta^2 (95\Sigma^2 - 178\Sigma - 40) \}.$

Type EES $\quad \int p \left(P_i^2(\mu) \mathfrak{S}_i^2(\phi) \right)^2 d\sigma$

$$= \frac{2\pi M}{2i+1} \frac{i+2!}{i-2!} \{ 1 + \tfrac{1}{2}\beta(\Sigma + 2) + \tfrac{1}{116}\beta^2 (29\Sigma^2 + 90\Sigma + 216) \}.$$

Type OES $\quad \int p \left(P_i^2(\mu) \mathbf{S}_i^2(\phi) \right)^2 d\sigma$

$$= \frac{2\pi M}{2i+1} \frac{i+2!}{i-2!} \{ 1 + \tfrac{1}{2}\beta(\Sigma + 2) + \tfrac{1}{116}\beta^2 (29\Sigma^2 + 106\Sigma + 136) \}.$$

Thirdly, when $s = 1$, Σ is infinite and we must use $j = i(i+1)$.

Type OOS $\quad \int p \left(\mathfrak{P}_i^1(\mu) \mathbf{S}_i^1(\phi) \right)^2 d\sigma = \frac{2\pi M}{2i+1} \frac{i+1!}{i-1!} \{ 1 + \tfrac{1}{6}\beta(j-2) + \tfrac{1}{144}\beta^2 (j^2 - 26j + 48) \}.$

Type EOS $\quad \int p\,(\mathfrak{P}_i{}^l(\mu)\,\mathfrak{S}_i{}^l(\phi))^2\,d\sigma = \dfrac{2\pi M}{2i+1}\dfrac{i+1\,!}{i-1\,!}\{1+\tfrac{1}{6}\beta\,(j+2)+\tfrac{1}{816}\beta^2(j^3+10j-96)\}.$

Type OOC $\quad \int p\,(\mathbf{P}_i{}^l(\mu)\,\mathfrak{C}_i{}^l(\phi))^2\,d\sigma$

$$= \dfrac{2\pi M}{2i+1}\dfrac{i+1\,!}{i-1\,!}\{1+\tfrac{1}{6}\beta\,(3j+10)+\tfrac{1}{816}\beta^2(29j^3+134j+384)\}.$$

Type EOC $\quad \int p\,(\mathbf{P}_i{}^l(\mu)\,\mathbf{O}_i{}^l(\phi))^2\,d\sigma = \dfrac{2\pi M}{2i+1}\dfrac{i+1\,!}{i-1\,!}\{1+\tfrac{2}{3}\beta(j+2)+\tfrac{1}{816}\beta^2(29j^3+74j+48)\}.$

Lastly, when $s=0$; $\Sigma=-i\,(i+1)=-j$. There are only two types—

Type EEC $\quad \int p\,(\mathfrak{P}_i\,(\mu)\,\mathfrak{C}_i\,(\phi))^2\,d\sigma = \dfrac{4\pi M}{2i+1}\,[1+\tfrac{1}{2}\beta j+\tfrac{1}{24}\beta^2\,(7j^2-10j)].$

Type OEC $\quad \int p\,(\mathfrak{P}_i\,(\mu)\,\mathbf{O}_i\,(\phi))^2\,d\sigma = \dfrac{4\pi M}{2i+1}\,[1+\tfrac{1}{2}\beta j+\tfrac{1}{24}\beta^2\,(7j^2-2j-24].$

PART III.

SUMMARY.

The symmetrical form in which LAMÉ presented the three functions whose product is a solid ellipsoidal harmonic is such as to render purely analytical investigations both elegant and convenient. But it seemed to me that facility for computation might be gained by the surrender of symmetry, and I have acted on this idea in the preceding paper.

Spheroidal analysis has been successfully employed where the ellipsoid is one of revolution, and it therefore seemed advisable to make that method the point of departure for the treatment of ellipsoids with three unequal axes. In spheroidal harmonics we start with a fundamental prolate ellipsoid of revolution, with imaginary semi-axes $k\sqrt{-1}, k\sqrt{-1}, 0$. The position of a point is then defined by three co-ordinates; the first of these, ν, is such that its reciprocal is the eccentricity of a meridional section of an ellipsoid confocal with the fundamental ellipsoid and passing through the point. Since that eccentricity diminishes as we recede from the origin, ν plays the part of a reciprocal to the radius vector. The second co-ordinate, μ, is the cosine of the auxiliary angle in the meridional ellipse measured from the axis of symmetry. It therefore plays the part of sine of latitude. The third co-ordinate is simply the longitude ϕ. The three co-ordinates may then be described as the radial, latitudinal, and longitudinal co-ordinates. The parameter k defines the absolute scale on which the figure is drawn.

It is equally possible to start with a fundamental oblate ellipsoid with real axes $k, k, 0$. We should then take the first co-ordinate, ζ, as such that $\zeta^2 = -\nu^2$. All

that follows would then be equally applicable; but, in order not to complicate the statement by continual reference to alternative forms, I shall adhere to the first form as a standard.

In this paper a closely parallel notation is adopted for the ellipsoid of three unequal axes. The squares of semi-axes of the fundamental ellipsoid are taken to be $-k^2 \frac{1+\beta}{1-\beta}$, $-k^2$, 0, and the three co-ordinates are still ν, μ, ϕ. Although their geometrical meanings are now by no means so simple, they may still be described as radial, latitudinal, and longitudinal co-ordinates. As before, we might equally well start with a fundamental ellipsoid whose squares of semi-axes are $k^2 \frac{1+\beta}{1-\beta}$, k^2, 0, and replace ν^2 by ζ^2, where $\zeta^2 = -\nu^2$. All possible ellipsoids are comprised in either of these types by making β vary from zero to infinity. But it is shown in § 2 that, by a proper choice of type, all possible ellipsoids are comprised in a range of β from zero to one-third. When β is zero we have the spheroids for which harmonic analysis already exists; and when $\beta = \frac{1}{3}$ the ellipsoid is such that the mean axis is the square root of mean square of the extreme axes. The harmonic analysis for this class of ellipsoid has not been yet worked out, but the method of this paper would render it possible to do so. We may then regard β as essentially less than $\frac{1}{3}$, and may conveniently make developments in powers of β.

In spheroidal analysis, for space internal to an ellipsoid ν_0, two of the three functions are the same P-functions that occur in spherical analysis; one P being a function of ν, the other of μ. The third function is a cosine or sine of a multiple of the longitude ϕ. In external space the P-function of ν is replaced by a Q-function, being a solution of the differential equation of the second kind.

The like is true in ellipsoidal analysis, and we have P- and Q-functions of ν for internal and external space, a P-function of μ, and a cosine- or sine-function of ϕ. I will now for a time set aside the Q-functions and consider them later.

There are eight cases to consider (§ 4); these are determined by the evenness or oddness of the degree i and of the order s of the harmonic, and by the alternative of whether they correspond with a cosine- or sine-function of ϕ. I indicate these eight types by the initials E, O, C, or S—for example, EOS means the type in which i is even, s is odd, and that there is association with a sine-function.

It appears that the new P-functions fall into two forms. The first form, which I write $\mathfrak{P}_i^s$, is found to be expressible in a finite series in terms of the $P_i^{s \pm 2u}$, where the P's are the ordinary functions of spherical analysis. The terms in this series are arranged in powers of β, so that the coefficient of $P_i^{s \pm 2u}$ has β^u as part of its coefficient. The second form, which I write P_i^s, is such that $\sqrt{\frac{\nu^2 - 1}{\nu^2 - \frac{1+\beta}{1-\beta}}}\, \mathsf{P}_i^s(\nu)$ or $\sqrt{\frac{1 - \mu^2}{\frac{1+\beta}{1-\beta} - \mu^2}}\, \mathsf{P}_i^s(\mu)$ is expressible by a series of the same kind as that for $\mathfrak{P}_i^s$. Amongst the eight types four involve $\mathfrak{P}$-functions and four P-functions; and if for

given s a $\mathfrak{P}_i^s$-function is associated with a cosine-function, the corresponding P_i^s is associated with a sine-function, and *vice versâ*.

Lastly, a $\mathfrak{P}$-function of ν is always associated with a $\mathfrak{P}$-function of μ; and the like is true of the P's.

Again, the cosine- and sine-functions fall into two forms. In the first form s and i are either both even or both odd, and the function, which I write $\mathfrak{C}_i^s$ or $\mathfrak{S}_i^s$, is expressed by a series of terms consisting of a coefficient multiplied by $\beta^k \cos$ or $\sin (s \pm 2k)\phi$. In the second form s and i differ as to evenness and oddness, and the function, written C_i^s or S_i^s, is expressed by a similar series multiplied by $(1 - \beta \cos 2\phi)^{\frac{1}{2}}$.

The combination of the two forms of P-function with the four forms of cosine- and sine-function gives the eight types of solid harmonic.

Corresponding to the two forms of P-function there are two forms of Q-function, such that $\mathfrak{Q}_i^s$ and $\mathsf{Q}_i^s \sqrt{\frac{\nu^2-1}{\nu^2-\frac{1+\beta}{1-\beta}}}$ are expansible in a series of ordinary Q-functions; but whereas the series for $\mathfrak{P}_i^s$ and P_i^s are terminable, because P_i^s vanishes when s is greater than i, this is not the case with the Q-functions. In fact the series for the Q-functions begins with Q_i or Q_i^1, and the order of the Q's increases by two at a time up to s when we have the principal or central term; it then goes on increasing up to $s = i$ or $i - 1$, and on to infinity.

In spherical and spheroidal analysis the differential equation satisfied by P_i^s involves the integer s, whereby the order is specified. So here also the differential equations, satisfied by $\mathfrak{P}_i^s$ or P_i^s and by $\mathfrak{C}_i^s$, $\mathfrak{S}_i^s$, or C_i^s, S_i^s, involve a constant, but it is no longer an integer. It seemed convenient to assume $s^2 - \beta\sigma$ as the form for this constant, where s is the known integer specifying the order of harmonic, and σ remains to be determined from the differential equations.

When the assumed forms for the P-function and for the cosine- and sine-functions are substituted in the differential equations, it is found (§ 6) that, in order to satisfy the equations, $\beta\sigma$ must be equal to the difference between two finite-continued fractions, each of which involves $\beta\sigma$. We thus have an equation for $\beta\sigma$, and the required root is that which vanishes when β vanishes.

For the harmonics of degrees 0, 1, 2, 3, and for all orders, σ may be found rigorously in algebraic form, but for higher degrees the equation can only be solved approximately, unless β should have a definite numerical value.

When $\beta\sigma$ has been determined, either rigorously or approximately, the successive coefficients of the series are determinable in such a way that the ratio of each coefficient to the preceding one is expressed by a continued fraction, which is, in fact, a portion of one of the two fractions involved in the equation for $\beta\sigma$.

Throughout the rest of the paper the greater part of the work is carried out with approximate forms, and, although it would be easy to attain to greater accuracy, I have thought it sufficient, in the first instance, to stop at β^2. With this

limitation the coefficients of the series assume simple forms (§ 8), and we have thus definite, if approximate, expressions for all the functions which can occur in ellipsoidal analysis.

In rigorous expressions, $\mathfrak{P}_i^s$ and P_i^s are essentially different from one another, but in approximate forms, when s is greater than a certain integer dependent on the degree of approximation, the two are the same thing in different shapes, except as to a constant factor. I have, therefore, in § 9 determined up to squares of β the factors whereby P_i^s is convertible into $\mathfrak{P}_i^s$, and C_i^s or S_i^s into $\mathfrak{C}_i^s$ or $\mathfrak{S}_i^s$. With the degree of approximation adopted there is no factor for converting the P's when $s = 3, 2, 1$. Similarly, down to $s = 3$ inclusive, the same factor serves for converting C_i^s into $\mathfrak{C}_i^s$ and S_i^s into $\mathfrak{S}_i^s$. But for $s = 2, 1, 0$ one form is needed for changing C into $\mathfrak{C}$, and another for changing S into $\mathfrak{S}$. It may be well to note that there is no sine-function when s is zero.

The use of these factors does much to facilitate the laborious reductions involved in the whole investigation.

It is well known that the Q-functions are expressible in terms of the P-functions by means of a definite integral. Hence $\mathfrak{Q}_i^s$ and Q_i^s must have a second form, which can only differ from the other by a constant factor. The factors connecting the two forms are determined in § 10.

The second part of the paper is devoted to applications of the harmonic method. In § 11 the perpendicular from the centre on to the tangent plane to an ellipsoid ν_0, and the area of an element of surface of the ellipsoid, are found in terms of the co-ordinates μ, ϕ, and the constant ν_0.

It is easy to form a function, continuous at the surface ν_0, which shall be a solid harmonic both for external and for internal space. Poisson's equation then enables us to determine the surface density of which this continuous function is the potential, and it is found to be a surface harmonic of μ, ϕ multiplied by the perpendicular on to the tangent plane. This application of Poisson's equation involves the use of the Q-function in its integral form. Accordingly, if the serial form for the Q-function is adopted as a standard, the expression for the potential of a layer of surface density involves the use of the factor for conversion between the two forms of Q-function.

This result may obviously be employed to determine the potential of an harmonic deformation of a solid ellipsoid.

The potential of the solid ellipsoid itself may be found by the consideration that it is externally equal to that of a focaloid shell of the same mass. It appears that in order to express the equivalent surface density in surface harmonics, it is only necessary to express the reciprocal of the square of the perpendicular on the tangent plane in that form. This result is attained by expressing x^2, y^2, z^2 in surface harmonics. When this done, an application of the preceding theorem enables us to write down the external potential of the solid ellipsoid at once. In § 12 the external potential of the solid ellipsoid is expressed rigorously in terms of solid harmonics of degrees zero and 2.

Since x^2, y^2, z^2 have been found in surface harmonics, we can also write down a rotation-potential about any one of the three axes in the same form.

The internal potential of a solid ellipsoid does not lend itself well to elliptic co-ordinates, but expressions for it are given in § 12.

If it be desired to express any arbitrary function of μ, ϕ in surface harmonics, it is necessary to know the integrals, over the surface of the ellipsoid, of the squares of the several surface harmonics, each multiplied by the perpendicular on to the tangent plane. The rest of the paper is devoted to the evaluation of these integrals. No attempt is made to carry the developments beyond β^3, although the methods employed would render it possible to do so.

When s is greater than unity, it appears that it is legitimate to develop the function to be integrated in powers of $\dfrac{1}{1-\mu^2}$; and when this is done, the integration, although laborious, does not present any great difficulty.

But when s is either 1 or 0, the method of development breaks down, because it would give rise to infinite elements in the integrals at the poles where μ^2 is unity. However, portions of the integrals in these cases can still be found by the former method of development. As to the residues which cannot be so treated, it appears that they depend on integrals of the forms

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\cos^{2s}\theta\,d\theta}{(1-\kappa'^2\sin^2\theta)^{\frac{1}{2}}} \text{ and } \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \cos^{2s}\theta\,(1-\kappa'^2\sin^2\theta)^{\frac{1}{2}}\,d\theta,$$

where κ'^2 is nearly equal to unity.

Development of the square-roots in powers of κ'^2 is useless on account of the slow convergence, and it is required to find series which proceed by powers of κ^2, where $\kappa^2 = 1 - \kappa'^2$.

By a somewhat difficult investigation, in respect to which I owe my special thanks to Mr. Hobson, the needed series are found (§ 19).

It appears that portions of the two integrals involve logarithms which become infinite when κ vanishes. Since, in the application of these integrals, the vanishing of κ implies the vanishing of β, we appear to be met by a difficulty. It is known that in spheroidal analysis no such terms appear, and we may feel confident that they cannot really exist in ellipsoidal analysis. In § 20 it is proved that the logarithmic terms do as a fact disappear. The residues of the integrals in the cases $s = 1$, 0 are thus found, and added to the previous portions to form the complete results.

The second part of the paper ends (§ 22) with a list of the integrals of the squares of the surface harmonics for all values of s, as far as the squares of β.

Finally, an appendix below contains a table of all the functions as far as $i = 5$, $s = 5$. It is probable that for the higher values of s the results would only be applicable when β is very small.

APPENDIX.

Table of the P- and Q-Functions.

$i=0$ (EEC) $\left\{\begin{matrix}\mathfrak{P}_0\\\mathfrak{Q}_0\end{matrix}\right\}=\ \left\{\begin{matrix}P_0\\Q_0\end{matrix}\right.+\tfrac{1}{4}\beta\left\{\begin{matrix}0\\Q_0^2\end{matrix}\right.+\tfrac{1}{144}\beta^2\left\{\begin{matrix}0\\Q_0^4\end{matrix}\right.\cdot$

$i=1$ (OEC) $\left\{\begin{matrix}\mathfrak{P}_1\\\mathfrak{Q}_1\end{matrix}\right\}=\ \left\{\begin{matrix}P_1\\Q_1\end{matrix}\right.+\tfrac{1}{4}\beta\left\{\begin{matrix}0\\Q_1^2\end{matrix}\right.+\tfrac{1}{144}\beta^2\left\{\begin{matrix}0\\Q_1^4\end{matrix}\right.$

(OOC) $\left\{\begin{matrix}P_1^1\\Q_1^1\end{matrix}\right\}=\Omega\left[\left\{\begin{matrix}P_1^1\\Q_1^1\end{matrix}\right.+\tfrac{3}{16}\beta(1+\tfrac{1}{6}\beta)\left\{\begin{matrix}0\\Q_1^3\end{matrix}\right.+\tfrac{5}{768}\beta^2\left\{\begin{matrix}0\\Q_1^5\end{matrix}\right.\right]$

(OOS) $\left\{\begin{matrix}\mathfrak{P}_1^1\\\mathfrak{Q}_1^1\end{matrix}\right\}=\ \left\{\begin{matrix}P_1^1\\Q_1^1\end{matrix}\right.+\tfrac{1}{16}\beta(1-\tfrac{1}{6}\beta)\left\{\begin{matrix}0\\Q_1^3\end{matrix}\right.+\tfrac{1}{768}\beta^2)\left\{\begin{matrix}0\\Q_1^5\end{matrix}\right.\cdot$

$i=2$ (EEC) $\left\{\begin{matrix}\mathfrak{P}_2\\\mathfrak{Q}_2\end{matrix}\right\}=\ \left\{\begin{matrix}P_2\\Q_2\end{matrix}\right.+\tfrac{1}{4}\beta\left\{\begin{matrix}P_2^2\\Q_2^2\end{matrix}\right.+\tfrac{1}{144}\beta^2\left\{\begin{matrix}0\\Q_2^4\end{matrix}\right.$

(EOC) $\left\{\begin{matrix}P_2^1\\Q_2^1\end{matrix}\right\}=\Omega\left[\left\{\begin{matrix}P_2^1\\Q_2^1\end{matrix}\right.+\tfrac{3}{16}\beta(1+\tfrac{2}{6}\beta)\left\{\begin{matrix}0\\Q_2^3\end{matrix}\right.+\tfrac{5}{768}\beta^2\left\{\begin{matrix}0\\Q_2^5\end{matrix}\right.\right]$

(EOS) $\left\{\begin{matrix}\mathfrak{P}_2^1\\\mathfrak{Q}_2^1\end{matrix}\right\}=\ \left\{\begin{matrix}P_2^1\\Q_2^1\end{matrix}\right.+\tfrac{1}{16}\beta(1-\tfrac{2}{6}\beta)\left\{\begin{matrix}0\\Q_2^3\end{matrix}\right.+\tfrac{1}{768}\beta^2\left\{\begin{matrix}0\\Q_2^5\end{matrix}\right.$

(EEC) $\left\{\begin{matrix}\mathfrak{P}_2^2\\\mathfrak{Q}_2^2\end{matrix}\right\}=\ -3\beta\left\{\begin{matrix}P_2\\Q_2\end{matrix}\right.+\left\{\begin{matrix}P_2^2\\Q_2^2\end{matrix}\right.+\tfrac{1}{4}\beta\left\{\begin{matrix}0\\Q_2^4\end{matrix}\right.+\tfrac{1}{144}\beta^2\left\{\begin{matrix}0\\Q_2^6\end{matrix}\right.$

(EES) $\left\{\begin{matrix}P_2^2\\Q_2^2\end{matrix}\right\}=\Omega\left[\qquad\left\{\begin{matrix}P_2^2\\Q_2^2\end{matrix}\right.+\tfrac{1}{12}\beta\left\{\begin{matrix}0\\Q_2^4\end{matrix}\right.+\tfrac{1}{518}\beta^2\left\{\begin{matrix}0\\Q_2^6\end{matrix}\right.\right]\cdot$

$i=3$ (OEC) $\left\{\begin{matrix}\mathfrak{P}_3\\\mathfrak{Q}_3\end{matrix}\right\}=\ \left\{\begin{matrix}P_3\\Q_3\end{matrix}\right.+\tfrac{1}{4}\beta\left\{\begin{matrix}P_3^2\\Q_3^2\end{matrix}\right.+\tfrac{1}{144}\beta^2\left\{\begin{matrix}0\\Q_3^4\end{matrix}\right.$

(OOC) $\left\{\begin{matrix}P_3^1\\Q_3^1\end{matrix}\right\}=\Omega\left[\left\{\begin{matrix}P_3^1\\Q_3^1\end{matrix}\right.+\tfrac{3}{16}\beta(1+\tfrac{3}{6}\beta)\left\{\begin{matrix}P_3^3\\Q_3^3\end{matrix}\right.+\tfrac{5}{768}\beta^2\left\{\begin{matrix}0\\Q_3^5\end{matrix}\right.\right]$

(OOS) $\left\{\begin{matrix}\mathfrak{P}_3^1\\\mathfrak{Q}_3^1\end{matrix}\right\}=\ \left\{\begin{matrix}P_3^1\\Q_3^1\end{matrix}\right.+\tfrac{1}{16}\beta(1-\tfrac{3}{6}\beta)\left\{\begin{matrix}P_3^3\\Q_3^3\end{matrix}\right.+\tfrac{1}{768}\beta^2\left\{\begin{matrix}0\\Q_3^5\end{matrix}\right.$

(OEC) $\left\{\begin{matrix}\mathfrak{P}_3^2\\\mathfrak{Q}_3^2\end{matrix}\right\}=\ -15\beta\left\{\begin{matrix}P_3\\Q_3\end{matrix}\right.+\left\{\begin{matrix}P_3^2\\Q_3^2\end{matrix}\right.+\tfrac{1}{14}\beta\left\{\begin{matrix}0\\Q_3^4\end{matrix}\right.+\tfrac{1}{1356}\beta^2\left\{\begin{matrix}0\\Q_3^6\end{matrix}\right.$

(OES) $\left\{\begin{matrix}P_3^2\\Q_3^2\end{matrix}\right\}=\Omega\left[\qquad\left\{\begin{matrix}P_3^2\\Q_3^2\end{matrix}\right.+\tfrac{1}{12}\beta\left\{\begin{matrix}0\\Q_3^4\end{matrix}\right.+\tfrac{1}{512}\beta^2\left\{\begin{matrix}0\\Q_3^6\end{matrix}\right.\right]$

(OOC) $\left\{\begin{matrix}P_3^3\\Q_3^3\end{matrix}\right\}=\Omega\left[-\tfrac{5}{4}\beta(1+\tfrac{3}{4}\beta)\left\{\begin{matrix}P_3^1\\Q_3^1\end{matrix}\right.+\left\{\begin{matrix}P_3^3\\Q_3^3\end{matrix}\right.+\tfrac{5}{16}\beta\left\{\begin{matrix}0\\Q_3^5\end{matrix}\right.+\tfrac{7}{7680}\beta^2\left\{\begin{matrix}0\\Q_3^7\end{matrix}\right.\right]$

(OOS) $\left\{\begin{matrix}\mathfrak{P}_3^3\\\mathfrak{Q}_3^3\end{matrix}\right\}=\ -\tfrac{15}{4}\beta(1-\tfrac{3}{4}\beta)\left\{\begin{matrix}P_3^1\\Q_3^1\end{matrix}\right.+\left\{\begin{matrix}P_3^3\\Q_3^3\end{matrix}\right.+\tfrac{1}{16}\beta\left\{\begin{matrix}0\\Q_3^5\end{matrix}\right.+\tfrac{1}{1500}\beta^2\left\{\begin{matrix}0\\Q_3^7\end{matrix}\right.$

$i=4$ (EEC) $\quad \begin{Bmatrix} \mathfrak{P}_4 \\ \mathfrak{Q}_4 \end{Bmatrix} = \begin{Bmatrix} P_4 \\ Q_4 \end{Bmatrix} + \tfrac{1}{8}\beta \begin{Bmatrix} P_4^2 \\ Q_4^2 \end{Bmatrix} + \tfrac{1}{136}\beta^2 \begin{Bmatrix} P_4^4 \\ Q_4^4 \end{Bmatrix}$

(EOC) $\quad \begin{Bmatrix} \mathfrak{P}_4^1 \\ \mathfrak{Q}_4^1 \end{Bmatrix} = \Omega \left[\begin{Bmatrix} P_4^1 \\ Q_4^1 \end{Bmatrix} + \tfrac{3}{16}\beta(1+\tfrac{4}{3}\beta) \begin{Bmatrix} P_4^3 \\ Q_4^3 \end{Bmatrix} + \tfrac{3}{784}\beta^2 \begin{Bmatrix} 0 \\ Q_4^5 \end{Bmatrix} \right]$

(EOS) $\quad \begin{Bmatrix} \mathfrak{P}_4^1 \\ \mathfrak{Q}_4^1 \end{Bmatrix} = \begin{Bmatrix} P_4^1 \\ Q_4^1 \end{Bmatrix} + \tfrac{1}{16}\beta(1-\tfrac{4}{3}\beta) \begin{Bmatrix} P_4^3 \\ Q_4^3 \end{Bmatrix} + \tfrac{3}{784}\beta^2 \begin{Bmatrix} 0 \\ Q_4^5 \end{Bmatrix}$

(EEC) $\quad \begin{Bmatrix} \mathfrak{P}_4^2 \\ \mathfrak{Q}_4^2 \end{Bmatrix} = -45\beta \begin{Bmatrix} P_4 \\ Q_4 \end{Bmatrix} + \begin{Bmatrix} P_4^2 \\ Q_4^2 \end{Bmatrix} + \tfrac{1}{24}\beta \begin{Bmatrix} P_4^4 \\ Q_4^4 \end{Bmatrix} + \tfrac{1}{1456}\beta^3 \begin{Bmatrix} P_4^6 \\ Q_4^6 \end{Bmatrix}$

(EES) $\quad \begin{Bmatrix} P_4^2 \\ Q_4^2 \end{Bmatrix} = \Omega \left[\begin{Bmatrix} P_4^2 \\ Q_4^2 \end{Bmatrix} + \tfrac{1}{18}\beta \begin{Bmatrix} P_4^4 \\ Q_4^4 \end{Bmatrix} + \tfrac{1}{8112}\beta^2 \begin{Bmatrix} 0 \\ Q_4^6 \end{Bmatrix} \right]$

(EOC) $\quad \begin{Bmatrix} P_4^3 \\ Q_4^3 \end{Bmatrix} = \Omega \left[-\tfrac{9}{4}\beta(1+\tfrac{1}{3}\beta) \begin{Bmatrix} P_4^1 \\ Q_4^1 \end{Bmatrix} + \begin{Bmatrix} P_4^3 \\ Q_4^3 \end{Bmatrix} + \tfrac{1}{96}\beta \begin{Bmatrix} 0 \\ Q_4^5 \end{Bmatrix} + \tfrac{7}{78780}\beta^2 \begin{Bmatrix} 0 \\ Q_4^7 \end{Bmatrix} \right]$

(EOS) $\quad \begin{Bmatrix} \mathfrak{P}_4^3 \\ \mathfrak{Q}_4^3 \end{Bmatrix} = -\tfrac{3}{4}\beta(1-\tfrac{1}{3}\beta) \begin{Bmatrix} P_4^1 \\ Q_4^1 \end{Bmatrix} + \begin{Bmatrix} P_4^3 \\ Q_4^3 \end{Bmatrix} + \tfrac{1}{32}\beta \begin{Bmatrix} 0 \\ Q_4^5 \end{Bmatrix} + \tfrac{1}{8800}\beta^2 \begin{Bmatrix} 0 \\ Q_4^7 \end{Bmatrix}$

(EEC) $\quad \begin{Bmatrix} \mathfrak{P}_4^4 \\ \mathfrak{Q}_4^4 \end{Bmatrix} = \tfrac{105}{2}\beta^2 \begin{Bmatrix} P_4 \\ Q_4 \end{Bmatrix} - \tfrac{1}{4}\beta \begin{Bmatrix} P_4^2 \\ Q_4^2 \end{Bmatrix} + \begin{Bmatrix} P_4^4 \\ Q_4^4 \end{Bmatrix} + \tfrac{1}{40}\beta \begin{Bmatrix} 0 \\ Q_4^6 \end{Bmatrix} + \tfrac{1}{38640}\beta^2 \begin{Bmatrix} 0 \\ Q_4^8 \end{Bmatrix}$

(EES) $\quad \begin{Bmatrix} P_4^4 \\ Q_4^4 \end{Bmatrix} = \Omega \left[-\tfrac{7}{8}\beta \begin{Bmatrix} P_4^2 \\ Q_4^2 \end{Bmatrix} + \begin{Bmatrix} P_4^4 \\ Q_4^4 \end{Bmatrix} + \tfrac{1}{80}\beta \begin{Bmatrix} 0 \\ Q_4^6 \end{Bmatrix} + \tfrac{1}{11520}\beta^2 \begin{Bmatrix} 0 \\ Q_4^8 \end{Bmatrix} \right].$

$i=5$ (OEC) $\quad \begin{Bmatrix} \mathfrak{P}_5 \\ \mathfrak{Q}_5 \end{Bmatrix} = \begin{Bmatrix} P_5 \\ Q_5 \end{Bmatrix} + \tfrac{1}{8}\beta \begin{Bmatrix} P_5^2 \\ Q_5^2 \end{Bmatrix} + \tfrac{1}{1188}\beta^2 \begin{Bmatrix} P_5^4 \\ Q_5^4 \end{Bmatrix}$

(OOC) $\quad \begin{Bmatrix} \mathfrak{P}_5^1 \\ \mathfrak{Q}_5^1 \end{Bmatrix} = \Omega \left[\begin{Bmatrix} P_5^1 \\ Q_5^1 \end{Bmatrix} + \tfrac{3}{16}\beta(1+\tfrac{1}{5}\beta) \begin{Bmatrix} P_5^3 \\ Q_5^3 \end{Bmatrix} + \tfrac{3}{788}\beta^2 \begin{Bmatrix} P_5^5 \\ Q_5^5 \end{Bmatrix} \right]$

(OOS) $\quad \begin{Bmatrix} \mathfrak{P}_5^1 \\ \mathfrak{Q}_5^1 \end{Bmatrix} = \begin{Bmatrix} P_5^1 \\ Q_5^1 \end{Bmatrix} + \tfrac{1}{16}\beta(1-\tfrac{1}{5}\beta) \begin{Bmatrix} P_5^3 \\ Q_5^3 \end{Bmatrix} + \tfrac{3}{788}\beta^2 \begin{Bmatrix} P_5^5 \\ Q_5^5 \end{Bmatrix}$

(OEC) $\quad \begin{Bmatrix} \mathfrak{P}_5^2 \\ \mathfrak{Q}_5^2 \end{Bmatrix} = -105\beta \begin{Bmatrix} P_5 \\ Q_5 \end{Bmatrix} + \begin{Bmatrix} P_5^2 \\ Q_5^2 \end{Bmatrix} + \tfrac{1}{24}\beta \begin{Bmatrix} P_5^4 \\ Q_5^4 \end{Bmatrix} + \tfrac{1}{1586}\beta^3 \begin{Bmatrix} 0 \\ Q_5^6 \end{Bmatrix}$

(OES) $\quad \begin{Bmatrix} P_5^2 \\ Q_5^2 \end{Bmatrix} = \Omega \left[\begin{Bmatrix} P_5^2 \\ Q_5^2 \end{Bmatrix} + \tfrac{1}{18}\beta \begin{Bmatrix} P_5^4 \\ Q_5^4 \end{Bmatrix} + \tfrac{1}{813}\beta^2 \begin{Bmatrix} 0 \\ Q_5^6 \end{Bmatrix} \right]$

(OOC) $\quad \begin{Bmatrix} P_5^3 \\ Q_5^3 \end{Bmatrix} = \Omega \left[-14\beta(1+\tfrac{1}{5}\beta) \begin{Bmatrix} P_5^1 \\ Q_5^1 \end{Bmatrix} + \begin{Bmatrix} P_5^3 \\ Q_5^3 \end{Bmatrix} + \tfrac{5}{96}\beta \begin{Bmatrix} P_5^5 \\ Q_5^5 \end{Bmatrix} + \tfrac{7}{78660}\beta^3 \begin{Bmatrix} 0 \\ Q_5^7 \end{Bmatrix} \right]$

(OOS) $\quad \begin{Bmatrix} \mathfrak{P}_5^3 \\ \mathfrak{Q}_5^3 \end{Bmatrix} = -42\beta(1-\tfrac{1}{5}\beta) \begin{Bmatrix} P_5^1 \\ Q_5^1 \end{Bmatrix} + \begin{Bmatrix} P_5^3 \\ Q_5^3 \end{Bmatrix} + \tfrac{1}{32}\beta \begin{Bmatrix} P_5^5 \\ Q_5^5 \end{Bmatrix} + \tfrac{1}{3318}\beta^2 \begin{Bmatrix} 0 \\ Q_5^7 \end{Bmatrix}$

(OEC) $\quad \begin{Bmatrix} \mathfrak{P}_5^4 \\ \mathfrak{Q}_5^4 \end{Bmatrix} = \tfrac{945}{2}\beta^2 \begin{Bmatrix} P_5 \\ Q_5 \end{Bmatrix} - 18\beta \begin{Bmatrix} P_5^2 \\ Q_5^2 \end{Bmatrix} + \begin{Bmatrix} P_5^4 \\ Q_5^4 \end{Bmatrix} + \tfrac{1}{40}\beta \begin{Bmatrix} 0 \\ Q_5^6 \end{Bmatrix} + \tfrac{1}{38640}\beta^2 \begin{Bmatrix} 0 \\ Q_5^8 \end{Bmatrix}$

(OES) $\quad \begin{Bmatrix} P_5^4 \\ Q_5^4 \end{Bmatrix} = \Omega \left[-9\beta \begin{Bmatrix} P_5^2 \\ Q_5^2 \end{Bmatrix} + \begin{Bmatrix} P_5^4 \\ Q_5^4 \end{Bmatrix} + \tfrac{1}{80}\beta \begin{Bmatrix} 0 \\ Q_5^6 \end{Bmatrix} + \tfrac{1}{11520}\beta^2 \begin{Bmatrix} 0 \\ Q_5^8 \end{Bmatrix} \right]$

(OOC) $\quad \begin{Bmatrix} P_5^5 \\ Q_5^5 \end{Bmatrix} = \Omega \left[\tfrac{63}{4}\beta^2 \begin{Bmatrix} P_5^1 \\ Q_5^1 \end{Bmatrix} - \tfrac{7}{2}\beta \begin{Bmatrix} P_5^3 \\ Q_5^3 \end{Bmatrix} + \begin{Bmatrix} P_5^5 \\ Q_5^5 \end{Bmatrix} + \tfrac{1}{240}\beta \begin{Bmatrix} 0 \\ Q_5^7 \end{Bmatrix} + \tfrac{3}{86860}\beta^2 \begin{Bmatrix} 0 \\ Q_5^9 \end{Bmatrix} \right]$

(OOS) $\quad \begin{Bmatrix} \mathfrak{P}_5^5 \\ \mathfrak{Q}_5^5 \end{Bmatrix} = \tfrac{21}{4}\beta^2 \begin{Bmatrix} P_5^1 \\ Q_5^1 \end{Bmatrix} - \tfrac{5}{8}\beta \begin{Bmatrix} P_5^3 \\ Q_5^3 \end{Bmatrix} + \begin{Bmatrix} P_5^5 \\ Q_5^5 \end{Bmatrix} + \tfrac{1}{14}\beta \begin{Bmatrix} 0 \\ Q_5^7 \end{Bmatrix} + \tfrac{1}{5788}\beta^2 \begin{Bmatrix} 0 \\ Q_5^9 \end{Bmatrix}.$

Note that in this table P_i^s denotes $\frac{(\nu^2-1)^{\frac{1}{2}s}}{2^i\,i!}\left(\frac{d}{d\nu}\right)^{i+s}(\nu^2-1)^i$, and Ω is $\left(\frac{\nu^2-\frac{1+\beta}{1-\beta}}{\nu^2-1}\right)^{\frac{1}{2}}$.

If the variable is μ, and if accordingly the factor $(\nu^2-1)^{\frac{1}{2}s}$ in P_i^s is replaced by $(1-\mu^2)^{\frac{1}{2}s}$, the signs of all the terms which have β as coefficient must be changed. Ω has still the same meaning, but must be written in the form $\left(\frac{\frac{1+\beta}{1-\beta}-\mu^2}{1-\mu^2}\right)^{\frac{1}{2}}$.

Table of the Cosine and Sine Functions.

$i=0$ (EEC) $\mathbb{C}_0=1.$

$i=1$ (OEC) $\mathbb{C}_1=\Phi$

(OOC)
(OOS) $\left\{\begin{matrix}\mathbb{C}_1^1\\\mathbb{S}_1^1\end{matrix}\right\}=\left\{\begin{matrix}\cos\\\sin\end{matrix}\right\}\phi.$

$i=2$ (EEC) $\mathbb{C}_2=1-\tfrac{3}{2}\beta\cos 2\phi$

(EOC)
(EOS) $\left\{\begin{matrix}\mathbb{C}_2^1\\\mathbb{S}_2^1\end{matrix}\right\}=\Phi\left(\left\{\begin{matrix}\cos\\\sin\end{matrix}\right\}\phi\right)$

(EEC)
(EES) $\left\{\begin{matrix}\mathbb{C}_2^2\\\mathbb{S}_2^2\end{matrix}\right\}=\tfrac{1}{2}\beta\left\{\begin{matrix}1\\0\end{matrix}\right.+\left\{\begin{matrix}\cos\\\sin\end{matrix}\right\}2\phi.$

$i=3$ (OEC) $\mathbb{C}_3=\Phi[1-3\beta\cos 2\phi]$

(OOC)
(OOS) $\left\{\begin{matrix}\mathbb{C}_3^1\\\mathbb{S}_3^1\end{matrix}\right\}=\left\{\begin{matrix}\cos\\\sin\end{matrix}\right\}\phi-\tfrac{5}{8}\beta(1\pm\tfrac{3}{5}\beta)\left\{\begin{matrix}\cos\\\sin\end{matrix}\right\}3\phi$

(OEC)
(OES) $\left\{\begin{matrix}\mathbb{C}_3^2\\\mathbb{S}_3^2\end{matrix}\right\}=\Phi\left[\tfrac{3}{2}\beta\left\{\begin{matrix}1\\0\end{matrix}\right.+\left\{\begin{matrix}\cos\\\sin\end{matrix}\right\}2\phi\right]$

(OOC)
(OOS) $\left\{\begin{matrix}\mathbb{C}_3^3\\\mathbb{S}_3^3\end{matrix}\right\}=\tfrac{3}{8}\beta(1\pm\tfrac{3}{4}\beta)\left\{\begin{matrix}\cos\\\sin\end{matrix}\right\}\phi+\left\{\begin{matrix}\cos\\\sin\end{matrix}\right\}3\phi.$

$i=4$ (EEC) $\mathbb{C}_4=1-5\beta\cos 2\phi+\tfrac{35}{16}\beta^2\cos 4\phi$

(EOC)
(EOS) $\left\{\begin{matrix}\mathbb{C}_4^1\\\mathbb{S}_4^1\end{matrix}\right\}=\Phi\left[\left\{\begin{matrix}\cos\\\sin\end{matrix}\right\}\phi-\tfrac{7}{8}\beta(1\pm\tfrac{1}{4}\beta)\left\{\begin{matrix}\cos\\\sin\end{matrix}\right\}3\phi\right]$

(EEC)
(EES) $\left\{\begin{matrix}\mathbb{C}_4^2\\\mathbb{S}_4^2\end{matrix}\right\}=\tfrac{6}{7}\beta\left\{\begin{matrix}1\\0\end{matrix}\right.+\left\{\begin{matrix}\cos\\\sin\end{matrix}\right\}2\phi-\tfrac{7}{12}\beta\left\{\begin{matrix}\cos\\\sin\end{matrix}\right\}4\phi.$

(EOC)
(EOS) $\left\{\begin{matrix}\mathbb{C}_4^3\\\mathbb{S}_4^3\end{matrix}\right\}=\Phi[\tfrac{7}{8}\beta(1\pm\tfrac{1}{4}\beta)\left\{\begin{matrix}\cos\\\sin\end{matrix}\right\}\phi+\left\{\begin{matrix}\cos\\\sin\end{matrix}\right\}3\phi]$

(EEC)
(EES) $\left\{\begin{matrix}\mathbb{C}_4^4\\\mathbb{S}_4^4\end{matrix}\right\}=\tfrac{3}{16}\beta^2\left\{\begin{matrix}1\\0\end{matrix}\right.+\tfrac{1}{3}\beta\left\{\begin{matrix}\cos\\\sin\end{matrix}\right\}2\phi+\left\{\begin{matrix}\cos\\\sin\end{matrix}\right\}4\phi.$

$i = 5$

(OEC) $\mathbf{C}_5 = \Phi[1 - 7\beta\cos 2\phi + \tfrac{91}{12}\beta^2\cos 4\phi]$

$\begin{matrix}(\text{OOC})\\(\text{OOS})\end{matrix}$ $\begin{Bmatrix}\mathfrak{C}_5^1\\\mathfrak{S}_5^1\end{Bmatrix} = \begin{Bmatrix}\cos\\\sin\end{Bmatrix}\phi - \tfrac{7}{4}\beta(1\pm\tfrac{14}{8}\beta)\begin{Bmatrix}\cos\\\sin\end{Bmatrix}3\phi + \tfrac{21}{16}\beta^2\begin{Bmatrix}\cos\\\sin\end{Bmatrix}5\phi$

$\begin{matrix}(\text{OEC})\\(\text{OES})\end{matrix}$ $\begin{Bmatrix}\mathbf{C}_5^2\\\mathbf{S}_5^2\end{Bmatrix} = \Phi\left[\tfrac{14}{4}\beta\begin{Bmatrix}1\\0\end{Bmatrix} + \begin{Bmatrix}\cos\\\sin\end{Bmatrix}2\phi - \tfrac{2}{4}\beta\begin{Bmatrix}\cos\\\sin\end{Bmatrix}4\phi\right]$

$\begin{matrix}(\text{OOC})\\(\text{OOS})\end{matrix}$ $\begin{Bmatrix}\mathfrak{C}_5^3\\\mathfrak{S}_5^3\end{Bmatrix} = \tfrac{3}{2}\beta(1\pm\tfrac{14}{8}\beta)\begin{Bmatrix}\cos\\\sin\end{Bmatrix}\phi + \begin{Bmatrix}\cos\\\sin\end{Bmatrix}3\phi - \tfrac{9}{16}\beta\begin{Bmatrix}\cos\\\sin\end{Bmatrix}5\phi$

$\begin{matrix}(\text{OEC})\\(\text{OES})\end{matrix}$ $\begin{Bmatrix}\mathbf{C}_5^4\\\mathbf{S}_5^4\end{Bmatrix} = \Phi\left[\tfrac{14}{6}\beta^2\begin{Bmatrix}1\\0\end{Bmatrix} + \beta\begin{Bmatrix}\cos\\\sin\end{Bmatrix}2\phi + \begin{Bmatrix}\cos\\\sin\end{Bmatrix}4\phi\right]$

$\begin{matrix}(\text{OOC})\\(\text{OOS})\end{matrix}$ $\begin{Bmatrix}\mathfrak{C}_5^5\\\mathfrak{S}_5^5\end{Bmatrix} = \tfrac{3}{32}\beta^2\begin{Bmatrix}\cos\\\sin\end{Bmatrix}\phi + \tfrac{5}{16}\beta\begin{Bmatrix}\cos\\\sin\end{Bmatrix}3\phi + \begin{Bmatrix}\cos\\\sin\end{Bmatrix}5\phi.$

Note that in this table

$$\Phi = (1 - \beta\cos 2\phi)^{\frac{1}{2}}.$$

A table of $P(\nu)$ and $Q(\nu)$ up to $i = 5$, $s = 5$ is contained in Professor BRYAN'S paper ('Proc. Camb. Phil. Soc.,' vol. vi., 1888, p. 297). The functions there tabulated as $T_n^s(\nu)$ and $U_n^s(\nu)$ in the notation here adopted, would be $P_n^s(\nu)$ (with the factor $(\nu^2 - 1)^{\frac{1}{2}s}$) and $(-)^s\frac{i-s!}{i+s!}Q_n^s(\nu)$.

The formula for $Q_i^s(\nu)$, where s is greater than i, is given in § 10 above.

INDEX

TO THE

PHILOSOPHICAL TRANSACTIONS,

SERIES A, VOL. 197.

4 C

ERRATA.

Page 183. In the fourth column of the table, *for* " 18 h. 19 m. 25 s." *read* " 16 h. 19 m. 2·5 s."
 „   187   17th line, *for* " taken to the position," *read* " taken to be the position. "
 „   190. 27th line, *insert* " *b* " after " trace of."
 „   194. Lines 1 and 2, *for* " inches" and "inch" *read* " seconds of arc "
 „   199. Last line, *for* " Hγ " *read* " Hν " ; and after " Hρ " read on to next page without stop.
 „   216   In the column " Remarks," opposite λ 4177·75, *for* " Straight " *read* " Strongest. "